OPDAM FUNCTIONS: PRODUCT FORMULA AND CONVOLUTION STRUCTURE IN DIMENSION 1

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ABSTRACT. Let $G_\lambda^{(\alpha,\beta)}(x)$ be the eigenfunctions of the Dunkl-Cherednik operator $T^{(\alpha,\beta)}$ on $\mathbb{R}$, with $\alpha \geq \beta \geq -\frac{1}{2}$. In this paper we express the product $G_\lambda^{(\alpha,\beta)}(x)G_\lambda^{(\alpha,\beta)}(y)$ as an integral in terms of $G_\lambda^{(\alpha,\beta)}(z)$ with an explicit kernel. In general this kernel is not positive. Furthermore, by taking the so-called rational limit, we recover the product formula for the Dunkl kernels proved in [13]. We then define and study a convolution structure associated to $G_\lambda^{(\alpha,\beta)}$.

1. INTRODUCTION

The Opdam hypergeometric functions $G_\lambda^{(\alpha,\beta)}$ on $\mathbb{R}$ are normalized eigenfunctions

$$\begin{cases} T^{(\alpha,\beta)}G_\lambda^{(\alpha,\beta)}(x) = i\lambda G_\lambda^{(\alpha,\beta)}(x) \\ G_\lambda^{(\alpha,\beta)}(0) = 1 \end{cases}$$

of the differential-difference operator

$$T^{(\alpha,\beta)}f(x) = f'(x) + \Big((2\alpha+1)\coth x + (2\beta+1)\tanh x\Big)\frac{f(x)-f(-x)}{2} - \rho f(-x). \tag{1.1}$$

Here $\alpha \geq \beta \geq -\frac{1}{2}$, $\alpha \neq -\frac{1}{2}$, $\rho = \alpha+\beta+1$ and $\lambda \in \mathbb{C}$. In Cherednik's notation, $T^{(\alpha,\beta)}$ writes

$$T(k_1,k_2)f(x) = f'(x) + \Big(\frac{2k_1}{1-e^{-2x}} + \frac{4k_2}{1-e^{-4x}}\Big)(f(x)-f(-x)) - (k_1+2k_2)f(x),$$

with $\alpha = k_1 + k_2 - \frac{1}{2}$ and $\beta = k_2 - \frac{1}{2}$. The function $G_\lambda^{(\alpha,\beta)}$ can be expressed as follows in terms of the Jacobi functions $\varphi_\lambda^{(\alpha,\beta)}$:

$$G_\lambda^{(\alpha,\beta)}(x) = \varphi_\lambda^{(\alpha,\beta)}(x) - \frac{1}{\rho - i\lambda}\frac{\partial}{\partial x}\varphi_\lambda^{(\alpha,\beta)}(x). \tag{1.2}$$

As main references we use the primary articles [11, 3] and the lecture notes [4, 12].

This paper deals with harmonic analysis for the eigenfunctions $G_\lambda^{(\alpha,\beta)}$. We derive mainly a product formula for $G_\lambda^{(\alpha,\beta)}$, which is analogous to the corresponding result of Flensted-Jensen and Koornwinder [6] for Jacobi functions, and of Ben Salem

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and Ould Ahmed Salem [1] for the Jacobi-Dunkl functions. The product formula is the key information needed in order to define an associated convolution structure on $\mathbb{R}$. More precisely, we deduce the product formula

$$G_\lambda^{(\alpha,\beta)}(x)G_\lambda^{(\alpha,\beta)}(y) = \int_{\mathbb{R}} G_\lambda^{(\alpha,\beta)}(z)d\mu_{x,y}^{(\alpha,\beta)}(z) \quad \forall x, y \in \mathbb{R}, \quad \forall \lambda \in \mathbb{C}, \tag{1.3}$$

from the corresponding formula for $\varphi_\lambda^{(\alpha,\beta)}$ on $\mathbb{R}^+$. Here $\mu_{x,y}^{(\alpha,\beta)}$ is an explicit real valued measure with compact support on $\mathbb{R}$, which may not be positive and which is uniformly bounded in $x, y \in \mathbb{R}$. We conclude the first part of the paper by recovering as a limit case the product formula for the Dunkl kernel proved in [13].

In the second part of the paper, we use the product formula (1.3) to define and study the translation operators

$$\tau_x^{(\alpha,\beta)}f(y) := \int_{\mathbb{R}} f(z)d\mu_{x,y}^{(\alpha,\beta)}(z).$$

We next define the convolution product of suitable functions f and g by

$$f *_{\alpha,\beta} g(x) = \int_{\mathbb{R}} \tau_x^{(\alpha,\beta)}f(-y)g(y)A_{\alpha,\beta}(|y|)dy,$$

where $A_{\alpha,\beta}(|y|) = \sinh(|y|)^{2\alpha+1}\cosh(y)^{2\beta+1}$. We show in particular that $f *_{\alpha,\beta} g = g *_{\alpha,\beta} f$ and that $\mathcal{F}(f *_{\alpha,\beta} g) = \mathcal{F}(f)\mathcal{F}(g)$, where $\mathcal{F}$ is the so-called Opdam-Cherednik transform. Eventually we prove an analog of the Kunze-Stein phenomenon for the $*_{\alpha,\beta}$-convolution product of L^p-spaces.

In the last part of the paper, we construct an orthonormal basis of the Hilbert space $L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$, generalizing the corresponding result of Koornwinder [10] for $L^2(\mathbb{R}^+, A_{\alpha,\beta}(x)dx)$.. As a limit case, we recover the Hermite functions constructed by Rosenblum [14] in $L^2(\mathbb{R}, |x|^{2\alpha+1}dx)$.

Our paper is organised as follows. In section 2, we recall some properties and formulas for Jacobi functions. In section 3, we give the proof of the product formula for $G_\lambda^{(\alpha,\beta)}$. Section 4 is devoted to the translation operators and the associated convolution product. Section 5 contains a Kunze-Stein type phenomenon. In the last section 6, we construct an orthonormal basis of $L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$.

2. Preliminaries

In this section we recall some properties of the Jacobi functions. See [6] and [7] for more details, as well as the survey [10].

For $\alpha \geq \beta \geq -\frac{1}{2}$ with $\alpha \neq -\frac{1}{2}$ and $\lambda \in \mathbb{C}$, let $\varphi_\lambda^{(\alpha,\beta)}$ be the Jacobi function defined by

$$\varphi_\lambda^{(\alpha,\beta)}(x) = {}_2F_1\Big(\frac{1}{2}(\rho + i\lambda), \frac{1}{2}(\rho - i\lambda); \alpha + 1; -\sinh^2(x)\Big),$$

where $\rho = \alpha+\beta+1$ and ${}_2F_1$ denotes the hypergeometric function. For $\alpha > \beta > -\frac{1}{2}$, and $x, y \geq 0$, the Jacobi functions satisfy the following product formula

$$\varphi_\lambda^{(\alpha,\beta)}(x)\varphi_\lambda^{(\alpha,\beta)}(y) = \int_0^1 \int_0^\pi \varphi_\lambda^{(\alpha,\beta)}(\arg\cosh|\gamma(x,y,r,\psi)|)dm_{\alpha,\beta}(r,\psi), \tag{2.1}$$

where

$$\gamma(x,y,r,\psi) = \cosh x \cosh y + re^{i\psi}\sinh x \sinh y,$$

and

$$dm_{\alpha,\beta}(r,\psi) = 2M_{\alpha,\beta}(1-r^2)^{\alpha-\beta-1}(r\sin\psi)^{2\beta}rdrd\psi \tag{2.2}$$

with

$$M_{\alpha,\beta} = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}.$$

When $\alpha = \beta > -\frac{1}{2}$, the product formula becomes

$$\varphi_\lambda^{(\alpha,\alpha)}(x)\varphi_\lambda^{(\alpha,\alpha)}(y) = M_\alpha \int_0^\pi \varphi_\lambda^{(\alpha,\alpha)}(\arg\cosh|\gamma(x,y,1,\psi)|)\sin^{2\alpha}(\psi)d\psi, \tag{2.3}$$

where $M_\alpha = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}$. Notice that the limit cases $\alpha > \beta = -\frac{1}{2}$ and $\alpha = \beta > -\frac{1}{2}$ are connected by the quadratic transformation $\varphi_\lambda^{(\alpha,-\frac{1}{2})}(x) = \varphi_{2\lambda}^{(\alpha,\alpha)}\big(\frac{x}{2}\big)$.

For $\alpha > \beta > -\frac{1}{2}$ and fixed $x, y \in \mathbb{R}^*$, we perform the change of variables $(r,\psi) \mapsto (z,\chi)$ defined by

$$\cosh z\ e^{i\chi} = \gamma(x,y,r,\psi), \tag{2.4}$$

i.e.

$$r\cos\psi = \frac{\cosh z\cos\chi - \cosh x\cosh y}{\sinh x\sinh y}, \qquad r\sin\psi = \frac{\cosh z\sin\chi}{\sinh x\sinh y}.$$

This implies in particular that

$$\cosh(|x|-|y|) \leq \cosh(z) \leq \cosh(|x|+|y|),$$

and therefore x, y, z satisfy the triangular inequality

$$||x|-|y|| \leq |z| \leq ||x|+|y||.$$

Moreover, an easy computation gives

$$1-r^2 = (\sinh x\sinh y)^{-2}g(x,y,z,\chi),$$

where

$$g(x,y,z,\chi) := 1-\cosh^2 x-\cosh^2 y-\cosh^2 z+2\cosh x\cosh y\cosh z\cos\chi. \tag{2.5}$$

Furthermore, the measure $\sinh^2 x\sinh^2 y\ rdrd\psi$ becomes $\cosh z\sinh z\ dzd\chi$, and therefore the measure $dm_{\alpha,\beta}$ given by (2.2) becomes

$$dm_{\alpha,\beta}(r,\psi) = 2M_{\alpha,\beta}\,g(x,y,z,\chi)^{\alpha-\beta-1}\big(\sinh x\sinh y\sinh z\big)^{-2\alpha}\sin^{2\beta}\chi\,A_{\alpha,\beta}(z)\,dzd\chi,$$

where

$$A_{\alpha,\beta}(z) := (\sinh z)^{2\alpha+1}(\cosh z)^{2\beta+1}. \tag{2.6}$$

Hence, the product formula (2.1) reads

$$\varphi_\lambda^{(\alpha,\beta)}(x)\varphi_\lambda^{(\alpha,\beta)}(y)=\int_0^\infty \varphi_\lambda^{(\alpha,\beta)}(z)W_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)dz,\qquad x,y>0,\tag{2.7}$$

where

$$W_{\alpha,\beta}(x,y,z):=2M_{\alpha,\beta}(\sinh x\sinh y\sinh z)^{-2\alpha}\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}\sin^{2\beta}\chi d\chi\tag{2.8}$$

if $x,y,z>0$ satisfy $|x-y|<z<x+y$ and $W_{\alpha,\beta}(x,y,z)=0$ otherwise. Here

$$g_+=\begin{cases} g & if \quad g>0,\\ 0 & if \quad g\le 0.\end{cases}$$

We point out that the function $W_{\alpha,\beta}(x,y,z)$ is nonnegative, symmetric in the variables x,y,z and that

$$\int_0^{+\infty} W_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)dz=1.$$

Furthermore, in [6, Formula (4.19)] the authors express $W_{\alpha,\beta}$ as follows in terms of the hypergeometric function ${}_2F_1$: For every $x,y,z>0$ satisfying the triangular inequality $|x-y|<z<x+y$,

$$\begin{aligned}W_{\alpha,\beta}(x,y,z)&=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac12)}(\cosh x\cosh y\cosh z)^{\alpha-\beta-1}(\sinh x\sinh y\sinh z)^{-2\alpha}\\ &\quad\times(1-B^2)^{\alpha-\frac12}{}_2F_1\Big(\alpha+\beta,\alpha-\beta;\alpha+\frac12;\frac{1-B}{2}\Big),\end{aligned}\tag{2.9}$$

where

$$B:=\frac{\cosh^2 x+\cosh^2 y+\cosh^2 z-1}{2\cosh x\cosh y\cosh z}.$$

Notice that

$$1\pm B=\frac{(\cosh(x+y)\pm\cosh z)(\cosh z\pm\cosh(x-y))}{2\cosh x\cosh y\cosh z},$$

hence

$$\begin{aligned}1-B^2 &= \frac{(\cosh 2(x+y)-\cosh 2z)(\cosh 2z-\cosh 2(x-y))}{16\cosh^2 x\cosh^2 y\cosh^2 z}\\ &= \frac{\sinh(x+y+z)\sinh(x+y-z)\sinh(x-y+z)\sinh(-x+y+z)}{4\cosh^2 x\cosh^2 y\cosh^2 z}.\end{aligned}\tag{2.10}$$

In the case $\alpha=\beta>-\frac12$, we use instead the change of variables

$$\cosh z=|\gamma(x,y,1,\psi)|=|\cosh x\cosh y+e^{i\psi}\sinh x\sinh y|,$$

and we obtain the same product formula (2.7), where $W_{\alpha,\alpha}$ is given by

$$\begin{aligned} W_{\alpha,\alpha}(x,y,z) = {} & \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} 2^{4\alpha+1}(\sinh 2x \sinh 2y \sinh 2z)^{-2\alpha} \\ & \times [\sinh(x+y+z)\sinh(x+y-z)]^{\alpha-1/2} \\ & \times [\sinh(x-y+z)\sinh(-x+y+z)]^{\alpha-1/2}. \qquad (2.11) \end{aligned}$$

In the case $\alpha > \beta = -\frac{1}{2}$, we use the quadratic transformation

$$\varphi_\lambda^{(\alpha,-\frac{1}{2})}(2x) = \varphi_{2\lambda}^{(\alpha,\alpha)}(x),$$

and we obtain again the product formula (2.7), with

$$W_{\alpha,-\frac{1}{2}}(x,y,z) = 2^{-2\alpha} W_{\alpha,\alpha}(\frac{x}{2},\frac{y}{2},\frac{z}{2}).$$

As noticed by Koornwinder [9] (see also [7]), the product formulas (2.1) and (2.7) are closely connected with the addition formula for the Jacobi functions, that we recall now for later use:

$$\varphi_\lambda^{(\alpha,\beta)}(\operatorname{arg\,cosh}|\gamma(x,y,r,\psi)|) = \sum_{0\le l\le k<\infty} \varphi_{\lambda,k,l}^{(\alpha,\beta)}(x)\varphi_{-\lambda,k,l}^{(\alpha,\beta)}(y)\chi_{k,l}^{(\alpha,\beta)}(r,\psi)\Pi_{k,l}^{(\alpha,\beta)}, \qquad (2.12)$$

where

$$\varphi_{\lambda,k,l}^{(\alpha,\beta)}(x) = \frac{c_{\alpha,\beta}(-\lambda)}{c_{\alpha+k+l,\beta+k-l}(-\lambda)}(2\sinh x)^{k-l}(2\cosh x)^{k+l}\varphi_\lambda^{(\alpha+k+l,\beta+k-l)}(x)$$

are modified Jacobi functions with

$$c_{\alpha,\beta}(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{\rho}{2}+i\frac{\lambda}{2})\Gamma(\frac{\alpha-\beta+1}{2}+i\frac{\lambda}{2})},$$

$$\chi_{k,l}^{(\alpha,\beta)}(r,\psi) = r^{k-l} R_l^{(\alpha-\beta-1,\beta+k-l)}(2r^2-1) R_{k-l}^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(\cos\psi),$$

with

$$R_n^{(\alpha,\beta)}(x) = {}_2F_1\Big(-n, n+\rho; \alpha+1; \frac{1-x}{2}\Big),$$

and

$$\Pi_{k,l}^{(\alpha,\beta)} = \Big(\int_0^1\int_0^\pi \chi_{k,l}^{(\alpha,\beta)}(r,\psi)^2 dm_{\alpha,\beta}(r,\psi)\Big)^{-1} \qquad (2.13)$$

are normalizing constants.

3. THE PRODUCT FORMULA FOR $G_\lambda^{(\alpha,\beta)}$

For $x,y,z\in\mathbb{R}$ and $\chi\in[0,\pi]$, let

$$\sigma_{x,y,z}^{\chi}=\begin{cases}\frac{\cosh x\cosh y-\cosh z\cos\chi}{\sinh x\sinh y} & if\ xy\neq 0,\\ 0, & if\ xy=0.\end{cases}\tag{3.1}$$

Furthermore, if $\alpha>\beta>-\frac{1}{2}$, let us define $\mathcal{K}_{\alpha,\beta}$ by

$$\begin{aligned}\mathcal{K}_{\alpha,\beta}(x,y,z) &= M_{\alpha,\beta}(\sinh|x|\sinh|y|\sinh|z|)^{-2\alpha}\int_0^{\pi} g(x,y,z,\chi)_+^{\alpha-\beta-1}\\ &\quad\times\Big[1-\sigma_{x,y,z}^{\chi}+\sigma_{z,x,y}^{\chi}+\sigma_{z,y,x}^{\chi}+\frac{\rho}{\beta+\frac{1}{2}}\coth x\coth y\coth z(\sin\chi)^2\Big]\\ &\quad\times(\sin\chi)^{2\beta}d\chi\end{aligned}$$

if $x,y,z\in\mathbb{R}^*$ satisfy the triangular inequality $||x|-|y||<|z|<|x|+|y|$, and $\mathcal{K}_{\alpha,\beta}(x,y,z)=0$ otherwise. Here $g(x,y,z,\chi)$ is as in (2.5).

Remark 3.1. *The following symmetry properties are easy to check:*

$$\begin{cases}\mathcal{K}_{\alpha,\beta}(x,y,z)=\mathcal{K}_{\alpha,\beta}(y,x,z),\\ \mathcal{K}_{\alpha,\beta}(x,y,z)=\mathcal{K}_{\alpha,\beta}(-z,y,-x),\\ \mathcal{K}_{\alpha,\beta}(x,y,z)=\mathcal{K}_{\alpha,\beta}(x,-z,-y).\end{cases}$$

Recall the eigenfunctions $G_\lambda^{(\alpha,\beta)}$ from the introduction. This section is devoted to the proof of the following main result.

Theorem 3.2. *Assume $\alpha>\beta>-\frac{1}{2}$. Then $G_\lambda^{(\alpha,\beta)}$ satisfies the following product formula*

$$G_\lambda^{(\alpha,\beta)}(x)G_\lambda^{(\alpha,\beta)}(y)=\int_{-\infty}^{+\infty}G_\lambda^{(\alpha,\beta)}(z)d\mu_{x,y}^{(\alpha,\beta)}(z),$$

for $x,y\in\mathbb{R}$ and $\lambda\in\mathbb{C}$. Here

$$d\mu_{x,y}^{(\alpha,\beta)}(z)=\begin{cases}\mathcal{K}_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(|z|)dz & if\ \ xy\neq 0,\\ \delta_x & if\ \ y=0,\\ \delta_y & if\ \ x=0,\end{cases}\tag{3.2}$$

and $A_{\alpha,\beta}$ is as in (2.6).

Let us split the eigenfunction $G_\lambda^{(\alpha,\beta)}$ as

$$G_\lambda^{(\alpha,\beta)}=G_{\lambda,e}^{(\alpha,\beta)}+G_{\lambda,o}^{(\alpha,\beta)}$$

into its even and odd parts:

$$G_{\lambda,e}^{(\alpha,\beta)}(x)=\varphi_\lambda^{(\alpha,\beta)}(x),$$

and

$$\begin{aligned} G_{\lambda,o}^{(\alpha,\beta)}(x) &= -\frac{1}{\rho-i\lambda}\frac{\partial}{\partial x}\varphi_\lambda^{(\alpha,\beta)}(x) \\ &= \frac{\rho+i\lambda}{4(\alpha+1)}\sinh 2x\ \varphi_\lambda^{(\alpha+1,\beta+1)}(x). \end{aligned}$$

For $x, y \in \mathbb{R}^*$, the product formula (2.7) for the Jacobi functions yields

$$\begin{aligned} G_{\lambda,e}^{(\alpha,\beta)}(x)G_{\lambda,e}^{(\alpha,\beta)}(y) &= \int_{||x|-|y||}^{|x|+|y|} G_{\lambda,e}^{(\alpha,\beta)}(z)W_{\alpha,\beta}(|x|,|y|,z)A_{\alpha,\beta}(z)dz \\ &= \frac{1}{2}\int_{I_{x,y}} G_\lambda^{(\alpha,\beta)}(z)W_{\alpha,\beta}(|x|,|y|,|z|)A_{\alpha,\beta}(|z|)dz, \end{aligned}$$

where

$$I_{x,y} := [-|x|-|y|, -||x|-|y||] \cup [||x|-|y||, |x|+|y|]. \tag{3.3}$$

Next let us turn to the mixed products. The following statement is Lemma 2.2 in [1].

Lemma 3.3. *For $\alpha > \beta > -\frac{1}{2}$ and $x, y \in \mathbb{R}^*$, we have*

$$\begin{aligned} G_{\lambda,o}^{(\alpha,\beta)}(x)G_{\lambda,e}^{(\alpha,\beta)}(y) &= M_{\alpha,\beta}\int_{I_{x,y}} G_\lambda^{(\alpha,\beta)}(z)(\sinh|x|\sinh|y|\sinh|z|)^{-2\alpha} \\ &\quad \times\Big\{\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}\sigma_{x,z,y}^\chi(\sin\chi)^{2\beta}d\chi\Big\}A_{\alpha,\beta}(|z|)dz \end{aligned}$$

where $g(x,y,z,\chi)$ is given by (2.5), *$\sigma_{x,z,y}^\chi$ is given by* (3.1), *and $I_{x,y}$ is as in* (3.3).

We consider now purely odd products, which is the most difficult case.

Lemma 3.4. *For $\alpha > \beta > -\frac{1}{2}$ and $x, y \in \mathbb{R}^*$, we have*

$$\begin{aligned} G_{\lambda,o}^{(\alpha,\beta)}(x)G_{\lambda,o}^{(\alpha,\beta)}(y) = M_{\alpha,\beta}\int_{I_{x,y}} & G_\lambda^{(\alpha,\beta)}(z)(\sinh|x|\sinh|y|\sinh|z|)^{-2\alpha} \\ \times\Big\{\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1} & \left(-\sigma_{x,y,z}^\chi - \frac{\rho}{(\beta+\frac{1}{2})}\coth x\coth y\coth z\sin^2\chi\right) \\ & \times(\sin\chi)^{2\beta}d\chi\Big\}A_{\alpha,\beta}(|z|)dz. \end{aligned}$$

Proof. For $x, y > 0$, we have

$$\begin{aligned} G_{\lambda,o}^{(\alpha,\beta)}(x)G_{\lambda,o}^{(\alpha,\beta)}(y) &= \frac{(\rho+i\lambda)^2}{16(\alpha+1)^2}\sinh(2x)\sinh(2y)\varphi_\lambda^{(\alpha+1,\beta+1)}(x)\varphi_\lambda^{(\alpha+1,\beta+1)}(y) \\ &= \mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y) + \mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x,y), \end{aligned}$$

where

$$\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y) := -\frac{\lambda^2+\rho^2}{16(\alpha+1)^2}\sinh(2x)\sinh(2y)\varphi_\lambda^{(\alpha+1,\beta+1)}(x)\varphi_\lambda^{(\alpha+1,\beta+1)}(y), \tag{3.4}$$

and

$$\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x,y) := \frac{\rho(\rho+i\lambda)}{8(\alpha+1)^2}\sinh(2x)\sinh(2y)\varphi_\lambda^{(\alpha+1,\beta+1)}(x)\varphi_\lambda^{(\alpha+1,\beta+1)}(y). \tag{3.5}$$

We can rewrite $\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}$ as

$$\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y) = -\int_0^1\int_0^\pi \varphi_\lambda^{(\alpha,\beta)}(\arg\cosh|\gamma(x,y,r,\psi)|)r\cos\psi dm_{\alpha,\beta}(r,\psi). \tag{3.6}$$

Indeed, by means of the addition formula (2.12) for the Jacobi functions, we have

$$\begin{aligned}
&\int_0^1\int_0^\pi \varphi_\lambda^{(\alpha,\beta)}(\arg\cosh|\gamma(x,y,r,\psi)|)\chi_{1,0}^{\alpha,\beta}(r,\psi)dm_{\alpha,\beta}(r,\psi)\\
&= \varphi_{\lambda,1,0}^{(\alpha,\beta)}(x)\varphi_{-\lambda,1,0}^{(\alpha,\beta)}(y)\Pi_{1,0}^{(\alpha,\beta)}\int_0^1\int_0^\pi \chi_{1,0}^{\alpha,\beta}(r,\psi)dm_{\alpha,\beta}(r,\psi)\\
&= \varphi_{\lambda,1,0}^{(\alpha,\beta)}(x)\varphi_{-\lambda,1,0}^{(\alpha,\beta)}(y),
\end{aligned}$$

where $\chi_{1,0}^{\alpha,\beta}(r,\psi) = r\cos\psi$, $\varphi_{\mu,1,0}^{(\alpha,\beta)}(x) = \frac{\rho+i\lambda}{4(\alpha+1)}\sinh 2x\varphi_\mu^{(\alpha+1,\beta+1)}(x)$ and $\Pi_{1,0}^{(\alpha,\beta)}$ is the constant (2.13). Now (3.6) follows from the fact that $\varphi_{-\lambda}^{(\alpha+1,\beta+1)}(x) = \varphi_\lambda^{(\alpha+1,\beta+1)}(x)$. Moreover, by performing the change of variables $(r,\psi)\to(z,\chi)$ defined by

$$\cosh ze^{i\chi} = \gamma(x,y,r,\psi),$$

and applying the same arguments as in the preliminary section for (2.1), the identity (3.6) becomes

$$\begin{aligned}
\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y) &= -2M_{\alpha,\beta}\int_{|x-y|}^{x+y} G_{\lambda,e}^{(\alpha,\beta)}(z)(\sinh x\sinh y\sinh z)^{-2\alpha}\\
&\quad\times\Big\{\int_0^\pi \sigma_{x,y,z}^\chi g(x,y,z,\chi)_+^{\alpha-\beta-1}(\sin\chi)^{2\beta}d\chi\Big\}A_{\alpha,\beta}(z)dz.
\end{aligned}$$

Using the evenness in (x,y,z) of $g(x,y,z,\chi)$, and the following symmetries

$$\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y) = \operatorname{sign}(xy)\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(|x|,|y|),$$

$$\sigma_{x,y,z}^\chi = \operatorname{sign}(xy)\sigma_{|x|,|y|,z}^\chi,$$

we get for all $x,y\in\mathbb{R}^*$

$$\begin{aligned}
\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y) &= -M_{\alpha,\beta}\int_{I_{x,y}} G_\lambda^{(\alpha,\beta)}(z)(\sinh|x|\sinh|y|\sinh|z|)^{-2\alpha}\\
&\quad\Big\{\int_0^\pi \sigma_{x,y,z}^\chi g(x,y,z,\chi)_+^{\alpha-\beta-1}(\sin\chi)^{2\beta}d\chi\Big\}A_{\alpha,\beta}(|z|)dz.
\end{aligned}$$

On the other hand, in order to handle the expression $\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x,y)$ for $x,y>0$, we use the product formulas (2.7) for $\varphi_\lambda^{(\alpha+1,\beta+1)}$ to get

$$\begin{aligned}
&\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x,y)\\
=&\frac{\rho(\rho+i\lambda)}{8(\alpha+1)^2}\sinh 2x\sinh 2y\;\varphi_\lambda^{(\alpha+1,\beta+1)}(x)\varphi_\lambda^{(\alpha+1,\beta+1)}(y)\\
=&\frac{\rho M_{\alpha+1,\beta+1}}{(\alpha+1)}\int_{|x-y|}^{x+y}\sinh(2x)\sinh(2y)(\sinh x\sinh y\sinh z)^{-2\alpha-2}\frac{\rho+i\lambda}{4(\alpha+1)}\varphi_\lambda^{(\alpha+1,\beta+1)}(z)\\
&\times\Big\{\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}(\sin\chi)^{2\beta+2}d\chi\Big\}A_{\alpha+1,\beta+1}(z)dz\\
=&2\frac{\rho}{\beta+\frac12}M_{\alpha,\beta}\int_{|x-y|}^{x+y}G_{\lambda,o}^{(\alpha,\beta)}(z)(\sinh x\sinh y\sinh z)^{-2\alpha}\coth x\coth y\coth z\\
&\times\Big\{\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}(\sin\chi)^{2\beta+2}d\chi\Big\}A_{\alpha,\beta}(z)dz.
\end{aligned}$$

Arguing again by evenness and oddness, and using the fact that $\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x,y)=\operatorname{sign}(xy)\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(|x|,|y|)$ for $x,y\in\mathbb{R}^*$, we conclude that for all $x,y\in\mathbb{R}^*$

$$\begin{aligned}
\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x,y)=\frac{\rho}{\beta+\frac12}M_{\alpha,\beta}&\int_{I_{x,y}}G_\lambda^{(\alpha,\beta)}(z)(\sinh|x|\sinh|y|\sinh|z|)^{-2\alpha}\\
&(\coth x\coth y\coth z)\Big\{\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}(\sin\chi)^{2\beta+2}d\chi\Big\}A_{\alpha,\beta}(|z|)dz.
\end{aligned}$$

This finishes the proof of Lemma 3.4, and therefore the proof of Theorem 3.2. ☐

Next we turn our attention to the case $\alpha=\beta>-\frac12$. For $x,y,z\in\mathbb{R}$, let

$$\sigma_{x,y,z}=\begin{cases}\frac{\cosh 2x\cosh 2y-\cosh 2z}{\sinh 2x\sinh 2y}, & if\ xy\neq 0,\\ 0, & if\ xy=0.\end{cases}\tag{3.7}$$

Moreover, we define the kernel $\mathcal{K}_\alpha$ by

$$\mathcal{K}_\alpha(x,y,z)=\frac12(1-\sigma_{x,y,z}+\sigma_{z,x,y}+\sigma_{z,y,x}+2(1-B^2)\coth x\coth y\coth z)W_{\alpha,\alpha}(x,y,z)$$

if $||x|-|y||<|z|<|x|+|y|$, and $\mathcal{K}_\alpha(x,y,z)=0$ otherwise. Recall the expressions of $1-B^2$ and $W_{\alpha,\alpha}$ from (2.10) and (2.11) respectively. The symmetry properties of $\mathcal{K}_{\alpha,\beta}$ (see Remark 3.1) remain true for $\mathcal{K}_\alpha$.

Theorem 3.5. *In the case $\alpha=\beta>-\frac12$ the product formula reads*

$$G_\lambda^{(\alpha,\alpha)}(x)G_\lambda^{(\alpha,\alpha)}(y)=\int_{-\infty}^{\infty}G_\lambda^{(\alpha,\alpha)}(z)d\mu_{x,y}^{(\alpha)}(z),\tag{3.8}$$

for $x, y \in \mathbb{R}$ *and* $\lambda \in \mathbb{C}$. *Here*

$$d\mu^{(\alpha)}_{x,y}(z) = \begin{cases} \mathcal{K}_\alpha(x,y,z)A_{\alpha,\alpha}(|z|)dz, & if \;\; xy \neq 0, \\ \delta_x, & if \;\; y = 0, \\ \delta_y, & if \;\; x = 0. \end{cases} \tag{3.9}$$

Proof. Applying the same arguments as for the case $\alpha > \beta$, we obtain

$$G^{(\alpha,\alpha)}_{\lambda,e}(x)G^{(\alpha,\alpha)}_{\lambda,e}(y) = 1/2 \int_{I_{x,y}} G^{(\alpha,\alpha)}_{\lambda}(z)W_{\alpha,\alpha}(|x|,|y|,|z|)A_{\alpha,\alpha}(|z|)dz,$$

and

$$G^{(\alpha,\alpha)}_{\lambda,o}(x)G^{(\alpha,\alpha)}_{\lambda,e}(y) = 1/2 \int_{I_{x,y}} G^{(\alpha,\alpha)}_{\lambda}(z)\sigma_{z,x,y}W_{\alpha,\alpha}(|x|,|y|,|z|)A_{\alpha,\alpha}(|z|)dz,$$

where $I_{x,y} = [-|x|-|y|, -||x|-|y||] \cup [||x|-|y||, |x|+|y|]$ and $\sigma_{x,y,z}$ is as in (3.7). On the other hand, for $x, y > 0$, we have

$$G^{(\alpha,\alpha)}_{\lambda,o}(x)G^{(\alpha,\alpha)}_{\lambda,o}(y) = \mathcal{I}^{(\alpha,\alpha)}_{\lambda,1}(x,y) + \mathcal{I}^{(\alpha,\alpha)}_{\lambda,2}(x,y),$$

where $\mathcal{I}^{(\alpha,\alpha)}_{\lambda,1}$ and $\mathcal{I}^{(\alpha,\alpha)}_{\lambda,2}$ are as in (3.4) and (3.5) with $\alpha = \beta$. Applying the same arguments as for $\mathcal{I}^{(\alpha,\beta)}_{\lambda,1}$, we can show that

$$\begin{aligned} \mathcal{I}^{(\alpha,\alpha)}_{\lambda,1}(x,y) &= -M_\alpha \int_0^\pi G^{(\alpha,\alpha)}_{\lambda,e}(\arg\cosh|\gamma(x,y,1,\psi)|)\cos\psi\sin^{2\alpha}\psi d\psi \\ &= -\int_{|x-y|}^{x+y} G^{(\alpha,\alpha)}_{\lambda,e}(z)\sigma_{x,y,z}W_{\alpha,\alpha}(x,y,z)A_{\alpha,\alpha}(z)dz. \end{aligned}$$

Here we have used the change of variables $\psi \mapsto z$ defined by $\cosh z = |\gamma(x,y,1,\psi)|$. Thus, for $x, y \in \mathbb{R}^*$, we have

$$\mathcal{I}^{(\alpha,\alpha)}_{\lambda,1}(x,y) = -\frac{1}{2}\int_{I_{x,y}} G^{(\alpha,\alpha)}_{\lambda}(z)\sigma_{x,y,z}W_{\alpha,\alpha}(|x|,|y|,|z|)A_{\alpha,\alpha}(|z|)dz.$$

On the other hand, by using the product formula (2.7) for $\varphi^{(\alpha+1,\alpha+1)}_\lambda$, we get for $x, y > 0$

$$\mathcal{I}^{(\alpha,\alpha)}_{\lambda,2}(x,y) = \frac{2\alpha+1}{2(\alpha+1)}\int_{|x-y|}^{x+y} G^{(\alpha,\alpha)}_{\lambda,o}(z)\frac{\sinh 2x \sinh 2y}{\sinh 2z}W_{\alpha+1,\alpha+1}(x,y,z)A_{\alpha+1,\alpha+1}(z)dz.$$

Since $W_{\alpha+1,\alpha+1}(x,y,z) = \left(\frac{\alpha+1}{\alpha+1/2}\right)\frac{1-B^2}{(\sinh x\sinh y\sinh z)^2}W_{\alpha,\alpha}(x,y,z)$, it follows that for $x, y > 0$

$$\begin{aligned} &\mathcal{I}^{(\alpha,\alpha)}_{\lambda,2}(x,y) \\ &= \frac{2\alpha+1}{(\alpha+1/2)}\int_{|x-y|}^{x+y} G^{(\alpha,\alpha)}_{\lambda,o}(z)(1-B^2)\coth x\coth y\coth z\ W_{\alpha,\alpha}(x,y,z)A_{\alpha,\alpha}(z)dz \\ &= \int_{I_{x,y}} G^{(\alpha,\alpha)}_{\lambda}(z)(1-B^2)\coth x\coth y\coth|z|W_{\alpha,\alpha}(x,y,|z|)A_{\alpha,\alpha}(|z|)dz. \end{aligned}$$

This finishes the proof of the theorem. □

For $\varepsilon > 0$, we have

$$G^{(\alpha,\alpha)}_{\lambda/\varepsilon}(\varepsilon x) = \varphi^{(\alpha,\alpha)}_{\lambda/\varepsilon}(\varepsilon x) + \frac{2\alpha+1+i\lambda/\varepsilon}{4(\alpha+1)}\sinh(2\varepsilon x)\varphi^{(\alpha,\alpha)}_{\lambda/\varepsilon}(\varepsilon x).$$

Using the fact that

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\{1+\mathcal{O}(z^{-1})\} \qquad \text{as} \qquad z \to \infty,$$

and the fact

$$\varphi^{(\alpha,\alpha)}_{\lambda/\varepsilon}(\varepsilon x) = {}_2F_1\big(\frac{1}{2}(2\alpha+1+\frac{i\lambda}{\varepsilon}), \frac{1}{2}(2\alpha+1-\frac{i\lambda}{\varepsilon}); \alpha+1; -\sinh^2\varepsilon x\big),$$

we deduce that

$$\lim_{\varepsilon\to 0}\varphi^{(\alpha,\alpha)}_{\lambda/\varepsilon}(\varepsilon x) = j_\alpha(x\lambda),$$

where $j_\alpha(x)$ is the normalized Bessel function

$$j_\alpha(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\alpha+1+k)k!}(\frac{x}{2})^{2k}.$$

Thus

$$\lim_{\varepsilon\to 0} G^{(\alpha,\alpha)}_{\lambda/\varepsilon}(\varepsilon x) = j_\alpha(x\lambda) + \frac{i\lambda x}{2(\alpha+1)}j_{\alpha+1}(x\lambda). \tag{3.10}$$

The right hand side of (3.10) is the so-called Dunkl kernel $E_\alpha(i\lambda, x)$ in dimension 1. Moreover, it is proved in [13] that

$$E_\alpha(i\lambda, x)E_\alpha(i\lambda, y) = \int_{\mathbb{R}} E_\alpha(i\lambda, z)k_\alpha(x,y,z)|z|^{2\alpha+1}dz, \tag{3.11}$$

where

$$\begin{aligned} k_\alpha(x,y,z) &= 2^{-2\alpha}\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}[1-\varsigma_{x,y,z}+\varsigma_{z,x,y}+\varsigma_{z,y,x}] \\ &\quad \times\frac{\big(((x+y)^2-z^2)(z^2-(x-y)^2)\big)^{\alpha-\frac{1}{2}}}{(xyz)^{2\alpha}}, \end{aligned}$$

and

$$\varsigma_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy} & if\ xy \neq 0, \\ 0 & if\ xy = 0. \end{cases} \tag{3.12}$$

Proposition 3.6. *For $\alpha > -\frac{1}{2}$, the following limit holds*

$$\lim_{\varepsilon\to 0}\varepsilon^{-2\alpha-2}\mathcal{K}_\alpha(\varepsilon x, \varepsilon y, \varepsilon z) = k_\alpha(x,y,z).$$

Proof. For $\varepsilon > 0$, we have

$$\begin{aligned}\mathcal{K}_\alpha(\varepsilon x, \varepsilon y, \varepsilon z) &= \frac{1}{2}\Big(1 - \sigma_{\varepsilon x,\varepsilon y,\varepsilon z} + \sigma_{\varepsilon z,\varepsilon x,\varepsilon y} + \sigma_{\varepsilon z,\varepsilon y,\varepsilon x} + \\ &\quad +2(1-B_\varepsilon^2)\coth\varepsilon x\coth\varepsilon y\coth\varepsilon z\Big)W_{\alpha,\alpha}(\varepsilon x,\varepsilon y,\varepsilon z),\end{aligned}$$

where

$$1 - B_\varepsilon^2 = \frac{\sinh\epsilon(x+y+z)\sinh\epsilon(x+y-z)\sinh\epsilon(x-y+z)\sinh\epsilon(-x+y+z)}{4\cosh^2\epsilon x\cosh^2\epsilon y\cosh^2\epsilon z}.$$

We may rewrite $\sigma_{x,y,z}$ as:

$$\sigma_{x,y,z} = \frac{\sinh^2 x + \sinh^2 y + 2\sinh^2 x\sinh^2 y - \sinh^2 z}{2\cosh x\cosh y\sinh x\sinh y}.$$

Thus

$$\lim_{\varepsilon\to 0}\sigma_{\varepsilon x,\varepsilon y,\varepsilon z} = \frac{x^2+y^2-z^2}{2xy} = \varsigma_{x,y,z},$$

where $\varsigma_{x,y,z}$ is as in (3.12). Moreover, it is easy to see that

$$1 - B_\varepsilon^2 \sim \frac{\varepsilon^4}{4}((x+y)^2 - z^2)(z^2-(x-y)^2) \quad \text{as} \quad \varepsilon \to 0$$

and therefore

$$\lim_{\varepsilon\to 0}(1-B_\varepsilon^2)\coth\varepsilon x\coth\varepsilon y\coth\varepsilon z = 0.$$

By using the expression (2.11) of $W_{\alpha,\alpha}$, we conclude that

$$W_{\alpha,\alpha}(\varepsilon x,\varepsilon y,\varepsilon z) \sim 2^{-2\alpha}\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\varepsilon^{-2\alpha-2}\frac{\big(((x+y)^2-z^2)(z^2-(x-y)^2)\big)^{\alpha-\frac{1}{2}}}{(xyz)^{2\alpha}}$$

as $\varepsilon \to 0$. □

Remark 3.7. *If we replace λ by λ/ε and (x, y) by $(\varepsilon x, \varepsilon y)$ in Theorem 3.5, then we recover the product formula* (3.11) *for the Dunkl kernels from* (3.8) *as $\varepsilon \to 0$.*

Theorem 3.8. *Let $x, y \in \mathbb{R}$.*

(i) *For $\alpha \geq \beta \geq -\frac{1}{2}$ with $\alpha \neq -\frac{1}{2}$, we have* $\operatorname{supp}(\mu_{x,y}^{(\alpha,\beta)}) \subset I_{x,y}$.

(ii) *For $\alpha \geq \beta \geq -\frac{1}{2}$ with $\alpha \neq -\frac{1}{2}$, we have $\mu_{x,y}^{(\alpha,\beta)}(\mathbb{R}) = 1$.*

(iii) *For $\alpha > \beta > -\frac{1}{2}$, we have $\|\mu_{x,y}^{(\alpha,\beta)}\| \leq 4 + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})}\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta+1)}$.*

(iv) *For $\alpha = \beta > -\frac{1}{2}$, we have$\|\mu_{x,y}^{(\alpha,\alpha)}\| \leq \frac{5}{2}$.*

Proof. (i) is obvious.

(ii) This claim follows from Theorems 3.2 and 3.5 and the fact that $G_{i\rho}^{(\alpha,\beta)} \equiv 1$.

(iii) From the proof of Theorem 3.2, we may rewrite the product formula for $G_\lambda^{(\alpha,\beta)}$ as follows:

$$G_\lambda^{(\alpha,\beta)}(x)G_\lambda^{(\alpha,\beta)}(y) = \int_{I_{x,y}} G_\lambda^{(\alpha,\beta)}(z)\tilde{\mathcal{K}}_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(|z|)dz + \mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x,y),$$

where $\mathcal{I}^{(\alpha,\beta)}_{\lambda,2}$ is given by (3.5) and

$$\begin{aligned}\tilde{\mathcal{K}}_{\alpha,\beta}(x,y,z) :=& M_{\alpha,\beta}(\sinh|x|\sinh|y|\sinh|z|)^{-2\alpha}\\ & \int_0^{\pi}(1-\sigma^{\chi}_{x,y,z}+\sigma^{\chi}_{z,y,x}+\sigma^{\chi}_{z,x,y})g(x,y,z,\chi)_+^{\alpha-\beta-1}(\sin\chi)^{2\beta}d\chi.\end{aligned}$$

By [1, Proposition 2.7], we have

$$\int_{I_{x,y}}|\tilde{\mathcal{K}}_{\alpha,\beta}(x,y,z)|A_{\alpha,\beta}(|z|)dz \le 4.$$

On the other hand, using the product formula (2.1) for the Jacobi functions, we may rewrite $\mathcal{I}^{(\alpha,\beta)}_{\lambda,2}$ as follows :

$$\begin{aligned}&\mathcal{I}^{(\alpha,\beta)}_{\lambda,2}(x,y)\\ &= \frac{\rho(\rho+i\lambda)}{8(\alpha+1)^2}\sinh(2x)\sinh(2y)\varphi^{(\alpha+1,\beta+1)}_{\lambda}(x)\varphi^{(\alpha+1,\beta+1)}_{\lambda}(y)\\ &= \frac{\rho(i\lambda+\rho)}{8(\alpha+1)^2}\sinh(2x)\sinh(2y)\int_0^1\int_0^{\pi}\varphi^{(\alpha+1,\beta+1)}_{\lambda}(\arg\cosh(|\gamma(x,y,r,\psi)|)dm_{\alpha+1,\beta+1}(r,\psi)\\ &= \frac{\rho}{4(\alpha+1)}\sinh(2x)\sinh(2y)\int_0^1\int_0^{\pi}\frac{G^{(\alpha,\beta)}_{\lambda,o}(\arg\cosh(|\gamma(x,y,r,\psi)|))}{|\gamma(x,y,r,\psi)|\sqrt{|\gamma(x,y,r,\psi)|^2-1}}dm_{\alpha+1,\beta+1}(r,\psi),\end{aligned}$$

where $\gamma(x,y,r,\psi)=\cosh x\cosh y+re^{i\psi}\sinh x\sinh y$. In order to conclude, it remains for us to prove the following inequality

$$\frac{\rho}{4(\alpha+1)}\sinh 2x\sinh 2y\int_0^1\int_0^{\pi}\frac{dm_{\alpha+1,\beta+1}(r,\psi)}{|\gamma(x,y,r,\psi)|\sqrt{|\gamma(x,y,r,\psi)|^2-1}} \le \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})}\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta+1)}.$$

By expressing $|\gamma(x,y,r,\psi)|$ and $dm_{\alpha+1,\beta+1}$, the left hand side becomes

$$\begin{aligned}&\frac{\rho}{4(\alpha+1)}\sinh 2x\sinh 2y\int_0^1\int_0^{\pi}\frac{dm_{\alpha+1,\beta+1}(r,\psi)}{|\gamma(x,y,r,\psi)|\sqrt{|\gamma(x,y,r,\psi)|^2-1}}\\ =&\frac{\rho}{4(\alpha+1)}\Big[\frac{2\Gamma(\alpha+2)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{3}{2})}\Big]\sinh 2x\sinh 2y\int_0^1\int_0^{\pi}(1-r^2)^{\alpha-\beta-1}\\ &\times r^{2\beta+3}(\sin\psi)^{2\beta+2}\frac{1}{\sqrt{(\cosh x\cosh y+r\cos\psi\sinh x\sinh y)^2+(r\sin\psi\sinh x\sinh y)^2}}\\ &\times\frac{1}{\sqrt{(\cosh x\cosh y+r\cos\psi\sinh x\sinh y)^2+(r\sin\psi\sinh x\sinh y)^2-1}}drd\psi\\ =&\frac{\rho\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+\frac{3}{2})}\int_0^1\int_0^{\pi}(1-r^2)^{\alpha-\beta-1}(r\sin\psi)^{2\beta+2}\frac{drd\psi}{\sqrt{U+\cos\psi}\sqrt{V+\cos\psi}},\end{aligned}$$

where

$$U=\frac{\cosh^2 x\cosh^2 y+r^2\sinh^2 x\sinh^2 y}{2r\cosh x\cosh y\sinh x\sinh y},$$

and

$$V=\frac{\cosh^2 x\cosh^2 y+r^2\sinh^2 x\sinh^2 y-1}{2r\cosh x\cosh y\sinh x\sinh y}.$$

Since

$$U-1>V-1=\frac{(\cosh x\cosh y-r\sinh x\sinh y)^2-1}{2r\cosh x\cosh y\sinh x\sinh y}\geq 0,$$

we can estimate

$$\begin{aligned}
&\frac{\rho}{4(\alpha+1)}\sinh 2x\sinh 2y\int_0^1\int_0^\pi\frac{dm_{\alpha+1,\beta+1}}{|\gamma(x,y,r,\psi)|\sqrt{|\gamma(x,y,r,\psi)|^2-1}}\\
\leq\ &\frac{\rho\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+\frac{3}{2})}\int_0^1\int_0^\pi(1-r^2)^{\alpha-\beta-1}(r\sin\psi)^{2\beta+2}(1+\cos\psi)^{-1}drd\psi\\
=\ &\frac{\rho}{2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{3}{2})}\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta+1)}\leq\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})}\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta+1)}
\end{aligned}$$

using classical formulas for the Beta and Gamma functions.
(iv) is proved in a similar way, using use the product formula (2.3) for $\varphi_\lambda^{(\alpha,\alpha)}$ instead of (2.1). □

Remark 3.9. *For all $\alpha\geq\beta>-\frac{1}{2}$, the measures $\mu_{x,y}^{(\alpha,\beta)}$ are not positive in general. Indeed, we may rewrite the kernel $\mathcal{K}_{\alpha,\beta}$ as*

$$\mathcal{K}_{\alpha,\beta}(x,y,z)\ =\ M_{\alpha,\beta}(\sinh|x|\sinh|y|\sinh|z|)^{-2\alpha}\Big[\mathcal{K}^{(1)}_{\alpha,\beta}(x,y,z)+\mathcal{K}^{(2)}_{\alpha,\beta}(x,y,z)\Big],$$

where

$$\mathcal{K}^{(1)}_{\alpha,\beta}(x,y,z)=\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}\varrho^\chi(x,y,z)(\sin\chi)^{2\beta}d\chi$$

with

$$\varrho^\chi(x,y,z)=1-\sigma^\chi_{x,y,z}+\sigma^\chi_{z,x,y}+\sigma^\chi_{z,y,x},$$

and

$$\mathcal{K}^{(2)}_{\alpha,\beta}(x,y,z)=\frac{\rho}{\beta+\frac{1}{2}}\coth x\coth y\coth z\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}(\sin\chi)^{2\beta+2}d\chi.$$

Assume that $x>0$. It is easy to see that $\frac{\partial}{\partial\chi}\varrho^\chi(x,x,-\frac{x}{2})\leq 0$, for all $\chi\in[0,\pi]$, and therefore $\varrho^\chi(x,x,-\frac{x}{2})\leq\varrho^0(x,x,-\frac{x}{2})$. Moreover, by studying the growth of the function $x\mapsto\varrho^0(x,x,-\frac{x}{2})$, on can check that $\varrho^0(x,x,-\frac{x}{2})\leq-\frac{3}{8}$. Then $\mathcal{K}^{(1)}_{\alpha,\beta}(x,x,-\frac{x}{2})<0$. Obviously $\mathcal{K}^{(2)}_{\alpha,\beta}(x,x,-\frac{x}{2})\leq 0$. Hence, there exists a neighborhood of $-\frac{x}{2}$ in the support of $\mu_{x,x}^{(\alpha,\beta)}$ for which $z\mapsto\mathcal{K}_{\alpha,\beta}(x,x,z)$ is strictly negative.

4. The convolution product

For $\alpha \geq \beta \geq -\frac{1}{2}$ and $f \in \mathcal{C}_c(\mathbb{R})$, the Opdam-Cherednik Fourier transform is defined by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x) G_\lambda^{(\alpha,\beta)}(-x) A_{\alpha,\beta}(|x|)dx \qquad \forall \lambda \in \mathbb{C}, \tag{4.1}$$

where $A_{\alpha,\beta}(x) = (\sinh x)^{2\alpha+1}(\cosh x)^{2\beta+1}$. The inverse Fourier transform of a suitable function g on $\mathbb{R}$ is given by:

$$\mathcal{J}g(x) \;=\; \int_{\mathbb{R}} g(\lambda) G_\lambda^{(\alpha,\beta)}(x)(1 - \frac{\rho}{i\lambda})\frac{d\lambda}{8\pi|c_{\alpha,\beta}(\lambda)|^2},$$

where

$$c_{\alpha,\beta}(\lambda) := \frac{2^{\rho - i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\lambda))}, \qquad \lambda \in \mathbb{C} \setminus i\mathbb{N}.$$

We refer to [11] for more details on $\mathcal{F}$.

We may express the Fourier transform $\mathcal{F}$ in terms of the Jacobi transform

$$\mathcal{F}_{\alpha,\beta}(f)(\lambda) = \int_0^\infty f(x)\varphi_\lambda^{(\alpha,\beta)}(x) A_{\alpha,\beta}(x)dx. \tag{4.2}$$

More precisely:

Lemma 4.1. *For $\lambda \in \mathbb{C}$ and $f \in C_c(\mathbb{R})$, we have*

$$\mathcal{F}(f)(\lambda) = 2\mathcal{F}_{\alpha,\beta}(f_e)(\lambda) + 2(\rho + i\lambda)\mathcal{F}_{\alpha,\beta}(Jf_o)(\lambda),$$

where f_e (resp. f_o) denotes the even (resp. odd) part of f, and

$$Jf_o(x) := \int_{-\infty}^x f_o(t)dt.$$

Proof. Write $f = f_e + f_o$. Firstly, if $\lambda = -i\rho$, then

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)A_{\alpha,\beta}(|x|)dx = 2\mathcal{F}_{\alpha,\beta}(f_e)(i\rho).$$

Secondly, if $\lambda \neq -i\rho$, we have

$$\mathcal{F}(f)(\lambda) = 2\mathcal{F}_{\alpha,\beta}(f_e)(\lambda) + \frac{2}{\rho - i\lambda}\int_0^\infty f_o(x)\frac{d}{dx}\varphi_\lambda^{(\alpha,\beta)}(x)A_{\alpha,\beta}(x)dx.$$

Let $\Delta_{\alpha,\beta}$ be the Jacobi operator given by

$$\Delta_{\alpha,\beta} f(x) = \frac{1}{A_{\alpha,\beta}(x)}\frac{d}{dx}\Big[A_{\alpha,\beta}(x)\frac{d}{dx}f(x)\Big].$$

By integration by parts, we obtain

$$\begin{aligned}
& \int_0^\infty f_o(x) \frac{d}{dx} \varphi_\lambda^{(\alpha,\beta)}(x) A_{\alpha,\beta}(x) dx \\
= & -\int_0^\infty \varphi_\lambda^{(\alpha,\beta)}(x) \frac{1}{A_{\alpha,\beta}(x)} \frac{d}{dx} \Big[A_{\alpha,\beta}(x) \frac{d}{dx} (J f_o(x)) \Big] A_{\alpha,\beta}(x) dx \\
= & -\mathcal{F}_{\alpha,\beta}(\Delta_{\alpha,\beta} J f_o)(\lambda) \\
= & (\lambda^2 + \rho^2) \mathcal{F}_{\alpha,\beta}(J f_o)(\lambda).
\end{aligned}$$

□

The following Plancherel formula was proved by Opdam [11, Theorem 9.13(3)]:

$$\begin{aligned}
\int_{\mathbb{R}} |f(x)|^2 A_{\alpha,\beta}(|x|) dx &= \int_{\mathbb{R}^+} (|\mathcal{F}(f)(\lambda)|^2 + |\mathcal{F}(\check{f})(\lambda)|^2) \frac{d\lambda}{16\pi |c(\lambda)|^2} \\
&= \int_{\mathbb{R}} \mathcal{F}(f)(\lambda) \overline{\mathcal{F}(\check{f})(-\lambda)} (1 - \frac{\rho}{i\lambda}) \frac{d\lambda}{8\pi |c(\lambda)|^2},
\end{aligned}$$

where $\check{f}(x) := f(-x)$. The following result can be proved by specializing [16, Theorem 4.1].

Theorem 4.2. *The Opdam-Cherednik Fourier transform* $\mathcal{F}$ *and its inverse* $\mathcal{J}$ *are topological isomorphisms between the Schwartz space* $\mathcal{S}_{\alpha,\beta}(\mathbb{R}) = (\cosh x)^{-\rho} \mathcal{S}(\mathbb{R})$ *and the Schwartz space* $\mathcal{S}(\mathbb{R})$. *Recall that* $\rho = \alpha + \beta + 1$.

Definition 4.3. *Let* f *be a suitable function on* $\mathbb{R}$ *and let* $x \in \mathbb{R}$. *For* $\alpha \geq \beta \geq -\frac{1}{2}$, *with* $\alpha \neq -\frac{1}{2}$, *we define the generalized translation operator* $\tau_x^{(\alpha,\beta)}$ *by*

$$\tau_x^{(\alpha,\beta)} f(y) = \int_{\mathbb{R}} f(z) d\mu_{x,y}^{(\alpha,\beta)}(z),$$

where $d\mu_{x,y}^{(\alpha,\beta)}$ *is given by* (3.2) *for* $\alpha > \beta$, *and by* (3.9) *for* $\alpha = \beta$.

The following proposition is clear. However for completeness we will sketch its proof.

Proposition 4.4. *For a suitable function* f *on* $\mathbb{R}$, *we have*

(i) $\tau_x^{(\alpha,\beta)} f(y) = \tau_y^{(\alpha,\beta)} f(x)$.
(ii) $\tau_0^{(\alpha,\beta)} f(y) = f(y)$.
(iii) $\tau_x^{(\alpha,\beta)} \tau_y^{(\alpha,\beta)} = \tau_y^{(\alpha,\beta)} \tau_x^{(\alpha,\beta)}$.
(iv) $\tau_x^{(\alpha,\beta)} G_\lambda^{(\alpha,\beta)}(y) = G_\lambda^{(\alpha,\beta)}(x) G_\lambda^{(\alpha,\beta)}(y)$.
(v) $\mathcal{F}(\tau_x^{(\alpha,\beta)} f)(\lambda) = G_\lambda^{(\alpha,\beta)}(x) \mathcal{F}(f)(\lambda)$, *where* $\mathcal{F}$ *is given by* (4.1).
(vi) $T^{(\alpha,\beta)}(\tau_x^{(\alpha,\beta)}) f = \tau_x^{(\alpha,\beta)}(T^{(\alpha,\beta)} f)$, *where* $T^{(\alpha,\beta)}$ *is the Dunkl-Cherednik operator* (1.1).

Proof. (i) follows from the property $\mathcal{K}_{\alpha,\beta}(x,y,z)=\mathcal{K}_{\alpha,\beta}(y,x,z)$.
(ii) follows from the fact that $\mathcal{K}_{\alpha,\beta}(0,y,z)=\delta_y$.
(iii) follows from the fact that the function

$$H(x_1,y_1,x_2,y_2):=\int_{\mathbb{R}}\mathcal{K}_{\alpha,\beta}(x_1,y_1,z)\mathcal{K}_{\alpha,\beta}(x_2,y_2,z)A_{\alpha,\beta}(|z|)dz$$

is symmetric in the four variables.
(iv) follows from the product formula for $G_\lambda^{(\alpha,\beta)}$.
(v) For $f\in\mathcal{C}_c(\mathbb{R})$, we have

$$\begin{aligned}\mathcal{F}(\tau_x^{(\alpha,\beta)}f)(\lambda) &= \int_{\mathbb{R}}\tau_x^{(\alpha,\beta)}f(z)G_\lambda^{(\alpha,\beta)}(-z)A_{\alpha,\beta}(|z|)dz\\ &= \int_{\mathbb{R}}\Big[\int_{\mathbb{R}}f(t)\mathcal{K}_{\alpha,\beta}(x,z,t)A_{\alpha,\beta}(|t|)dt\Big]G_\lambda^{(\alpha,\beta)}(-z)A_{\alpha,\beta}(|z|)dz\\ &= \int_{\mathbb{R}}f(t)\Big[\int_{\mathbb{R}}G_\lambda^{(\alpha,\beta)}(-z)\mathcal{K}_{\alpha,\beta}(x,z,t)A_{\alpha,\beta}(|z|)dz\Big]A_{\alpha,\beta}(|t|)dt.\end{aligned}$$

Since $\mathcal{K}_{\alpha,\beta}(x,z,t)=\mathcal{K}_{\alpha,\beta}(x,-t,-z)$, it follows from Theorem 3.2 that

$$\begin{aligned}\mathcal{F}(\tau_x^{(\alpha,\beta)}f)(\lambda) &= G_\lambda^{(\alpha,\beta)}(x)\int_{\mathbb{R}}f(t)G_\lambda^{(\alpha,\beta)}(-t)A_{\alpha,\beta}(|t|)dt\\ &= G_\lambda^{(\alpha,\beta)}(x)\mathcal{F}(f)(\lambda).\end{aligned}$$

(vi) This property follows from the injectivity of $\mathcal{F}$ and the fact that $\tau_x^{(\alpha,\beta)}(T^{(\alpha,\beta)}f)$ and $T^{(\alpha,\beta)}(\tau_x^{(\alpha,\beta)}f)$ have the same Fourier transform, namely

$$\lambda\mapsto i\lambda\, G_\lambda^{(\alpha,\beta)}(x)\mathcal{F}(f)(\lambda).$$

□

Lemma 4.5. *For* $1\le p\le\infty$, $f\in L^p(\mathbb{R},A_{\alpha,\beta}(|t|)dt)$, *and* $x\in\mathbb{R}$, *we have*

$$\|\tau_x^{(\alpha,\beta)}f\|_p\le C_{\alpha,\beta}\|f\|_p, \tag{4.3}$$

where

$$C_{\alpha,\beta}=\begin{cases}4+\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac12)}\frac{\Gamma(\beta+\frac12)}{\Gamma(\beta+1)}, & if \quad \alpha>\beta>-\frac12,\\ \\ \frac52, & if \quad \alpha=\beta>-\frac12.\end{cases} \tag{4.4}$$

Proof. For $p=\infty$, the inequality (4.3) follows from Theorem 3.7. For $1\le p<\infty$, we use again the boundness of $\int_{\mathbb{R}}|\mathcal{K}_{\alpha,\beta}(x,y,t)|A_{\alpha,\beta}(|y|)dy$ and Hölder's inequality to estimate

$$\begin{aligned}\|\tau_x^{(\alpha,\beta)}f\|_p^p &\le C_{\alpha,\beta}^{p-1}\int_{\mathbb{R}}\int_{\mathbb{R}}|f(t)|^p|\mathcal{K}_{\alpha,\beta}(x,y,t)|A_{\alpha,\beta}(|t|)A_{\alpha,\beta}(|y|)dtdy\\ &\le C_{\alpha,\beta}^p\|f\|_p^p.\end{aligned}$$

□

Definition 4.6. *For suitable functions f and g, we define the convolution product $f *_{\alpha,\beta} g$ by*

$$f *_{\alpha,\beta} g(x) = \int_{\mathbb{R}} \tau_x^{(\alpha,\beta)} f(-y) g(y) A_{\alpha,\beta}(|y|) dy.$$

Remark 4.7. *It is clear that this convolution product is both commutative and associative:*

(i) $f *_{\alpha,\beta} g = g *_{\alpha,\beta} f$.
(ii) $(f *_{\alpha,\beta} g) *_{\alpha,\beta} h = f *_{\alpha,\beta} (g *_{\alpha,\beta} h)$.

Proposition 4.8. *Let $\mathcal{D}_a(\mathbb{R})$ be the space of smooth functions on $\mathbb{R}$ supported in $[-a,a]$. For $f \in \mathcal{D}_a(\mathbb{R})$ and $g \in \mathcal{D}_b(\mathbb{R})$, we have $f *_{\alpha,\beta} g \in \mathcal{D}_{a+b}(\mathbb{R})$ and*

$$\mathcal{F}(f *_{\alpha,\beta} g)(\lambda) = \mathcal{F}(f)(\lambda)\mathcal{F}(g)(\lambda).$$

Proof. By definition we have

$$\mathcal{F}(f *_{\alpha,\beta} g)(\lambda) = \int_{\mathbb{R}}\int_{\mathbb{R}} \tau_x^{(\alpha,\beta)} f(-y) g(y) G_\lambda^{(\alpha,\beta)}(-x) A_{\alpha,\beta}(|x|) A_{\alpha,\beta}(|y|) dx dy.$$

Using the product formula for $G_\lambda^{(\alpha,\beta)}$ and Remark 3.1, we deduce that

$$\begin{aligned}\mathcal{F}(f *_{\alpha,\beta} g)(\lambda) &= \int_{\mathbb{R}} f(z) \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} G_\lambda^{(\alpha,\beta)}(-x) \mathcal{K}_{\alpha,\beta}(-z,-y,-x) A_{\alpha,\beta}(|x|) dx \\ &\quad \times A_{\alpha,\beta}(|y|) dy \; A_{\alpha,\beta}(|z|) dz \\ &= \Big\{ \int_{\mathbb{R}} f(z) G_\lambda^{(\alpha,\beta)}(-z) A_{\alpha,\beta}(|z|) dz \Big\}\Big\{ \int_{\mathbb{R}} g(y) G_\lambda^{(\alpha,\beta)}(-y) A_{\alpha,\beta}(|y|) dy \Big\} \\ &= \mathcal{F}(f)(\lambda)\mathcal{F}(g)(\lambda).\end{aligned}$$

□

By standard arguments, the following statement follows from Lemma 4.5.

Proposition 4.9. *Assume that $1 \le p,q,r \le \infty$ satisfy $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$. Then, for every $f \in L^p(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$ and $g \in L^q(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$, we have $f *_{\alpha,\beta} g \in L^r(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$, and*

$$\|f *_{\alpha,\beta} g\|_r \le C_{\alpha,\beta} \|f\|_p \|g\|_q,$$

where $C_{\alpha,\beta}$ is as in (4.4).

5. The Kunze-Stein phenomenon

This remarkable phenomenon was first observed by Kunze and Stein [8] for the group $G = SL(2,\mathbb{R})$ equipped with its Haar measure. They proved that

$$L^p(G) * L^2(G) \subset L^2(G) \quad \forall 1 \le p < 2.$$

By such inclusion, we mean the existence of a constant $C_p > 0$ such that the following inequality holds:

$$\|f * g\|_2 \le C_p \|f\|_p \|g\|_2 \quad \forall f \in L^p(G),\ \forall g \in L^2(G).$$

This result was generalized by Cowling [2] to connected non compact semisimple Lie groups with finite center. We prove the following analog in our setting (we understand that Trimèche has recently extended this result to higher dimensions).

Theorem 5.1. *Let* $1 \leq p < 2 < q \leq \infty$. *Then*

$$L^p(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) *_{\alpha,\beta} L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) \subset L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) \tag{5.1}$$

and

$$L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) *_{\alpha,\beta} L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) \subset L^q(\mathbb{R}, A_{\alpha,\beta}(|x|)dx). \tag{5.2}$$

Proof. (i) Let $f, g \in \mathcal{C}_c(\mathbb{R})$. Then, by the Plancherel formula, we have

$$\begin{aligned}
&\int_{\mathbb{R}} |(f *_{\alpha,\beta} g)|^2(x) A_{\alpha,\beta}(|x|)dx \\
&= \int_{\mathbb{R}^+} |\mathcal{F}(f *_{\alpha,\beta} g)(\lambda)|^2 \frac{d\lambda}{16\pi|c(\lambda)|^2} + \int_{\mathbb{R}^+} |\mathcal{F}(f *_{\alpha,\beta} g)\check{}(\lambda)|^2 \frac{d\lambda}{16\pi|c(\lambda)|^2} \\
&\leq \sup_{\lambda \in \mathbb{R}, w \in \{\pm 1\}} |\mathcal{F}(w \cdot g)(\lambda)|^2 \Big[\int_{\mathbb{R}^+} |\mathcal{F}(f)(\lambda)|^2 \frac{d\lambda}{16\pi|c(\lambda)|^2} + \int_{\mathbb{R}^+} |\mathcal{F}(\check{f})(\lambda)|^2 \frac{d\lambda}{16\pi|c(\lambda)|^2} \Big] \\
&= \sup_{\lambda \in \mathbb{R}, w \in \{\pm 1\}} |\mathcal{F}(w \cdot g)(\lambda)|^2 \|f\|_2^2.
\end{aligned}$$

Here we have used the fact that $\mathcal{F}(f *_{\alpha,\beta} g)\check{} = \mathcal{F}(\check{f})\mathcal{F}(\check{g})$. Next, if p and q are dual indices, we estimate

$$\begin{aligned}
|\mathcal{F}(w \cdot g)(\lambda)| &\leq \int_{\mathbb{R}} |g(wx)||G_\lambda^{(\alpha,\beta)}(-x)| A_{\alpha,\beta}(x)dx \\
&\leq \|g\|_p \|G_\lambda^{(\alpha,\beta)}\|_q
\end{aligned}$$

using Hölder's inequality. Thus it remains for us to show that $\|G_\lambda^{(\alpha,\beta)}\|_q$ is bounded uniformly in $\lambda \in \mathbb{R}$. This follows from Schapira's inequality [16]

$$|G_\lambda^{\alpha,\beta}(x)| \leq G_0^{(\alpha,\beta)}(x) \qquad \forall x \in \mathbb{R},\ \forall \lambda \in \mathbb{R},$$

and the well-known estimate [5]

$$\varphi_0^{(\alpha,\beta)}(x) \leq C(1 + |x|)e^{-\rho|x|} \quad \forall x \in \mathbb{R},$$

which yield

$$|G_\lambda^{(\alpha,\beta)}(x)| \leq C(1 + |x|)e^{-\rho|x|} \qquad \forall x \in \mathbb{R}, \qquad \forall \lambda \in \mathbb{R}.$$

As a conclusion

$$\|f *_{\alpha,\beta} g\|_2 \leq c_p \|f\|_2 \|g\|_p.$$

(ii) Let $f, g, k \in \mathcal{C}_c(\mathbb{R})$. Using the Cauchy-Schwartz inequality and (5.1), we get

$$\begin{aligned}
\Big| \int_{\mathbb{R}} (f *_{\alpha,\beta} g)(x) k(x) A_{\alpha,\beta}(|x|)dx \Big| &\leq C \|g\|_2 \|f * \check{k}\|_2 \\
&\leq c_p \|f\|_2 \|g\|_2 \|k\|_p.
\end{aligned}$$

Hence $\|f *_{\alpha,\beta} g\|_q \leq c_q \|f\|_2 \|g\|_2$. □

The following results are deduced by interpolation and duality from Theorem 5.1 and Proposition 4.8.

Corollary 5.2. (i) *Let* $1 \leq p < q \leq 2$. *Then*

$$L^p(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) *_{\alpha,\beta} L^q(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) \subset L^q(\mathbb{R}, A_{\alpha,\beta}(|x|)dx).$$

(ii) *Let* $1 < p < 2$ *and* $p < q \leq \frac{p}{2-p}$. *Then*

$$L^p(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) *_{\alpha,\beta} L^p(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) \subset L^q(\mathbb{R}, A_{\alpha,\beta}(|x|)dx).$$

(iii) *Let* $2 < p,\ q < \infty$ *such that* $\frac{q}{2} \leq p < q$. *Then*

$$L^p(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) *_{\alpha,\beta} L^{q'}(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) \subset L^q(\mathbb{R}, A_{\alpha,\beta}(|x|)dx).$$

6. A SPECIAL ORTHOGONAL SYSTEMS

In this section we will construct an orthonormal basis for $L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$. We will also prove that by applying a certain limit argument we recover the Hermite functions for $L^2(\mathbb{R}, |x|^{2\alpha+1}dx)$, constructed by Rosenblum in [13].

Proposition 6.1. *Fix* α, β *and* δ *such that* $\alpha \geq \beta \geq -\frac{1}{2}$ *and* $\delta > -1$. *We set*

$$\begin{cases} H^{\delta}_{2n}(t) := (\cosh t)^{-\alpha-\beta-\delta-2} P_n^{(\alpha,\delta)}(1 - 2\tanh^2 t), \\ H^{\delta}_{2n+1}(t) := (\cosh t)^{-\alpha-\beta-\delta-2} P_n^{(\alpha+1,\delta)}(1 - 2\tanh^2 t)\tanh t, \end{cases} \tag{6.1}$$

where $P_n^{(\alpha,\delta)}(r) = {}_2F_1\big(-n, n+\alpha+\delta+1; \alpha+1; \frac{1-r}{2}\big)$ *denotes the Jacobi polynomial. Then* $\{H^{\delta}_n\}_{n=0}^{+\infty}$ *is an orthogonal basis of* $L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$.

Proof. Firstly, by oddness we have

$$\int_{\mathbb{R}} H^{\delta}_{2n}(t) H^{\delta}_{2n+1}(t)(\sinh|t|)^{2\alpha+1}(\cosh t)^{2\beta+1}dt = 0.$$

Secondly, using the change of variable $x = \tanh t$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}} H^{\delta}_{2n}(t) H^{\delta}_{2m}(t)(\sinh|t|)^{2\alpha+1}(\cosh t)^{2\beta+1}dt \\ &= 2\int_0^1 P_n^{(\alpha,\delta)}(1-2x^2) P_m^{(\alpha,\delta)}(1-2x^2)(1-x^2)^{\delta} x^{2\alpha+1}dx \\ &= 2^{-\delta-\alpha}\int_{-1}^1 P_n^{(\alpha,\delta)}(r) P_m^{(\alpha,\delta)}(r)(1+r)^{\delta}(1-r)^{\alpha}dr \\ &= 2^{-\delta-\alpha-1}\frac{\Gamma(\alpha+n+1)\Gamma(\delta+n+1)}{(\alpha+\delta+2n+1)\Gamma(n+1)\Gamma(\alpha+\delta+n+1)}\delta_{n,m}. \end{aligned}$$

Above we have used the orthogonality of the Jacobi polynomials. The same argument gives

$$\begin{aligned}&\int_{\mathbb{R}} H^\delta_{2n+1}(t)H^\delta_{2m+1}(t)|\sinh t|^{2\alpha+1}(\cosh t)^{2\beta+1}dt\\ =\;& 2^{-\delta-\alpha-2}\frac{\Gamma(\alpha+n+2)\Gamma(\delta+n+1)}{(\alpha+\delta+2n+2)\Gamma(n+1)\Gamma(\alpha+\delta+n+2)}\delta_{n,m}.\end{aligned}$$

Further, it is known that $\{P_n^{(\alpha,\delta)}\}_{n=0}^{+\infty}$ span a dense subspace of $L^2(]-1,1[,(1-r)^\alpha(1+r)^\delta dr)$ (see for instance [15]). Hence $\{H^\delta_{2n}\}_{n=0}^{+\infty}$ and $\{H^\delta_{2n+1}\}_{n=0}^{+\infty}$ span a dense subspace of $L^2(\mathbb{R},A_{\alpha,\beta}(|x|)dx)_e$ and $L^2(\mathbb{R},A_{\alpha,\beta}(|x|)dx)_o$ respectively. □

Proposition 6.2. *The Opdam-Cherednik transform of the basis* $\{H^\delta_n\}_n$ *is given by*

$$\begin{aligned}\mathcal{F}(H^\delta_{2n})(\lambda) &= \frac{(-1)^n}{n!}\frac{\Gamma(\alpha+1)\Gamma(\frac12(\delta-i\lambda+1))\Gamma(\frac12(\delta+i\lambda+1))}{\Gamma(\frac12(\delta+\alpha+\beta+2)+n)\Gamma(\frac12(\delta+\alpha-\beta+2)+n)}\\ &\quad\times P_n\Big(-\frac{\lambda^2}{4};\frac{(\delta+1)}{2},\frac{(\delta+1)}{2},\frac{(\alpha+\beta+1)}{2},\frac{(\alpha-\beta+1)}{2}\Big),\end{aligned}\tag{6.2}$$

and

$$\begin{aligned}\mathcal{F}(H^\delta_{2n+1})(\lambda) &= \frac{i\lambda+\rho}{2}\frac{(-1)^{n+1}}{n!}\frac{\Gamma(\alpha+1)\Gamma(\frac12(\delta-i\lambda+1))\Gamma(\frac12(\delta+i\lambda+1))}{\Gamma(\frac12(\delta+\alpha+\beta+4)+n)\Gamma(\frac12(\delta+\alpha-\beta+2)+n)}\\ &\quad\times P_n\Big(-\frac{\lambda^2}{4};\frac{(\delta+1)}{2},\frac{(\delta+1)}{2},\frac{(\alpha+\beta+3)}{2},\frac{(\alpha-\beta+1)}{2}\Big),\end{aligned}\tag{6.3}$$

where

$$\begin{aligned}P_n(x^2;a,b,c,d) &= (a+b)_n(a+c)_n(a+d)_n\\ &\quad {}_4F_3\left(\begin{matrix}-n, & n+a+b+c+d-1, & a-x, & a+x;\\ a+b, & a+c, & a+d; & \end{matrix}\middle|\,1\right)\end{aligned}$$

denotes Wilson polynomials.

Proof. Since H^δ_{2n} are even functions, and therefore $\mathcal{F}(H^\delta_{2n})$ coincides with $\mathcal{F}_{\alpha,\beta}(H^\delta_{2n})$ (see (4.2) for the definition of $\mathcal{F}_{\alpha,\beta}$), it follows that (6.2) is nothing other than the identity (9.4) given in [10] without details. However, for completeness and since we shall use the same arguments in the proof of $\mathcal{F}(H^\delta_{2n+1})$, we will give all the details. For (6.2), we have

$$\begin{aligned}H^\delta_{2n}(t) &= (-1)^n(\cosh t)^{-\alpha-\beta-\delta-2}P_n^{(\delta,\alpha)}(2\tanh^2 t-1)\\ &= (-1)^n(\cosh t)^{-\alpha-\beta-\delta-2}\frac{(\delta+1)_n}{n!}{}_2F_1(-n,\delta+\alpha+n+1;\delta+1;1-\tanh^2 t)\\ &= (-1)^n\frac{(\delta+1)_n}{n!}(\cosh t)^{-\alpha-\beta-\delta-2}{}_2F_1(-n,\delta+\alpha+n+1;\delta+1;\cosh^{-2}t)\\ &= (-1)^n\frac{(\delta+1)_n}{n!}\sum_{m=0}^{n}\frac{(-n)_m(\delta+\alpha+n+1)_m}{(\delta+1)_m m!}(\cosh t)^{-2m-\alpha-\beta-\delta-2}.\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{F}(H^{\delta}_{2n})(\lambda) &= \int_{\mathbb{R}} H^{\delta}_{2n}(t)\varphi^{(\alpha,\beta)}_{\lambda}(t)|\sinh t|^{2\alpha+1}(\cosh t)^{2\beta+1}dt\\ &= (-1)^n\frac{(\delta+1)_n}{n!}\sum_{m=0}^{n}\frac{(-n)_m(\delta+\alpha+n+1)_m}{(\delta+1)_m m!}\\ &\quad\times\int_{\mathbb{R}}(\cosh t)^{-2m-\alpha-\beta-\delta-2}\varphi^{(\alpha,\beta)}_{\lambda}(t)|\sinh t|^{2\alpha+1}(\cosh t)^{2\beta+1}dt.\end{aligned}$$

By [10, formula (9.1)], we have:

$$\begin{aligned}&\int_0^{\infty}(\cosh t)^{-\mu-\alpha-\beta-1}\varphi^{(\alpha,\beta)}_{\lambda}(t)(\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1}dt\\ &= \frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2}(\mu-i\lambda))\Gamma(\frac{1}{2}(\mu+i\lambda))}{2\Gamma(\frac{1}{2}(\alpha+\beta+\mu+1))\Gamma(\frac{1}{2}(\alpha-\beta+\mu+1))}.\end{aligned}$$

Thus

$$\begin{aligned}&\int_{\mathbb{R}} H^{\delta}_{2n}(t)\varphi^{(\alpha,\beta)}_{\lambda}(t)|\sinh t|^{2\alpha+1}(\cosh t)^{2\beta+1}dt\\ &= (-1)^n\frac{(\delta+1)_n}{n!}\Gamma(\alpha+1)\sum_{m=0}^{n}\frac{(-n)_m(n+\delta+\alpha+1)_m}{(\delta+1)_m m!}\\ &\quad\times\frac{\Gamma(m+\frac{1}{2}(\delta+1-i\lambda))\Gamma(m+\frac{1}{2}(\delta+1+i\lambda))}{\Gamma(m+\frac{1}{2}(\alpha+\beta+\delta+2))\Gamma(m+\frac{1}{2}(\alpha-\beta+\delta+2))}\\ &= (-1)^n\frac{(\delta+1)_n}{n!}\frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2}(\delta-i\lambda+1)\Gamma(\frac{1}{2}(\delta+i\lambda+1))}{\Gamma(\frac{1}{2}(\delta+\alpha+\beta+2))\Gamma(\frac{1}{2}(\delta+\alpha-\beta+2))}\\ &\quad\times{}_4F_3\left(\begin{matrix}-n, & \delta+\alpha+n+1, & \frac{1}{2}(\delta-i\lambda+1), & \frac{1}{2}(\delta+i\lambda+1),\\ \delta+1, & \frac{1}{2}(\delta+\alpha+\beta+2), & \frac{1}{2}(\delta+\alpha-\beta+2), & \end{matrix}\middle|\, 1\right).\end{aligned}$$

Similarly :

$$\begin{aligned}H^{\delta}_{2n+1}(t) &= (-1)^n(\cosh t)^{-\alpha-\beta-\delta-2}P^{(\delta,\alpha+1)}_{n}(2\tanh^2 t-1)\tanh t\\ &= (-1)^n\frac{(\delta+1)_n}{n!}(\cosh t)^{-\alpha-\beta-\delta-2}\tanh t\\ &\quad\times{}_2F_1(-n,n+\delta+\alpha+1;\delta+1;\cosh^{-2}t)\\ &= (-1)^n\frac{(\delta+1)_n}{n!}\sum_{m=0}^{n}\frac{(-n)_m(n+\delta+\alpha+2)_m}{(\delta+1)_m m!}(\cosh t)^{-2m-\alpha-\beta-\delta-3}\sinh t.\end{aligned}$$

Thus

$$\begin{aligned}
&\mathcal{F}(H^{\delta}_{2n+1})(\lambda)\\
=\;& -\frac{\rho+i\lambda}{4(\alpha+1)}\int_{\mathbb{R}} H^{\delta}_{2n+1}(t)\sinh(2t)\varphi_\lambda^{(\alpha+1,\beta+1)}(t)|\sinh t|^{2\alpha+1}(\cosh t)^{2\beta+1}dt\\
=\;& -4\frac{\rho+i\lambda}{4(\alpha+1)}\int_0^\infty H^{\delta}_{2n+1}(t)\varphi_\lambda^{(\alpha+1,\beta+1)}(t)(\sinh t)^{2\alpha+2}(\cosh t)^{2\beta+2}dt\\
=\;& -\frac{\rho+i\lambda}{\alpha+1}(-1)^n\frac{(\delta+1)_n}{n!}\sum_{m=0}^{n}\frac{(-n)_m(n+\delta+\alpha+2)_m}{(\delta+1)_m m!}\\
&\times\int_0^\infty(\cosh t)^{-\alpha-\beta-2-(2m+\delta+1)-1}\varphi_\lambda^{(\alpha+1,\beta+1)}(t)(\sinh t)^{2\alpha+3}(\cosh t)^{2\beta+3}dt\\
=\;& -\frac{\rho+i\lambda}{\alpha+1}(-1)^n\frac{(\delta+1)_n}{n!}\sum_{m=0}^{n}\frac{(-n)_m(n+\delta+\alpha+2)_m}{(\delta+1)_m m!}\\
&\times\frac{\Gamma(\alpha+2)\Gamma(\frac{1}{2}(2m+\delta+1-i\lambda))\Gamma(\frac{1}{2}(2m+\delta+1+i\lambda))}{2\Gamma(\frac{1}{2}(\alpha+\beta+2m+\delta+4))\Gamma(\frac{1}{2}(\alpha-\beta+2m+\delta+2))}\\
=\;& -2^{-1}(-1)^n\Gamma(\alpha+1)(\rho+i\lambda)\frac{(\delta+1)_n}{n!}\sum_{m=0}^{n}\frac{(-n)_m(n+\delta+\alpha+2)_m}{(\delta+1)_m m!}\\
&\times\frac{\Gamma(\frac{1}{2}(2m+\delta+1-i\lambda))\Gamma(\frac{1}{2}(2m+\delta+1+i\lambda))}{\Gamma(\frac{1}{2}(\alpha+\beta+2m+\delta+4))\Gamma(\frac{1}{2}(\alpha-\beta+2m+\delta+2))}\\
=\;& -(-1)^n(\rho+i\lambda)\frac{(\delta+1)_n}{n!}\frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2}(\delta+1-i\lambda))\Gamma(\frac{1}{2}(\delta+1+i\lambda))}{\Gamma(\frac{1}{2}(\alpha+\beta+\delta+4))\Gamma(\frac{1}{2}(\alpha-\beta+\delta+2))}\\
&\times{}_4F_3\left(\begin{matrix}-n, & \delta+\alpha+n+2, & \frac{1}{2}(\delta-i\lambda+1), & \frac{1}{2}(\delta+i\lambda+1);\\ \delta+1, & \frac{1}{2}(\delta+\alpha+\beta+4), & \frac{1}{2}(\delta+\alpha-\beta+2); & \end{matrix}\middle|\,1\right).
\end{aligned}$$

□

Remark 6.3. *Let $\varepsilon>0$. In the definition of $H^{\delta}_n(t)$ we replace δ by ε^{-2} and we make the change of variable $t\to\varepsilon t$. Then (6.1) becomes*

$$\begin{cases} H^{\varepsilon^{-2}}_{2n}(\varepsilon t)=(\cosh\varepsilon t)^{-\alpha-\beta-\varepsilon^{-2}-2}P_n^{(\alpha,\varepsilon^{-2})}(1-2\tan^2\varepsilon t),\\ H^{\varepsilon^{-2}}_{2n+1}(\varepsilon t)=(\cosh\varepsilon t)^{-\alpha-\beta-\varepsilon^{-2}-2}P_n^{(\alpha,\varepsilon^{-2})}(1-2\tan^2\varepsilon t)\tan\varepsilon t.\end{cases}$$

Using the fact that

$$P_n^{(a,b)}(1-2\tanh^2 t)=\frac{(a+1)_n}{n!}\,{}_2F_1\big(-n,n+a+b+1;a+1;\tanh^2 t\big),$$

then we may express $H^{\varepsilon^{-2}}_{2n}(\varepsilon t)$ and $H^{\varepsilon^{-2}}_{2n+1}(\varepsilon t)$ in terms of the hypergeometric function ${}_2F_1\big(-n,n+\alpha+\varepsilon^{-2}+1;\alpha+1;\tanh^2\varepsilon t\big)$. Moreover, it is easy to check that

$$\cosh(\varepsilon t)^{\varepsilon^{-2}}\sim e^{-\frac{t^2}{2}}\quad as\ \varepsilon\to 0,$$

and that

$$\lim_{p\to\infty} {}_2F_1(a,p;b;\frac{z}{p}) = {}_1F_1(a;b;z).$$

Hence

$$\begin{aligned}\lim_{\varepsilon\to 0} H_{2n}^{\varepsilon^{-2}}(\varepsilon t) &= \frac{(\alpha+1)_n}{n!} e^{\frac{t^2}{2}} {}_1F_1(-n,\alpha+1;t^2)\\ &= e^{-\frac{t^2}{2}} L_n^{\alpha}(t^2),\end{aligned}$$

and

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} H_{2n+1}^{\varepsilon^{-2}}(\varepsilon t) = e^{-\frac{t^2}{2}} t L_n^{\alpha+1}(t^2),$$

where L_n^α denotes the Laguerre polynomial. Thus we recover the even and the odd Hermite functions constructed by Rosenblum in the rational Dunkl setting [13, Definition 3.4].

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