

5ECM Amsterdam 2008

Minisymposium HAGFI

Hypoellipticity, Analysis on Groups and Functional Inequalities

Evolution equations on homogeneous spaces

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July 15, 2008

3 evolution equations

Heat equation

$$\begin{cases} \partial_t u(t, x) - \Delta_x u(t, x) = g(t, x) \\ u(0, x) = f(x) \end{cases}$$

Schrödinger equation

$$\begin{cases} i \partial_t u(t, x) \pm \Delta_x u(t, x) = g(t, x) \\ u(0, x) = f(x) \end{cases}$$

Wave equation

$$\begin{cases} \partial_t^2 u(t, x) - \Delta_x u(t, x) = g(t, x) \\ u(0, x) = f_0(x), \partial_t|_{t=0} u(t, x) = f_1(x) \end{cases}$$

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- homogeneous \rightsquigarrow linear case : $g=0$
- possible nonlinearities : $g(t, x) = \Gamma(u(t, x))$

$$\text{e.g. } \Gamma(u) = \begin{cases} u |u|^{2\sigma} \\ |u|^{1+2\sigma} \end{cases} \quad \text{with } \sigma > 0$$

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Symmetric spaces

type	constant curvature	rank one	general case
Euclidean	$\mathbb{R}^n, \mathbb{T}^n$	$\mathbb{R}_{(\text{radial})}^{(n)}, \mathbb{T}$	$\mathfrak{p} \rtimes K/K$
compact	$\mathbb{S}^{n-1} = \mathbb{S}(\mathbb{R}^n)$	$\mathbb{S}(\mathbb{F}^n)$	U/K
non compact	$\mathbb{H}^n = \mathbb{H}^n(\mathbb{R})$	$\mathbb{H}^n(\mathbb{F})$	G/K
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Symmetric spaces

Other interesting cases :

- Cartan–Hadamard manifolds
- Lie groups $\left\{ \begin{array}{l} \text{polynomial growth (compact, nilpotent)} \\ \text{semisimple} \\ \text{amenable (solvable) exponential growth} \end{array} \right.$
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Heat equation on symmetric spaces

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Solution

$$u(t, x) = \underbrace{e^{t\Delta_x} f(x)}_{\text{homogeneous}} + \underbrace{\int_0^t ds e^{(t-s)\Delta_x} g(s, x)}_{\text{inhomogeneous}}$$

Homogeneous solution :

$$u(t, x) = e^{t\Delta_x} f(x) = f * \underbrace{h_t}_{\text{heat kernel}}(x)$$

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Heat kernel on \mathbb{H}^n

Explicit expression for \mathbb{H}^n [Debiard–Gaveau–Mazet 1976]

$$h_t(r) = 2^{-\frac{n+1}{2}} \pi^{-\frac{n}{2}} t^{-\frac{1}{2}} e^{-(\frac{n-1}{2})^2 t} \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r}\right)^{\frac{n-1}{2}} e^{-\frac{r^2}{4t}}$$

where

$$\left(-\frac{1}{\sinh r} \frac{\partial}{\partial r}\right)^{\frac{n-1}{2}} f(r) = \frac{1}{\sqrt{\pi}} \int_r^{+\infty} ds \frac{\sinh s}{\sqrt{\cosh s - \cosh r}} \left(-\frac{1}{\sinh s} \frac{\partial}{\partial s}\right)^{\frac{n}{2}} f(s)$$

if n is even.

Similar expression for

- all hyperbolic spaces $\mathbb{H}^n(\mathbb{F}) = \mathbb{H}^n(\mathbb{R}), \mathbb{H}^n(\mathbb{C}), \mathbb{H}^n(\mathbb{H}), \mathbb{H}^2(\mathbb{O})$
[Lohoué–Rychener 1982]
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Heat kernel on \mathbb{H}^n

Sharp global estimate [Davies–Mandouvalos 1988]

$$h_t(r) \asymp t^{-\frac{n}{2}} (1+r) (1+t+r)^{\frac{n-3}{2}} e^{-(\frac{n-1}{2})^2 t} e^{-\frac{n-1}{2} r} e^{-\frac{r^2}{4t}}$$

$\forall t > 0$ and $\forall r \geq 0$

Comments :

- spectral gap $\rightsquigarrow e^{-(\frac{n-1}{2})^2 t}$
- Gaussian $e^{-\frac{r^2}{4t}}$
- $t \leq 1+r \rightsquigarrow t^{-\frac{n}{2}} j(r)^{-\frac{1}{2}}$
where $j(r) = \left(\frac{\sinh r}{r}\right)^{n-1}$ jacobian of exponential map
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Heat kernel on G/K

- $h_t(x)$ bi- K -invariant Schwartz function on G
- no explicit expression in general
- tool = inverse spherical Fourier transform :

$$h_t(x) = \text{const.} \int_{\mathfrak{a}} d\lambda \underbrace{|\mathbf{c}(\lambda)|^{-2}}_{\text{Plancherel measure}} e^{-t(|\rho|^2 + |\lambda|^2)} \underbrace{\phi_\lambda(x)}_{\text{spherical functions}}$$

- Cartan decomposition $G = K(\exp \overline{\mathfrak{a}^+})K$
 \rightsquigarrow estimate h_t on the positive Weyl chamber $\overline{\mathfrak{a}^+}$

Sharp global estimate [A–Ji 1999 ; A–Ostellari 2003]

$$h_t(\exp X) \asymp t^{-\frac{n}{2}} \left\{ \prod_{\alpha \in \Sigma_{\text{red}}^+} (1 + \langle \alpha, X \rangle) (1 + t + \langle \alpha, X \rangle)^{\frac{m_\alpha + m_{2\alpha} - 1}{2}} \right\} \\ \times e^{-|\rho|^2 t} e^{-\langle \rho, X \rangle} e^{-\frac{|X|^2}{4t}}$$

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Heat kernel on G/K

In particular

- $h_t(\exp X) \asymp t^{-\frac{n}{2}} e^{-|\rho|^2 t} j(X)^{-\frac{1}{2}} e^{-\frac{|X|^2}{4t}}$
if $t \lesssim 1 + \langle \alpha, X \rangle$ for all simple \rightsquigarrow positive roots α
- $h_t(x) \asymp t^{-\frac{d}{2}} e^{-|\rho|^2 t} \phi_0(x) e^{-\frac{|x|^2}{4t}}$
if $t \gtrsim 1 + |x|$

where $n = \dim G/K$ and $d = \text{rank } G/K + 2 |\Sigma_{\text{red}}^+|$

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Green function on G/K

$$\begin{aligned} g_\zeta(x) &= \text{integral kernel of } (-\Delta - |\rho|^2 + \zeta^2)^{-1} \\ &= \int_0^{+\infty} ds e^{(\rho^2 - \zeta^2)t} h_t(x) \end{aligned}$$

Behavior at infinity [A–Ji 1999]

$$g_\zeta(x) \asymp \begin{cases} |x|^{\frac{1-d}{2}} \phi_0(x) e^{-\zeta|x|} & \text{if } \zeta > 0 \\ |x|^{2-d} \phi_0(x) & \text{if } \zeta = 0 \end{cases}$$

Consequence [Guivarc’h–Ji–Taylor 1998] :
Martin compactification of G/K

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[Ostellari 2003]

$\Delta = \mathfrak{p}$ -component of Casimir + anything in \mathfrak{k}
Same estimate as for G/K when $t \geq 1 + |x|$

[Alexopoulos–Lohoué 2003]

$\Delta = \sum_j X_j^2$ any invariant subLaplacian on G

- Upper estimate for $t \geq 1$ and $|x| = O(t)$:

$$h_t(x) \leq C t^{-\frac{d}{2}} e^{-\Lambda t} \phi_0(x) e^{-c \frac{|x|^2}{t}}$$

- Lower estimate for $t \geq 1$ and $|x| = O(\sqrt{t})$:

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Further results

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- [Bruno Schapira 2008] Heat kernel associated to the Heckman–Opdam Laplacian
- [...; A–Bougerol–Jeulin 2002] Brownian motion on G/K
- [Lalley 1991] Sharp global estimate for random walks on homogeneous trees
- [Parkinson 2007] Local and central limit theorems on affine buildings
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- [Steger–Trojan] Asymptotic behavior of random walks on affine buildings

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Schrödinger equation on symmetric spaces

Schrödinger equation

$$\begin{cases} i \partial_t u(t, x) + \Delta_x u(t, x) = g(t, x) \\ u(0, x) = f(x) \end{cases}$$

Solution

$$u(t, x) = \underbrace{e^{it\Delta_x} f(x)}_{\text{homogeneous}} + \underbrace{\int_0^t ds e^{i(t-s)\Delta_x} g(s, x)}_{\text{inhomogeneous}}$$

Homogeneous solution :

$$u(t, x) = e^{it\Delta_x} f(x) = f * \underbrace{S_t}_{\text{Schrödinger kernel}}(x)$$

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Schrödinger kernel on \mathbb{H}^n

Explicit expression

$$s_t(r) = 2^{-\frac{n+1}{2}} \pi^{-\frac{n}{2}} \underbrace{(it)^{-\frac{1}{2}}}_{e^{-i \operatorname{sign}(t) \frac{\pi}{4}} |t|^{-\frac{1}{2}}} e^{-i(\frac{n-1}{2})^2 t} \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r}\right)^{\frac{n-1}{2}} e^{\frac{i}{4} \frac{r^2}{t}}$$

Global estimate

$$|s_t(r)| \lesssim \begin{cases} |t|^{-n/2} j(r)^{-1/2} & \text{if } |t| \leq 1+r \\ |t|^{-3/2} \varphi_0(r) & \text{if } |t| \geq 1+r \end{cases}$$

2 fundamental inequalities

Dispersive estimate [A–Pierfelice]

Let $2 \leq q, \tilde{q} \leq \infty$

- $0 < |t| \leq 1$: $\|e^{it\Delta}\|_{L^{q'} \rightarrow L^{\tilde{q}}} \lesssim |t|^{-\max\{\frac{1}{2} - \frac{1}{q}, \frac{1}{2} - \frac{1}{\tilde{q}}\}n}$
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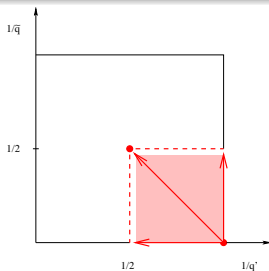
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2 fundamental inequalities

Hint : The crucial diagonal inequality

$$\|e^{it\Delta}\|_{L^{q'} \rightarrow L^q} \lesssim |t|^{-\frac{3}{2}} \quad \text{when} \quad \begin{cases} |t| \geq 1 \\ q > 2 \end{cases}$$

follows from the estimate

$$\int_0^{+\infty} dr |s_t(r)|^{\frac{q}{2}} \phi_0(r) \lesssim |t|^{-\frac{3}{2}}$$

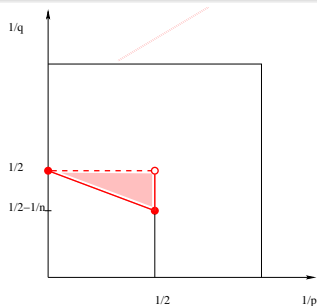
and from the sharp Kunze–Stein phenomenon [Ionescu 2000]

2 fundamental inequalities

Strichartz estimate

$$\|u(t, x)\|_{L_t^{\tilde{p}} L_x^{\tilde{q}}} \lesssim \|f(x)\|_{L_x^2} + \|g(t, x)\|_{L_t^{p'} L_x^{q'}}$$

$\forall (p, q), (\tilde{p}, \tilde{q})$ in the *admissible triangle*



Application to NLS

Wellposedness \sim existence & uniqueness

Consider the NLS

$$\begin{cases} i \partial_t u(t, x) + \Delta_x u(t, x) = \Gamma(u(t, x)) \\ u(0, x) = f(x) \end{cases}$$

with powerlike nonlinearities

$$\begin{cases} |\Gamma(u)| \lesssim |u|^{1+2\sigma} \\ |\Gamma(u) - \Gamma(v)| \lesssim |u - v| \{ |u|^{2\sigma} + |v|^{2\sigma} \} \end{cases}$$

Then we have global wellposedness for small initial data

$$\begin{cases} \text{in } L^2(\mathbb{H}^n) & \text{if } \sigma \leq \frac{2}{n} \\ \text{in } H^1(\mathbb{H}^n) & \text{if } \sigma \leq \frac{2}{n-2} \end{cases}$$

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LWP \rightsquigarrow GWP for arbitrary data and subcritical σ

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Better dispersion in $\mathbb{H}^n \rightsquigarrow$ stronger results than in \mathbb{R}^n

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Further results

- Previous results under radial assumptions
[Banica 2007 ; Pierfelice ; Banica–Carles–Staffinali]
- [Ionescu–Staffilani] Related results in the defocusing case
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Wave equation on symmetric spaces

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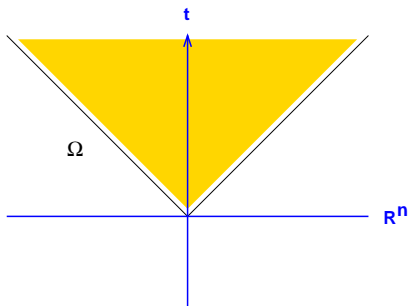
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- [Georgiev–Lindblat–Sogge 1997] GWP for NLW in \mathbb{R}^n
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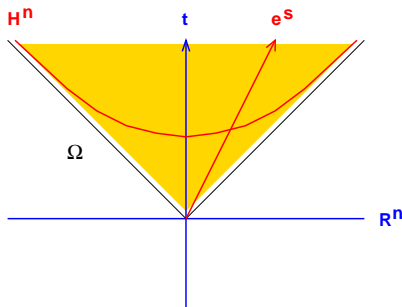
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$$\begin{cases} t^2 - x_1^2 - \cdots - x_n^2 > 0 \\ t > 0 \end{cases}$$



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Wave operator (D'Alembertian)

$$\square = \partial_t^2 - \Delta_{\mathbb{R}^n} = e^{-\frac{n+3}{2}s} \circ \underbrace{\left\{ \partial_s^2 - \Delta_{\mathbb{H}^n} - \left(\frac{n-1}{2}\right)^2 \right\}}_{-\mathcal{L}_x} \circ e^{\frac{n-1}{2}s}$$

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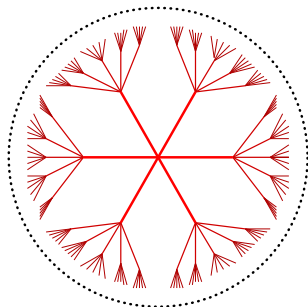
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Homogeneous trees

$X = \mathbb{T}_Q$ homogeneous tree with $Q+1$ edges

Example : $Q=5$



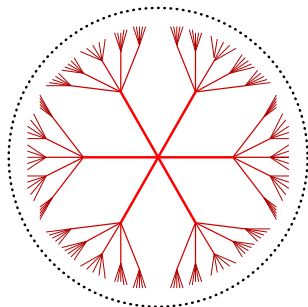
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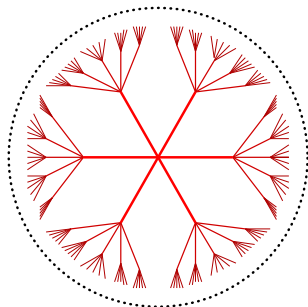
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Wave equation on \mathbb{T}_Q

$$\gamma \Delta_n^{\mathbb{Z}} u(n, x) - \underbrace{\{\Delta_x^X + 1 - \gamma\}}_{\mathcal{L}} u(n, x) = g(n, x)$$

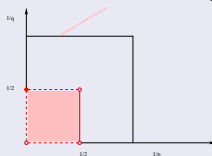
$$\text{Laplacian on } \mathbb{Z} : \Delta^{\mathbb{Z}} f(n) = \frac{f(n+1) + f(n-1) - 2f(n)}{2}$$

$$\text{Laplacian on } X : \Delta^X f(x) = \frac{1}{Q+1} \sum_{d(y,x)=1} f(y) - f(x)$$

[A–Pierfelice–Vallarino]

Dispersive estimate : exponential decay in n

Strichartz estimate : valid for the full square



NLW : no critical power

Further results

- [Ionescu 2000]
 $L^q \rightarrow L^q$ estimates for $\cos t\sqrt{-\mathcal{L}}$ and $\frac{\sin t\sqrt{-\mathcal{L}}}{\sqrt{-\mathcal{L}}}$
on hyperbolic spaces
- [Cowling–Giulini–Meda 2001 ; Cowling–Meda 2002]
 $L^{q_1} \rightarrow L^{q_2}$ estimates for $e^{-t\sqrt{-\Delta-|\rho|^2+\zeta^2}}$ on G/K
for restricted complex time t