DFG–JSPS Joint Seminar

Infinite Dimensional Harmonic Analysis IV

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The Schrödinger equation on symmetric spaces

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1. Introduction

Schrödinger equation :

$$\begin{cases} i\,\partial_t u(t,x) + \Delta_x u(t,x) = g(t,x) \\ u(0,x) = f(x) \end{cases} \tag{1}$$

Wave equation :

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = g(t,x) \\ u(0,x) = f_0(x),\ \partial_t|_{t=0} u(t,x) = f_1(x) \end{cases} \tag{2}$$

Homogeneous equations : $g = 0 \rightsquigarrow$ linear

Typical nonlinearities :

$$g(t,x) \doteqdot \begin{cases} u(t,x)\,|u(t,x)|^\gamma \\ |u(t,x)|^{1+\gamma} \end{cases}$$

Setting : Riemannian manifold

e.g. symmetric space $\begin{cases} \mathbb{R}^n \\ \mathbb{S}^n \text{ sphere} \\ \mathbb{H}^n \text{ hyperbolic space} \end{cases}$

2. Schrödinger in $\mathbb{R}^n$

Solution to the Schrödinger equation (1) :

$$u(t,x) = \underbrace{e^{it\Delta} f(x)}_{F(t,x)} + \underbrace{\int_0^t e^{i(t-s)\Delta_x} g(s,x)\,ds}_{G(t,x)}$$

Homogeneous solution

$$F(t,x) = e^{it\Delta} f(x) = (f * s_t)(x)$$

given by explicit kernel

$$s_t(x) = 2^{-n} \pi^{-\frac{n}{2}} e^{\mathrm{sign}(t)\, i \frac{\pi}{4} n} |t|^{-\frac{n}{2}} e^{-\frac{i}{4}\frac{|x|^2}{t}}$$

(heat kernel with imaginary time)

Two main estimates :

- **Dispersive estimates :**

$$\|e^{it\Delta}\|_{L^1 \to L^\infty} = \|s_t\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}}$$

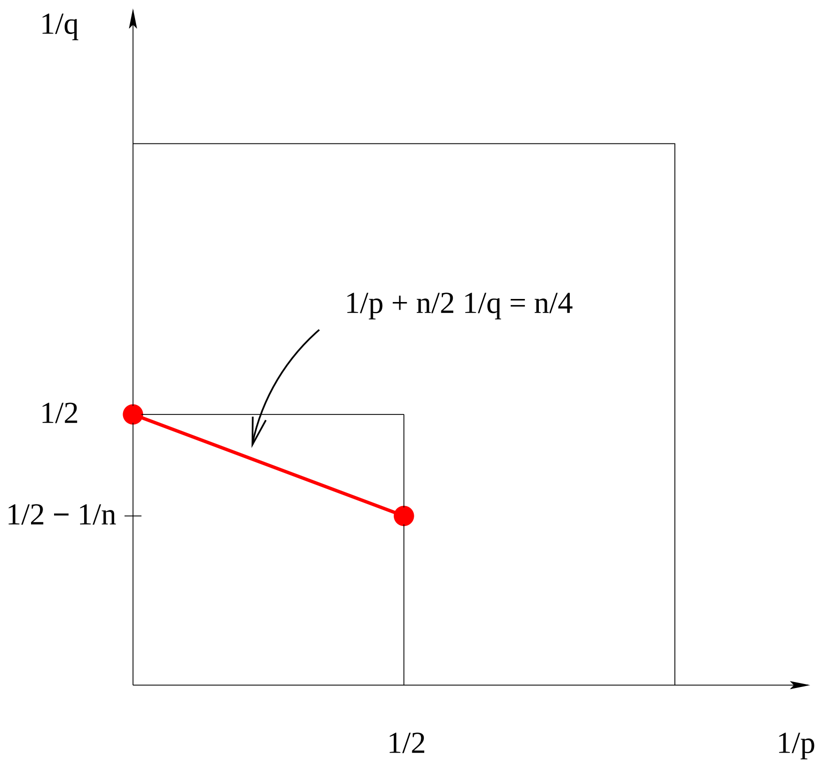
$$\Rightarrow \|e^{it\Delta}\|_{L^{q'} \to L^q} \lesssim |t|^{-n(\frac{1}{2}-\frac{1}{q})} \quad \forall\, 2 \le q \le \infty$$

$$\Rightarrow \|e^{it\Delta}\|_{L^{q'} \to L^{\tilde{q}}} \lesssim |t|^{-\frac{n}{2}(1-\frac{1}{q}-\frac{1}{\tilde{q}})} \quad \forall\, 2 \le q, \tilde{q} \le \infty$$

- **Strichartz type estimates :**

$$\|u(t,x)\|_{L_t^{\tilde{p}} L_x^{\tilde{q}}} \lesssim \|f(x)\|_{L_x^2} + \|g(t,x)\|_{L_t^{p'} L_x^{q'}} \quad (3)$$

$\forall$ (p,q), $(\tilde{p},\tilde{q})$ in the *admissible* interval



In dimension $n > 2$, the estimate () holds true at the endpoint $(\frac{1}{2}, \frac{1}{2} - \frac{1}{n})$

References : [Ginibre–Velo], [Keel–Tao]

Wellposedness (existence and uniqueness)
for NLS (NonLinear Schrödinger equation):

- $g(t,x) \doteqdot u(t,x)\,|u(t,x)|^{\gamma} \quad (\gamma > 0)$

conservation of $\begin{cases} L^2 \text{ mass} \\ H^1 \text{ energy} \end{cases}$

wellposedness	in $L^2(\mathbb{R}^n)$	in $H^1(\mathbb{R}^n)$
global	$\gamma < \frac{4}{n}$	$\gamma < \frac{4}{n-2}$
global	$\gamma = \frac{4}{n}$ s.i.d.	$\gamma = \frac{4}{n-2}$ s.i.d.

s.i.d. = small initial data

- $g(t,x) \doteqdot |u(t,x)|^{1+\gamma} \quad (\gamma > 0)$

no conservation!

wellposedness	in $L^2(\mathbb{R}^n)$	in $H^1(\mathbb{R}^n)$
local	$\gamma < \frac{4}{n}$	$\gamma < \frac{4}{n-2}$ s.i.d.
global	$\gamma = \frac{4}{n}$ s.i.d.	$\gamma = \frac{4}{n-2}$ s.i.d.

Reference: [Cazenave]

Scattering for NLS

Strichartz estimates
$\rightsquigarrow$ global solution $u(t,x) \in L_t^p L_x^q$
$t \to \pm\infty$: $u(t,x) \to 0$ in $L_t^p L_x^q$
$\rightsquigarrow$ NLS approaches HS
$\rightsquigarrow$ $u(t,x)$ close to $e^{it\Delta} u_\pm(x)$

Scattering for NLS in $L^2(\mathbb{R}^n)$:
For any global solution $u \in C(\mathbb{R}, L^2(\mathbb{R}^n))$,
do there exist $u_\pm \in L^2(\mathbb{R}^n)$ such that

$$\| u(t,x) - e^{it\Delta} u_\pm(x) \|_{L_x^2} \to 0 \quad \text{as} \quad t \to \pm\infty \ ?$$

Idem for H^1 scattering

- $g(t,x) \doteqdot u(t,x)\,|u(t,x)|^\gamma \quad (\gamma > 0)$

$$\begin{cases} L^2 \text{ scattering for } \gamma = \frac{4}{n} \text{ and s.i.d} \\ H^1 \text{ scattering for } \frac{4}{n} < \gamma < \frac{4}{n-2} \\ \text{no scattering for } 0 < \gamma < \frac{2}{n} \end{cases}$$

- $g(t,x) \doteqdot |u(t,x)|^{1+\gamma} \quad (\gamma > 0)$

no result (to our knowledge)

Reference : [Tao–Visan–Zhang]

3. NLS on manifolds

Aim : extend theory from Euclidean space to Riemannian manifold e.g. symmetric space

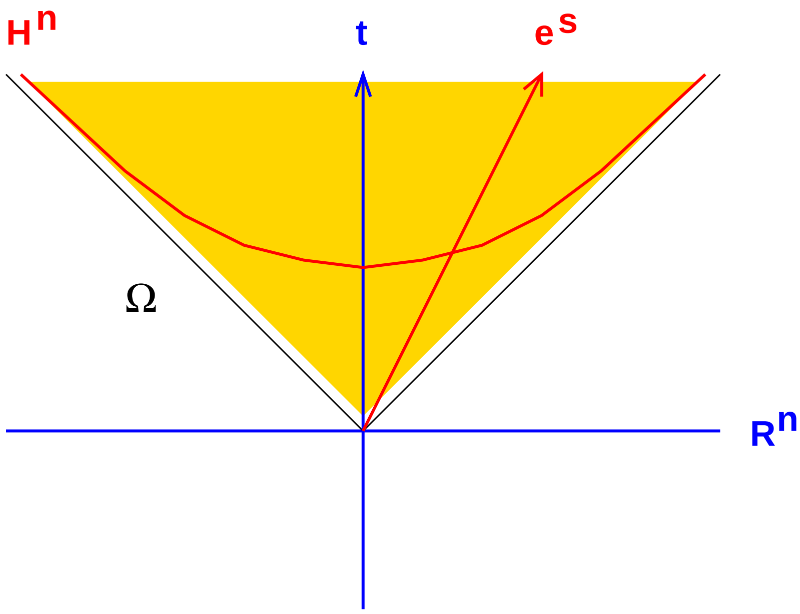
Motivations :

- understand influence of geometry
  - ○ compact case :
    expect less dispersion $\rightsquigarrow$ weaker results
  - ○ noncompact case :
    expect more dispersion $\rightsquigarrow$ stronger results
- transfer results (in both directions)
  between the flat case and the curved case

Parenthesis :

- ○ [Georgiev–Lindblat–Sogge]
  GWP for NLW in $\mathbb{R}^n$
- ○ [Tataru] shortcut via $\mathbb{H}^n$

Light cone Ω $\begin{cases} t^2 - x_1^2 - \dots - x_n^2 > 0 \\ t > 0 \end{cases}$



$$\begin{aligned} \Box &= \partial_t^2 - \Delta_{\mathbb{R}^n} \\ &= e^{-\frac{n+3}{2}s} \circ \Big\{ \partial_s^2 \underbrace{- \Delta_{\mathbb{H}^n} - \Big(\frac{n-1}{2}\Big)^2}_{-\mathcal{L}} \Big\} \circ e^{\frac{n-1}{2}s} \end{aligned}$$

dispersive estimates on $\mathbb{H}^n$
with exponential decay

$\rightsquigarrow$ GWP for NLW on $\mathbb{R}^n$

weighted Strichartz estimates on $\mathbb{H}^n \rightsquigarrow \mathbb{R}^n$

NLS on compact manifolds M

∘ [Burq–Gérard–Tzvetkov]
local Strichartz estimates
with loss of derivatives

$$\|e^{it\Delta}f(x)\|_{L^p(I;L^q(M))} \leq C_I \|f\|_{H^{1/p}(M)}$$

∘ [Bourgain]
LWP on $H^s(\mathbb{T}^n)$ when $s > \frac{n-2}{2}$

∘ [Burq–Gérard–Tzvetkov]

$$\begin{cases} \text{LWP on } H^s(M) \text{ when } s > \frac{n-1}{2} \\ \text{LWP on } H^s(\mathbb{S}^n) \text{ when } s > \frac{n-2}{2} \end{cases}$$

∘ [Gérard–Pierfelice]
GWP on $H^1(M)$
for 4–dimensional manifolds

NLS on hyperbolic space $\mathbb{H}^n$

Previous results :

∘ [Fontaine]

∘ [Tataru]

∘ [Banica] weighted dispersive estimate

$$\begin{cases} \text{dimension } n \geq 3 \\ \text{radial functions} \\ \text{weight } \; w(x) = \frac{\sinh|x|}{|x|} \end{cases}$$

$$\|w(x)u(t,x)\|_{L^\infty_x} \lesssim \left\{|t|^{-\frac{n}{2}} + |t|^{-\frac{3}{2}}\right\} \|w(x)^{-1} f(x)\|_{L^1_x}$$

∘ [Pierfelice] weighted Strichartz estimate

$$\begin{cases} \text{Damek–Ricci spaces} \\ \text{radial functions} \\ \text{weight } \; w(x) = \left(\frac{\sinh|x|}{|x|}\right)^{m_1} \left(\frac{\sinh 2|x|}{2|x|}\right)^{m_2} \\ \text{same admissible interval as } \mathbb{R}^n \end{cases}$$

$$\| w(x)^{\frac{1}{2}-\frac{1}{\tilde{q}}} u(t,x) \|_{L^{\tilde{p}}_t L^{\tilde{q}}_x} \lesssim \| f(x) \|_{L^2_x} + \| w(x)^{\frac{1}{q}-\frac{1}{2}} g(t,x) \|_{L^{p'}_t L^{q'}_x}$$

∘ [Banica–Carles–Staffilani]
application to NLS and scattering
for radial functions

∘ [Chanillo]
partial results for G/K with G complex

Kernel expression : $r=|x|$

- n odd :

$$s_t(r)=2^{-\frac{n+1}{2}}\pi^{-\frac{n}{2}}\,e^{i\,\mathrm{sign}(t)\frac{\pi}{4}}\,|t|^{-\frac{1}{2}}\,e^{i(\frac{n-1}{2})^2t}$$
$$\times\,\Big(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\Big)^{\frac{n-1}{2}}e^{-\frac{i}{4}\frac{r^2}{t}}$$

- n even :

$$s_t(r)=2^{-\frac{n+1}{2}}\pi^{-\frac{n+1}{2}}e^{i\,\mathrm{sign}(t)\frac{\pi}{4}}\,|t|^{-\frac{1}{2}}\,e^{i(\frac{n-1}{2})^2t}$$
$$\times\int_r^\infty ds\frac{\sinh s}{\sqrt{\cosh s-\cosh r}}\Big(-\frac{1}{\sinh s}\frac{\partial}{\partial s}\Big)^{\frac{n}{2}}e^{-\frac{i}{4}\frac{s^2}{t}}$$

Kernel estimates : $t\neq 0$

$$|s_t(r)|\lesssim\begin{cases}|t|^{-n/2}\,j(r)^{-1/2} & \text{if } |t|\leq 1+r\\ |t|^{-3/2}\,\varphi_0(r) & \text{if } |t|\geq 1+r\end{cases}$$

where $\begin{cases}j(r)^{-1/2}\asymp(1+r)^{\frac{n-1}{2}}\,e^{-\frac{n-1}{2}r}\\ \varphi_0(r)\asymp(1+r)\;e^{-\frac{n-1}{2}r}\end{cases}$

$$\Rightarrow\;\|s_t\|_{L^q}\lesssim\begin{cases}|t|^{-n/2} & \text{if } 0<|t|\leq 1\\ |t|^{-3/2} & \text{if } |t|\geq 1\end{cases}\qquad\forall\,q>2$$

Idem for $\|s_t\|_{L^{q,a}}$ $\forall\,q>2$ and $\forall\,a\geq 1$

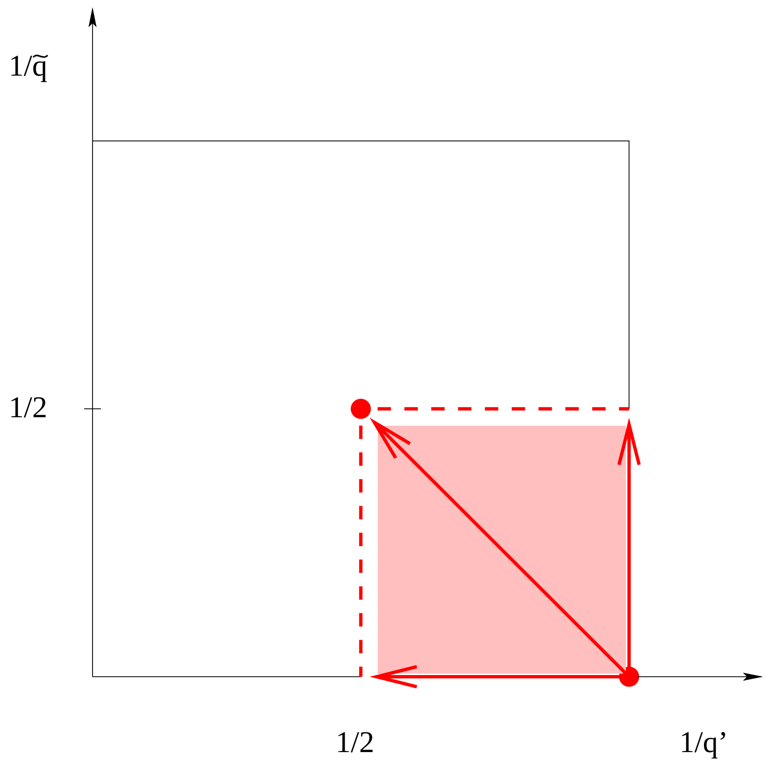
Dispersive estimates :

- $0<|t|\leq 1$:

$$\|e^{it\Delta}\|_{L^{q'}\to L^{\tilde{q}}} \lesssim \begin{cases} 1 & \text{if } q=\tilde{q}=2 \\ |t|^{-\max\{\frac{1}{2}-\frac{1}{q},\frac{1}{2}-\frac{1}{\tilde{q}}\}n} & \text{if } 2<q,\tilde{q}\leq\infty \end{cases}$$

- $|t|\geq 1$:

$$\|e^{it\Delta}\|_{L^{q'}\to L^{\tilde{q}}} \lesssim \begin{cases} 1 & \text{if } q=\tilde{q}=2 \\ |t|^{-\frac{3}{2}} & \text{if } 2<q,\tilde{q}\leq\infty \end{cases}$$



Proof : Interpolation

Crucial estimate : $\|e^{it\Delta}\|_{L^{q'}\to L^q} \lesssim |t|^{-\frac{3}{2}} \begin{cases} |t| \geq 1 \\ q > 2 \end{cases}$

follows from

- $\|s_t\|_{L^{q,1}} \lesssim |t|^{-\frac{3}{2}}$
- the sharp Kunze–Stein phenomenon :
  [Ionescu 2000] $L^{2,1}(G) * L^{2,1}(G) \subset L^{2,\infty}(G)$

$\Rightarrow$ [Cowling 1997] $L^{q',a}(G) * L^{q',b}(G) \subset L^{q',c}(G)$
if $1 \leq q' < 2$ and $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} \geq 1$

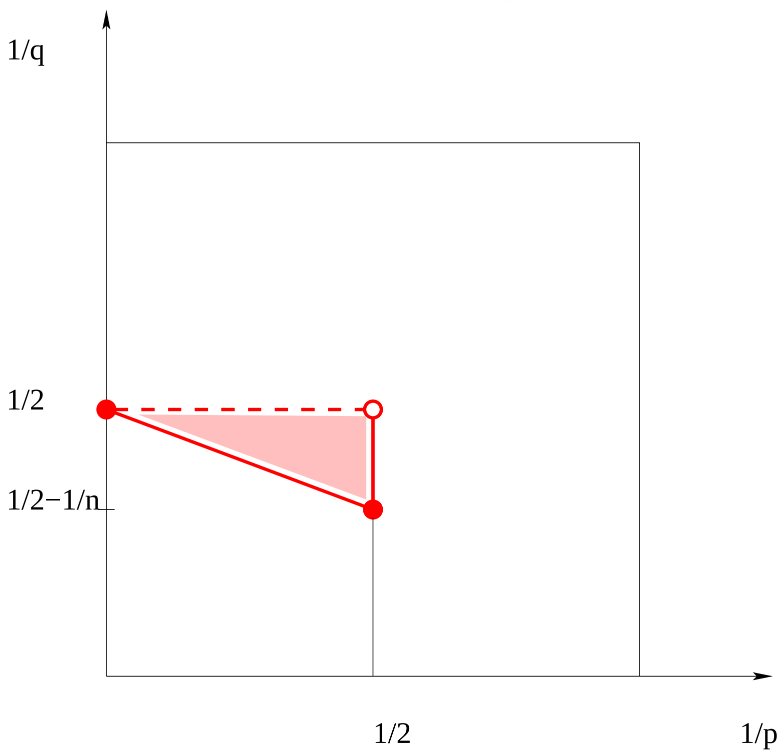
$\Rightarrow$ $L^{q'}(K\backslash G/K) * L^{q'}(G/K) \subset L^{q',c'}(K\backslash G/K)$
if $1 \leq q' < 2$ and $\frac{1}{c'} \leq \frac{2}{q'} - 1$

$\Rightarrow$ $L^{q'}(G/K) * L^{q,c}(K\backslash G/K) \subset L^q(G/K)$
if $q > 2$ and $c \leq \frac{q}{2}$

Strichartz estimates :

$$\|u(t,x)\|_{L_t^{\tilde{p}} L_x^{\tilde{q}}} \lesssim \|f(x)\|_{L_x^2} + \|g(t,x)\|_{L_t^{p'} L_x^{q'}}$$

$\forall\ (p,q),\ (\tilde{p},\tilde{q})$ in the *admissible* triangle



Wellposedness

Same results as in $\mathbb{R}^n$
with the following improvement

Theorem :
Assume $|g(t,x)| \lesssim |u(t,x)|^{1+\gamma}$ with $\gamma>0$
Then global wellposedness for s.i.d.

$$\begin{cases} \text{in } L^2(\mathbb{H}^n) \text{ if } \gamma\leq\frac{4}{n} \\ \text{in } H^1(\mathbb{H}^n) \text{ if } \gamma\leq\frac{4}{n-2} \end{cases}$$

Scattering

Theorem : Same assumption. Then

$$\begin{cases} L^2 \text{ scattering for } \gamma\leq\frac{4}{n} \text{ and s.i.d.} \\ H^1 \text{ scattering for } \gamma\leq\frac{4}{n-2} \text{ and s.i.d.} \end{cases}$$

In contrast with $\mathbb{R}^n$, no long range effect
$\rightsquigarrow$ scattering for $\gamma>0$ arbitrarily small

4. Further results and problems

NLW (NonLinear Wave equation)

Smoothing properties

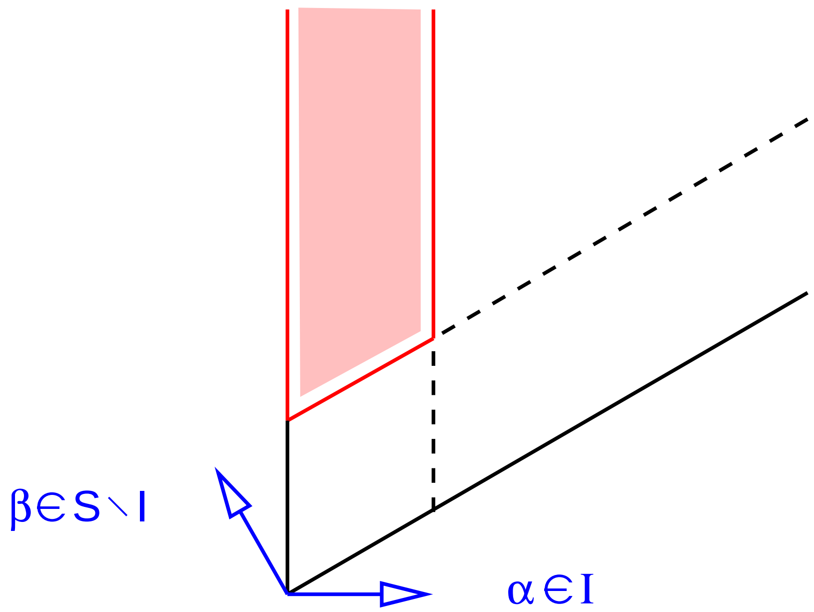
NLS and NLW

- Damek–Ricci spaces
- higher rank noncompact symmetric spaces
- compact symmetric spaces
- discrete setting
  - trees, hyperbolic groups, ...
  - buildings
- special functions related to root systems
  - Dunkl (rational)
  - Heckman–Opdam (trigonometric)
  - Macdonald–Cherednik (q–theory)

Conjectural kernel estimate in higher rank :

Let $t \neq 0$ and $x \in \overline{\mathfrak{a}^+}$

Assume $\begin{cases} |t| \geq 1 + \langle \alpha, x \rangle & \forall\, \alpha \in I \\ |t| \leq 1 + \langle \alpha, x \rangle & \forall\, \alpha \in S \smallsetminus I \end{cases}$



Then $|s_t(x)| \lesssim |t|^{-d_I/2}\, \varphi_{I,0}(x_I) \left\{ \frac{j_I(x)}{j_S(x)} \right\}^{1/2}$

Here : $I \subset S$ (simple roots)

$d_I = \ell/2 - |R^+_{I,\mathrm{red}}| - \sum_{\alpha \in R^+ \smallsetminus R^+_I} m_\alpha/2$

$\varphi_{I,0} =$ basic spherical function of (G_I, K_I)

$j_I =$ jacobian of the exponential map

$\mathfrak{p}_I \longrightarrow G_I/K_I$