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Nonlinear dispersive equations on hyperbolic spaces

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Joint work
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Evolution equations

Heat equation

$$\begin{cases} \partial_t u(t,x) - \Delta_x u(t,x) = g(t,x) \\ u(0,x) = f(x) \end{cases}$$

Schrödinger equation

$$\begin{cases} i\,\partial_t u(t,x) \pm \Delta_x u(t,x) = g(t,x) \\ u(0,x) = f(x) \end{cases}$$

Wave equation

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = g(t,x) \\ u(0,x) = f_0(x),\ \partial_t|_{t=0} u(t,x) = f_1(x) \end{cases}$$

Evolution equations

- homogeneous $\rightsquigarrow$ linear case : $g=0$
- possible nonlinearities : $g(t,x)=\Gamma(u(t,x))$

  e.g. $\Gamma(u)=\begin{cases}|u|^{\gamma}\\ u\,|u|^{\gamma-1}\end{cases}$ with $\gamma>1$

Symmetric spaces

type	constant curvature	rank one	general case
Euclidean	$\mathbb{R}^n$, $\mathbb{T}^n$	$\mathbb{R}^{(n)}_{(\text{radial})}$, $\mathbb{T}$	$\mathfrak{p} \rtimes K/K$
compact	$\mathbb{S}^{n-1} = \mathsf{S}(\mathbb{R}^n)$	$\mathsf{S}(\mathbb{F}^n)$	U/K
non compact	$\mathbb{H}^n = \mathsf{H}^n(\mathbb{R})$	$\mathsf{H}^n(\mathbb{F})$	G/K
p–adic		trees	buildings

Symmetric spaces

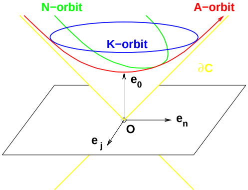
Other interesting cases :

- Cartan–Hadamard manifolds
- Lie groups $\begin{cases} \text{polynomial growth (compact, nilpotent)} \\ \text{semisimple} \\ \text{amenable (solvable) exponential growth} \end{cases}$
- locally symmetric spaces

Real hyperbolic spaces

3 models :

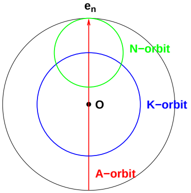
- Hyperboloid $\mathbb{H}^n$: $\begin{cases} x_0^2 - x_1^2 - \ldots - x_n^2 = 1 \\ x_0 > 0 \end{cases}$



metric : $ds^2 = -\,dx_0^2 + dx_1^2 + \ldots + dx_n^2$
Laplacian : $\Delta f = \left(-\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^n}{\partial x_n^2}\right) \widetilde{f}\,\Big|_{\mathbb{H}^n}$
where f is extended by homogeneity of degree 0
to the light cone C : $\begin{cases} x_0^2 - x_1^2 - \ldots - x_n^2 > 0 \\ x_0 > 0 \end{cases}$

Real hyperbolic spaces

- Ball $\mathbb{B}^n$

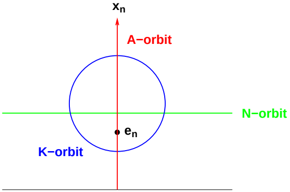


metric : $ds^2 = \frac{4}{(1-|x|^2)^2}\,|dx|^2$

Laplacian : $\Delta = \frac{(1-|x|^2)^2}{4}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} + (n-2)\,\frac{1-|x|^2}{2}\sum_{j=1}^{n} x_j\,\frac{\partial}{\partial x_j}$

Real hyperbolic spaces

- Half–space $\mathbb{R}^n_+$



metric : $ds^2 = x_n^{-2}\,|dx|^2$

Laplacian : $\Delta = x_n^2 \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} - (n-2)\, x_n \frac{\partial}{\partial x_n}$

Schrödinger equation on $\mathbb{H}^n$

Assume $n \geq 2$

Schrödinger equation

$$\begin{cases} i\,\partial_t u(t,x) + \Delta_x u(t,x) = g(t,x) \\ u(0,x) = f(x) \end{cases}$$

Fourier transform :

$$\begin{cases} i\,\partial_t \widehat{u}(t,\lambda) - \left\{ (\frac{n-1}{2})^2 + \lambda^2 \right\} \widehat{u}(t,\lambda) = \widehat{g}(t,\lambda) \\ \widehat{u}(0,\lambda) = \widehat{f}(\lambda) \end{cases}$$

Solution :

$$\widehat{u}(t,\lambda) = \underbrace{e^{-i\{(\frac{n-1}{2})^2+\lambda^2\}\,t}\,\widehat{f}(\lambda)}_{\text{homogeneous}} + \underbrace{\int_0^t ds\; e^{-i\{(\frac{n-1}{2})^2+\lambda^2\}(t-s)}\,\widehat{g}(s,\lambda)}_{\text{inhomogeneous}}$$

Schrödinger equation on $\mathbb{H}^n$

Assume $n \geq 2$

Schrödinger equation

$$\begin{cases} i\,\partial_t u(t,x) + \Delta_x u(t,x) = g(t,x) \\ u(0,x) = f(x) \end{cases}$$

Solution

$$u(t,x) = \underbrace{e^{\,i\,t\,\Delta_x} f(x)}_{\text{homogeneous}} + \underbrace{\int_0^t ds\; e^{\,i(t-s)\Delta_x} g(s,x)}_{\text{inhomogeneous}}$$

Homogeneous solution :

$$u(t,x) = e^{\,i\,t\,\Delta_x} f(x) = f * \underbrace{s_t}_{\text{Schrödinger kernel}}(x)$$

Schrödinger kernel on $\mathbb{H}^n$

Explicit expression

$$s_t(r) = 2^{-\frac{n+1}{2}} \pi^{-\frac{n}{2}} \underbrace{(it)^{-\frac{1}{2}}}_{e^{-i\,\mathrm{sign}(t)\frac{\pi}{4}}\,|t|^{-\frac{1}{2}}} e^{-i(\frac{n-1}{2})^2 t} \left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{\frac{n-1}{2}} e^{\frac{i}{4}\frac{r^2}{t}}$$

where

$$\left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{\frac{n-1}{2}} f(r) = \frac{1}{\sqrt{\pi}} \int_r^{+\infty} ds\, \frac{\sinh s}{\sqrt{\cosh s - \cosh r}} \left(-\frac{1}{\sinh s}\frac{\partial}{\partial s}\right)^{\frac{n}{2}} f(s)$$

if n is even.

Similar expression for

- all hyperbolic spaces $\mathrm{H}^n(\mathbb{F}) = \mathrm{H}^n(\mathbb{R}), \mathrm{H}^n(\mathbb{C}), \mathrm{H}^n(\mathbb{H}), \mathrm{H}^2(\mathbb{O})$
- Damek–Ricci spaces

Heat kernel :

- $\mathbb{H}^n$ [Debiard–Gaveau–Mazet 1976]
- $\mathrm{H}^n(\mathbb{F})$ [Lohoué–Rychener 1982]

Schrödinger kernel on $\mathbb{H}^n$

Global estimate [A–P 2009]

$$|s_t(r)| \lesssim \begin{cases} |t|^{-n/2}\, j(r)^{-1/2} & \text{if } |t| \le 1+r \\ |t|^{-3/2}\, \varphi_0(r) & \text{if } |t| \ge 1+r \end{cases}$$

$\forall\ t>0$ and $\forall\ r\ge 0$,

where $j(r) = \left(\frac{\sinh r}{r}\right)^{n-1}$ jacobian of exponential map

and $\varphi_0(r) \asymp (1+r)\, e^{-\frac{n-1}{2} r}$ fundamental spherical function

Heat kernel estimate for complex time [Davies–Mandouvalos 1988, Mandouvalos] :

$$|h_t(r)| \lesssim |t|^{-\frac{n}{2}} (1+r)(1+r+|t|)^{\frac{n-3}{2}} e^{-\frac{n-1}{2} r} e^{-\operatorname{Re}\{(\frac{n-1}{2})^2 t + \frac{r^2}{4t}\}}$$

$\forall\ \operatorname{Re} t \ge 0,\ \forall\ r \ge 0$

2 fundamental inequalities

Dispersive estimate [A–P 2009]

Let $2 < q \leq \infty$

Then

$$\| e^{it\Delta} \|_{L^{q'} \to L^q} \lesssim \begin{cases} |t|^{-(\frac{1}{2}-\frac{1}{q})n} & \text{if } 0 < |t| < 1 \\ |t|^{-\frac{3}{2}} & \text{if } |t| \geq 1 \end{cases}$$

2 fundamental inequalities

Hint : The crucial large time inequality

$$\| e^{it\Delta} \|_{L^{q'} \to L^q} \lesssim |t|^{-\frac{3}{2}}$$

follows from the estimate

$$\Big\{ \int_0^{+\infty} dr \, |s_t(r)|^{\frac{q}{2}} \, \phi_0(r) \Big\}^{\frac{2}{q}} \lesssim |t|^{-\frac{3}{2}}$$

and from the sharp Kunze–Stein phenomenon [Ionescu 2000]

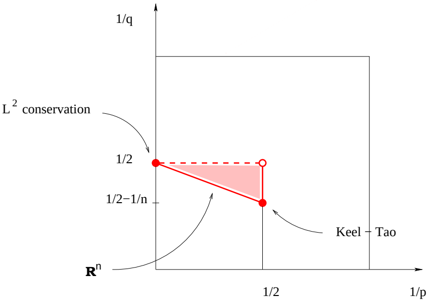
$$L^{2,1}(G) * L^{2,1}(G) \subset L^{2,\infty}(G)$$

2 fundamental inequalities

Strichartz estimate [A–P 2009]

$$\|u(t,x)\|_{L_t^\infty L_x^2} + \|u(t,x)\|_{L_t^{\tilde p} L_x^{\tilde q}} \lesssim \|f(x)\|_{L_x^2} + \|g(t,x)\|_{L_t^{p'} L_x^{q'}}$$

$\forall\ (p,q),\ (\tilde p, \tilde q)$ in the following triangle



Application to NLS

Wellposedness $\sim$ existence & uniqueness [A–P 2009]

Consider the NLS

$$\begin{cases} i\,\partial_t u(t,x) + \Delta_x u(t,x) = \Gamma(u(t,x)) \\ u(0,x) = f(x) \end{cases}$$

with powerlike nonlinearities

$$\begin{cases} |\Gamma(u)| \lesssim |u|^\gamma \\ |\Gamma(u)-\Gamma(v)| \lesssim |u-v|\,\{|u|^{\gamma-1}+|v|^{\gamma-1}\} \end{cases}$$

Then we have global wellposedness for small initial data

$$\begin{cases} \text{in } L^2(\mathbb{H}^n) & \text{if } \gamma \leq 1+\frac{4}{n} \\ \text{in } H^1(\mathbb{H}^n) & \text{if } \gamma \leq 1+\frac{4}{n-2} \end{cases}$$

Application to NLS

Particular case : $\Gamma(u) = u\,|u|^{\gamma-1}$ defocusing
L^2 and H^1 conservation
LWP $\rightsquigarrow$ GWP for arbitrary large data and subcritical σ

Scattering

Same assumptions $\Rightarrow$ scattering for small initial data

$$\begin{cases} \text{in } L^2(\mathbb{H}^n) & \text{if } \gamma \leq 1 + \frac{4}{n} \\ \text{in } H^1(\mathbb{H}^n) & \text{if } \gamma \leq 1 + \frac{4}{n-2} \end{cases}$$

Comment :

Better dispersion in $\mathbb{H}^n$ $\rightsquigarrow$ stronger results than in $\mathbb{R}^n$

Other results

- Previous results under radial assumptions
  [Banica 2007 ; Pierfelice 2009 ; Banica–Carles–Staffinali]
- [Ionescu–Staffilani] Related results in the defocusing case
  - weaker dispersive and Strichartz estimates
  - Morawetz inequality
  - scattering for arbitrary data
- [Chanillo] Some results for G complex
- [A–P–V]
  - Damek–Ricci spaces (straightforward)
  - higher rank (in progress)
- [Kaizaku] Smoothing effects on hyperbolic spaces
- [Burq–Guillarmou–Hassell]
  Convex cocompact surfaces with constant negative curvature

Wave equation on $\mathbb{H}^n$

Assume $n \geq 3$

$\mathscr{L} = \Delta + (\frac{n-1}{2})^2$ shifted Laplacian (no spectral gap)

Wave equation

$$\begin{cases} \partial_t^2 u(t,x) - \mathscr{L}_x u(t,x) = g(t,x) \\ u(0,x) = f_0(x),\ \partial_t|_{t=0} u(t,x) = f_1(x) \end{cases}$$

Solution

$$u(t,x) = \overbrace{(\cos t\sqrt{-\mathscr{L}_x})\, f_0(x) + \frac{\sin t\sqrt{-\mathscr{L}_x}}{\sqrt{-\mathscr{L}_x}}\, f_1(x)}^{\text{homogeneous}}$$
$$+ \underbrace{\int_0^t ds\, \frac{\sin(t-s)\sqrt{-\mathscr{L}_x}}{\sqrt{-\mathscr{L}_x}}\, g(s,x)}_{\text{inhomogeneous}}$$

Dispersive estimate [A–P–V]

Set $D = \sqrt{-\Delta}$ and $\mathscr{D} = \sqrt{-\mathscr{L}}$

Then

$$\| D^{-\sigma+\tau} \mathscr{D}^{-\tau} e^{it\mathscr{D}} \|_{L^{q'} \to L^q} \lesssim \begin{cases} |t|^{-(n-1)(\frac{1}{2}-\frac{1}{q})} & \text{if } 0<|t|<1 \\ |t|^{-2+\tau} & \text{if } |t|\geq 1 \end{cases}$$

under the following assumptions :

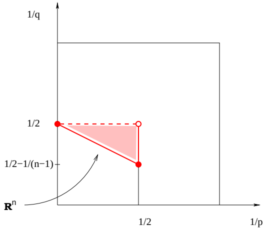
- $n \geq 3$
- $2 < q < \infty$
- $\sigma \geq (n+1)(\frac{1}{2}-\frac{1}{q})$
- $0 \leq \tau < 2$

Strichartz estimate [A–P–V]

$$\|\mathscr{D}^{\frac{1}{2}}u(t,x)\|_{L_t^\infty L_x^2}+\|\mathscr{D}^{-\frac{1}{2}}\partial_t u(t,x)\|_{L_t^\infty L_x^2}+\|D^{-\frac{\widetilde{\sigma}-1}{2}}u(t,x)\|_{L_t^{\tilde{p}}L_x^{\tilde{q}}}$$
$$\lesssim \|\mathscr{D}^{\frac{1}{2}}f_0(x)\|_{L_x^2}+\|\mathscr{D}^{-\frac{1}{2}}f_1(x)\|_{L_x^2}+\|D^{\frac{\sigma-1}{2}}g(t,x)\|_{L_t^{p'}L_x^{q'}}$$

under the following assumptions :

- $n\geq 3$
- (p,q) and $(\widetilde{p},\widetilde{q})$ belong to the following triangle



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$$\lesssim\|\mathscr{D}^{\frac{1}{2}}f_0(x)\|_{L_x^2}+\|\mathscr{D}^{-\frac{1}{2}}f_1(x)\|_{L_x^2}+\|D^{\frac{\sigma-1}{2}}g(t,x)\|_{L_t^{p'}L_x^{q'}}$$

under the following assumptions :

- $n\geq 3$
- (p,q) and $(\widetilde{p},\widetilde{q})$ are *admissible*
- $\sigma\geq(n+1)(\frac{1}{2}-\frac{1}{q})$ and $\widetilde{\sigma}\geq(n+1)(\frac{1}{2}-\frac{1}{\widetilde{q}})$

Application to NLW [A–P–V]

Consider the NLW

$$\begin{cases} \partial_t^2 u(t,x) - \mathscr{L}_x u(t,x) = \Gamma(u(t,x)) \\ u(0,x) = f_0(x),\ \partial_t|_{t=0}\, u(t,x) = f_1(x) \end{cases}$$

with powerlike nonlinearities

$$\begin{cases} |\Gamma(u)| \lesssim |u|^\gamma \\ |\Gamma(u) - \Gamma(v)| \lesssim |u-v|\,\{|u|^{\gamma-1} + |v|^{\gamma-1}\} \end{cases}$$

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with powerlike nonlinearities

Then we have global wellposedness under the following conditions :

- $f_0 \in H^{\frac{\sigma-1}{2},\frac{1}{2}}(\mathbb{H}^n)$ and $f_1 \in H^{\frac{\sigma-1}{2},-\frac{1}{2}}(\mathbb{H}^n)$ are small

  where $H^{\sigma,\tau}(\mathbb{H}^n) = D^{-\sigma}\mathscr{D}^{-\tau}L^2(\mathbb{H}^n)$

$$\begin{cases} = H^{\sigma}(\mathbb{H}^n) & \text{if } \tau = 0 \\ \supset H^{\sigma+\tau}(\mathbb{H}^n) & \text{if } \tau > 0 \\ \subset H^{\sigma+\tau}(\mathbb{H}^n) & \text{if } \tau < 0 \end{cases}$$

Application to NLW [A–P–V]

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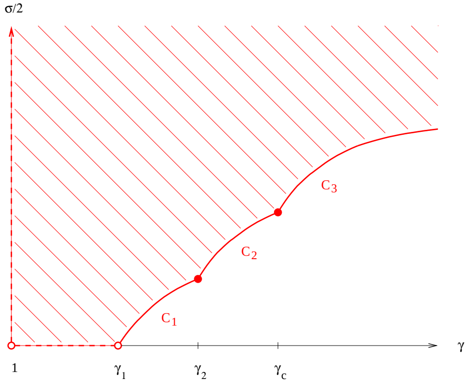
with powerlike nonlinearities

Then we have global wellposedness under the following conditions :

- $f_0 \in H^{\frac{\sigma-1}{2},\frac{1}{2}}(\mathbb{H}^n)$ and $f_1 \in H^{\frac{\sigma-1}{2},-\frac{1}{2}}(\mathbb{H}^n)$ are small
- $\gamma > 1$ and $\sigma > 0$ belong to the following region

Application to NLW [A–P–V]



Comments

- Same picture as for local wellposedness in $\mathbb{R}^n$
- Stronger results : in short,
  global wellposedness in $\mathbb{H}^n$ when local wellposedness in $\mathbb{R}^n$

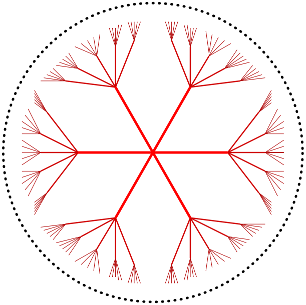
Other results

- [Tataru 2001]
  Dispersive estimates for $\cos t\mathscr{D}$ and $\frac{\sin t\mathscr{D}}{\mathscr{D}}$ on $\mathbb{H}^n$
  $\rightsquigarrow$ weighted Strichartz estimates
  $\rightsquigarrow$ NLW on $\mathbb{R}^n$
- [Ionescu 2000]
  $L^q \longrightarrow L^q$ estimates for $\cos t\mathscr{D}$ and $\frac{\sin t\mathscr{D}}{\mathscr{D}}$ on hyperbolic spaces
- [Cowling–Giulini–Meda 2001 ; Cowling–Meda 2002]
  $L^{q_1} \longrightarrow L^{q_2}$ estimates for $e^{-t\sqrt{-\mathscr{L}+\theta}}$ on G/K
  valid under restrictions on the complex time t
- [Hassani–Lohoué]
  Some results in higher rank

Homogeneous trees

$\mathbb{T} = \mathbb{T}_Q$ homogeneous tree with $Q+1$ edges

Example : $Q=5$



$\gamma = \frac{2}{Q^{1/2}+Q^{-1/2}}$

no local analysis $\rightsquigarrow$ no Sobolev spaces

Wave equation on $\mathbb{T}$

$$\gamma\,\Delta_n^{\mathbb{Z}} u(n,x) - \underbrace{\{\Delta_x^{\mathbb{T}} + 1 - \gamma\}}_{\mathscr{L}} u(n,x) = g(n,x)$$

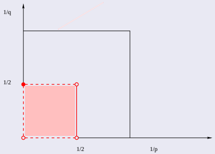
Laplacian on $\mathbb{Z}$: $\Delta^{\mathbb{Z}} f(n) = \frac{f(n+1) + f(n-1) - 2\,f(n)}{2}$

Laplacian on $\mathbb{T}$: $\Delta^{\mathbb{T}} f(x) = \frac{1}{Q+1} \sum_{d(y,x)=1} f(y) \;-\; f(x)$

Results

Dispersive estimate : exponential decay in n

Strichartz estimate : valid for the full square



NLW : any power $\gamma > 1$