

On entropy and Hausdorff dimension of measures defined through a non-homogeneous Markov process

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Abstract

In this work we study the Hausdorff dimension of measures whose weight distribution satisfies a markov non-homogeneous property. We prove, in particular, that the Hausdorff dimensions of this kind of measures coincide with their lower Rényi dimensions (entropy). Moreover, we show that the Tricot dimensions (packing dimension) equal the upper Rényi dimensions.

As an application we get a continuity property of the Hausdorff dimension of the measures, when it is seen as a function of the distributed weights under the ℓ^∞ norm.

1 Introduction

Let us consider the dyadic tree (even though all the results in this paper can be easily generalised to any ℓ -adic structure, $\ell \in \mathbb{N}$), let \mathbb{K} be its limit (Cantor) set and note $(\mathcal{F}_n)_{n \in \mathbb{N}}$ the associated filtration with the usual 0–1 encoding. We are interested in Borel measures μ on \mathbb{K} constructed in the following way: Take $(p_n, q_n)_{n \in \mathbb{N}}$ a sequence of couples of real numbers satisfying $0 \leq p_n, q_n \leq 1$.

Let $I = I_{\epsilon_1, \dots, \epsilon_n}$ be a cylinder of the n th generation and $IJ = I_{\epsilon_1, \dots, \epsilon_n, \epsilon_{n+1}}$ a subcylinder of the $(n+1)$ th generation, where $\epsilon_1, \dots, \epsilon_n, \epsilon_{n+1} \in \{0, 1\}$. The mass distribution of $\mu|_I$ will be as follows: $\mu(I_0) = p_0$, $\mu(I_1) = 1 - p_0$ et

$$\frac{\mu(IJ)}{\mu(I)} = \begin{cases} p_n \mathbf{1}_{\{\epsilon_{n+1}=0\}} + (1 - p_n) \mathbf{1}_{\{\epsilon_{n+1}=1\}}, & \text{if } \epsilon_n = 0 \\ q_n \mathbf{1}_{\{\epsilon_{n+1}=0\}} + (1 - q_n) \mathbf{1}_{\{\epsilon_{n+1}=1\}}, & \text{if } \epsilon_n = 1 \end{cases} \quad (1)$$

We use the notation $\dim_{\mathcal{H}}$ for the Hausdorff dimension and $\dim_{\mathcal{T}}$ for the packing (Tricot) dimension.

Definition 1.1 *If μ is a measure on \mathbb{K} , we will denote by $h_*(\mu)$ the lower entropy of the measure :*

$$h_*(\mu) = \liminf_{n \rightarrow \infty} \frac{-1}{n} \sum_{I \in \mathcal{F}_n} \log \mu(I) \cdot \mu(I),$$

by $h^(\mu)$ the upper entropy of the measure :*

$$h^*(\mu) = \limsup_{n \rightarrow \infty} \frac{-1}{n} \sum_{I \in \mathcal{F}_n} \log \mu(I) \cdot \mu(I),$$

by $\dim_*(\mu)$ the lower Hausdorff dimension of μ :

$$\dim_* \mu = \inf\{\dim_{\mathcal{H}} E ; E \subset \mathbb{K} \text{ and } \mu(E) > 0\}$$

and by $\dim^*(\mu)$ the upper Hausdorff dimension of μ :

$$\dim^* \mu = \inf\{\dim_{\mathcal{H}} E ; E \subset \mathbb{K} \text{ and } \mu(\mathbb{K} \setminus E) = 0\}.$$

In the same way we define the lower packing dimension (Tricot dimension) of μ :

$$\text{Dim}_* \mu = \inf\{\dim_{\mathcal{T}} E ; E \subset \mathbb{K} \text{ and } \mu(E) > 0\}$$

and by $\text{Dim}^*(\mu)$ the upper Hausdorff dimension of μ :

$$\text{Dim}^* \mu = \inf\{\dim_{\mathcal{T}} E ; E \subset \mathbb{K} \text{ and } \mu(\mathbb{K} \setminus E) = 0\}.$$

One can show that (see [Bat02],[BH02])

$$\dim_*(\mu) \leq h_*(\mu) \leq h^*(\mu) \leq \text{Dim}^*(\mu),$$

and there are examples of these inequalities being strict, even when the measure μ is rather “regular”.

It is also well known (cf [Fal97], [Bil65], [Mat95], [Fan94], [You82], [Rén70] and [Heu98]) that

$$\dim_*(\mu) = \inf \text{ess}_\mu \liminf_{n \rightarrow \infty} \frac{\log \mu(I_n(x))}{-n \log 2}$$

and

$$\dim^*(\mu) = \sup \text{ess}_\mu \liminf_{n \rightarrow \infty} \frac{\log \mu(I_n(x))}{-n \log 2},$$

where $I_n(x)$ is the dyadic cylinder of the n th generation containing x , $\inf \text{ess}_\mu$ is the essential infimum and $\sup \text{ess}_\mu$ is the essential supremum, taken over μ -almost all $x \in \mathbb{K}$.

In the case of measures defined by (1) we can use tools developed in [Bat96] and [Bat00] to prove they are exact, i.e. that $\dim_*(\mu) = \dim^*(\mu)$ or equivalently that

$$\liminf_{n \rightarrow \infty} \frac{\log \mu(I_n(x))}{-n \log 2} = \dim_*(\mu), \text{ for } \mu\text{-almost all } x \in \mathbb{K}$$

and therefore $\dim_*(\mu) = \dim^*(\mu)$. However, theorem 1.2 implies this statement.

In general, there is no trivial inequality relation between $h_*(\mu)$ and $\dim^*(\mu)$. Furthermore, it is easy to construct measures μ satisfying (1) such that $h_*(\mu) \neq h^*(\mu)$ which shows that the sequence of functions $\frac{\log \mu(I_n(x))}{-n \log 2}$ does not necessarily converge (in any space).

The proof of theorem 1.2 implies that there is a sequence $(c_n)_{n \in \mathbb{N}}$ of real numbers such that

$$\lim_{n \rightarrow \infty} \left[\frac{\log \mu(I_n(x))}{-n \log 2} - c_n \right] = 0,$$

where $c_n = \frac{-1}{n \log 2} \sum_{I \in \mathcal{F}_n} \log(\mu(I))\mu(I)$. This can be seen as a Shannon-McMillan-type theorem generalised to measures defined through non-homogeneous Markov chains.

Remark that the tools of [KP76] and [Kah87] can be applied to give the same results for “almost every” measure μ satisfying (1). Other results in this sense involving coloring of graphs are proposed in [Nas87].

A. Bisbas and C. Karanikas [BK94] have already partially proved the conclusions of theorem 1.2, for this kind of measures, under some assumptions on the sequences $(p_n, q_n)_{n \in \mathbb{N}}$. In particular they prove the theorem when the sequences $(p_n, q_n)_{n \in \mathbb{N}}$ are uniformly bounded away from 0 and 1, which is the case of a perturbation of an homogeneous Markov chain. We thank A. Bisbas for communicating to us this article.

Theorem 1.2 *If μ satisfies (1) then*

$$\dim_*(\mu) = \dim^*(\mu) = h_*(\mu) \text{ and } \text{Dim}_*(\mu) = \text{Dim}^*(\mu) = h^*(\mu).$$

Using the same type of arguments we also obtain the following continuity result.

Theorem 1.3 *Let μ and μ' be measures defined by (1) and the corresponding sequences $(p_n, q_n)_{n \in \mathbb{N}}$ and $(p'_n, q'_n)_{n \in \mathbb{N}}$ respectively. Then $|\dim_*(\mu) - \dim_*(\mu')|$ and $|\text{Dim}_*(\mu) - \text{Dim}_*(\mu')|$ go to 0 as $\|(p_n, q_n)_{n \in \mathbb{N}} - (p'_n, q'_n)_{n \in \mathbb{N}}\|_\infty$ tends to 0.*

2 Lemmas and preliminary results

Let us introduce some notation: for $p \in [0, 1]$ we note

$$h(p) = p \log p + (1 - p) \log(1 - p)$$

and if $I = I_{\epsilon_1, \dots, \epsilon_{n-1}} \in \mathcal{F}_n$, let us also set

$$\gamma(I, n) = \sum_{i=0,1} \log \left(\frac{\mu(II_i)}{\mu(I)} \right) \frac{\mu(II_i)}{\mu(I)}.$$

Remark that for $n \in \mathbb{N}$ and $I \in \mathcal{F}_{n-1}$, $\gamma(I, n)$ is equal to $h(p_n)$ if $\epsilon_{n-1} = 0$ and to $h(q_n)$ if $\epsilon_{n-1} = 1$ and therefore $\gamma(I, n)$ is absolutely bounded by $\log 2$.

Let us start with the following easy lemma.

Lemma 2.1 *For all $n, k \in \mathbb{N}$ and all $I \in \mathcal{F}_{n-1}$ we can write*

$$\sum_{K \in \mathcal{F}_k} \log \left(\frac{\mu(IK)}{\mu(I)} \right) \frac{\mu(IK)}{\mu(I)} = \gamma(I, n) + \sum_{i=0,1} \frac{\mu(II_i)}{\mu(I)} \sum_{K \in \mathcal{F}_{k-1}} \log \left(\frac{\mu(II_i K)}{\mu(II_i)} \right) \frac{\mu(II_i K)}{\mu(II_i)}. \quad (2)$$

where I_0 and I_1 are the two cylinders of the first generation.

Furthermore, if we note $a_n^k(I)$ and $b_n^k(I)$ respectively the quantities

$$a_n^k(I) = \sum_{K \in \mathcal{F}_{k-1}} \log \left(\frac{\mu(II_0 K)}{\mu(II_0)} \right) \frac{\mu(II_0 K)}{\mu(II_0)} \text{ and}$$

$$b_n^k(I) = \sum_{K \in \mathcal{F}_{k-1}} \log \left(\frac{\mu(II_1 K)}{\mu(II_1)} \right) \frac{\mu(II_1 K)}{\mu(II_1)}$$

then $a_n^k(I) = a_n^k(I')$ and $b_n^k(I) = b_n^k(I')$, for all $I, I' \in \mathcal{F}_n$.

Proof We have

$$\begin{aligned}
& \sum_{K \in \mathcal{F}_k} \log \left(\frac{\mu(IK)}{\mu(I)} \right) \frac{\mu(IK)}{\mu(I)} = \\
& \sum_{i=0,1} \sum_{K \in \mathcal{F}_{k-1}} \log \left(\frac{\mu(II_iK)}{\mu(I)} \right) \frac{\mu(II_iK)}{\mu(I)} = \\
& \sum_{i=0,1} \sum_{K \in \mathcal{F}_{k-1}} \log \left(\frac{\mu(II_iK)}{\mu(II_i)} \right) \frac{\mu(II_iK)}{\mu(I)} + \sum_{i=0,1} \log \left(\frac{\mu(II_i)}{\mu(I)} \right) \frac{\mu(II_i)}{\mu(I)} \quad (3)
\end{aligned}$$

Since we have set

$$\gamma(I, n) = \sum_{i=0,1} \log \left(\frac{\mu(II_i)}{\mu(I)} \right) \frac{\mu(II_i)}{\mu(I)},$$

the equalities (3) give

$$\sum_{K \in \mathcal{F}_k} \log \left(\frac{\mu(IK)}{\mu(I)} \right) \frac{\mu(IK)}{\mu(I)} = \gamma(I, n) + \sum_{i=0,1} \frac{\mu(II_i)}{\mu(I)} \sum_{K \in \mathcal{F}_{k-1}} \log \left(\frac{\mu(II_iK)}{\mu(II_i)} \right) \frac{\mu(II_iK)}{\mu(II_i)}.$$

It is immediate that $0 \leq -\gamma(I, n) \leq \log 2$. By the construction of the measure, the quantities $a_n^k(I)$ and $b_n^k(I)$ do not depend on the cylinder I but only on the cylinder's generation n and this ends the proof. •

Remark 2.2 Since the quantities $a_n^k(I)$ and $b_n^k(I)$ depend only on the generation of I and on k , we can note $a_n^k = a_n^k(I)$ and $b_n^k = b_n^k(I)$ for $I \in \mathcal{F}_n$. We also note $\Delta_n^k = \frac{1}{k} |a_n^k - b_n^k|$.

We also need the following technical estimates.

Lemma 2.3 For all $p, q \in [0, 1]$ we have $|h(p) - h(q)| \leq (1 - |p - q|) \log 2$. Furthermore, for all $k \in \mathbb{N}$ and all $\alpha > 0$,

$$\frac{(1 - |p - q|) \log 2}{k + 1} + |p - q| \left(1 - \frac{1}{k + 1} \right) \alpha \leq \max \left\{ \left(1 - \frac{1}{k + 1} \right) \alpha, \frac{\log 2}{k + 1} \right\}.$$

The proof uses elementary 2-dimensional calculus and is therefore omitted.

Proposition 2.4 Let I, I' be two cylinders of the n th generation. Then

$$\frac{1}{k} \left| \sum_{K \in \mathcal{F}_k} \log \left(\frac{\mu(IK)}{\mu(I)} \right) \frac{\mu(IK)}{\mu(I)} - \sum_{K \in \mathcal{F}_k} \log \left(\frac{\mu(I'K)}{\mu(I')} \right) \frac{\mu(I'K)}{\mu(I')} \right| < \eta(k)$$

where η is a positive function, not depending on n , such that $\eta(k)$ goes to 0 as k tends to ∞ .

Proof Take any two cylinders $I = I_{\epsilon_1, \dots, \epsilon_n}$, $I' = I_{\epsilon'_1, \dots, \epsilon'_n}$ of the n th generation. If $\epsilon_n = \epsilon'_n$ then by definition of the measure μ we get

$$\frac{1}{k} \left| \sum_{K \in \mathcal{F}_k} \log \left(\frac{\mu(IK)}{\mu(I)} \right) \frac{\mu(IK)}{\mu(I)} - \sum_{K \in \mathcal{F}_k} \log \left(\frac{\mu(I'K)}{\mu(I')} \right) \frac{\mu(I'K)}{\mu(I')} \right| = 0.$$

If $\epsilon_n \neq \epsilon'_n$, using lemma 2.3, lemma 2.1 and the notation therein we obtain:

$$\begin{aligned}
\Delta_{n-1}^{k+1} &= \left| \frac{1}{k+1} \sum_{K \in \mathcal{F}_{k+1}} \log \left(\frac{\mu(IK)}{\mu(I)} \right) \frac{\mu(IK)}{\mu(I)} - \frac{1}{k+1} \sum_{K \in \mathcal{F}_{k+1}} \log \left(\frac{\mu(I'K)}{\mu(I')} \right) \frac{\mu(I'K)}{\mu(I')} \right| = \\
&= \left| \frac{\gamma(I, n) - \gamma(I', n)}{k+1} + \frac{1}{k+1} \frac{\mu(II_0)}{\mu(I)} \sum_{K \in \mathcal{F}_k} \log \left(\frac{\mu(II_0K)}{\mu(II_0)} \right) \frac{\mu(II_0K)}{\mu(II_0)} + \right. \\
&+ \frac{1}{k+1} \frac{\mu(II_1)}{\mu(I)} \sum_{K \in \mathcal{F}_k} \log \left(\frac{\mu(II_1K)}{\mu(II_1)} \right) \frac{\mu(II_1K)}{\mu(II_1)} - \\
&- \frac{1}{k+1} \frac{\mu(I'I_0)}{\mu(I')} \sum_{K \in \mathcal{F}_k} \log \left(\frac{\mu(I'I_0K)}{\mu(I'I_0)} \right) \frac{\mu(I'I_0K)}{\mu(I'I_0)} - \\
&- \left. \frac{1}{k+1} \frac{\mu(I'I_1)}{\mu(I')} \sum_{K \in \mathcal{F}_k} \log \left(\frac{\mu(I'I_1K)}{\mu(I'I_1)} \right) \frac{\mu(I'I_1K)}{\mu(I'I_1)} \right| = \\
&= \left| \frac{h(p_n) - h(q_n)}{k+1} + \frac{1}{k+1} \left(\left(\frac{\mu(II_0)}{\mu(I)} - \frac{\mu(I'I_0)}{\mu(I')} \right) a_n^k + \left(\frac{\mu(II_1)}{\mu(I)} - \frac{\mu(I'I_1)}{\mu(I')} \right) b_n^k \right) \right| \leq \\
&\leq \frac{|h(p_n) - h(q_n)|}{k+1} + \left| \frac{1}{k+1} \left(\frac{\mu(II_0)}{\mu(I)} - \frac{\mu(I'I_0)}{\mu(I')} \right) (a_n^k - b_n^k) \right| \\
&\leq \frac{(1 - |p_n - q_n|) \log 2}{k+1} + |p_n - q_n| \frac{|a_n^k - b_n^k|}{k} \frac{k}{k+1} \tag{4}
\end{aligned}$$

We can rewrite relation (4) in the following way

$$\frac{|a_{n-1}^{k+1} - b_{n-1}^{k+1}|}{k+1} \leq \frac{(1 - |p_n - q_n|) \log 2}{k+1} + |p_n - q_n| \frac{|a_n^k - b_n^k|}{k} \left(1 - \frac{1}{k+1} \right)$$

and thus,

$$\Delta_{n-1}^{k+1} \leq \frac{(1 - |p_n - q_n|) \log 2}{k+1} + |p_n - q_n| \left(1 - \frac{1}{k+1} \right) \Delta_n^k \tag{5}$$

By lemma 2.3 we then obtain,

$$\Delta_{n-1}^{k+1} \leq \max \left\{ \left(1 - \frac{1}{k+1} \right) \Delta_n^k, \frac{\log 2}{k+1} \right\}. \tag{6}$$

We use a recursion argument to finish the proof the lemma. First observe that if for some $\ell \in \{0, \dots, k\}$ we have

$$\Delta_{n+\ell}^{k-\ell} < \frac{\log 2}{k-\ell} \tag{7}$$

then we will also have $\Delta_{n+\ell-1}^{k-\ell+1} < \frac{\log 2}{k-\ell+1}$, by relation (6), and therefore $\Delta_{n-1}^{k+1} \leq \frac{\log 2}{k+1}$.

On the other hand, if inequality (7) does not hold for any $\ell \in \{0, \dots, k\}$ then by (6) we get

$$\Delta_{n+\ell-1}^{k-\ell+1} \leq \left(1 - \frac{1}{k-\ell+1} \right) \Delta_{n+\ell}^{k-\ell}$$

and finally

$$\Delta_{n-1}^{k+1} \leq \prod_{\ell=1}^{k+1} \left(1 - \frac{1}{\ell+1}\right) \log 2 \leq \frac{e^2 \log 2}{k+1} \quad (8)$$

Take $\eta(k) = \frac{e^2 \log 2}{k+1}$. By equations (7) and (8) we get $\Delta_{n-1}^{k+1} \leq \max \left\{ \frac{\log 2}{k+1}, \frac{e^2 \log 2}{k} \right\} = \frac{e^2 \log 2}{k+1} = \eta(k)$ and the proof is complete. •

We will also use the following two theorems of [BH02] that we include without proof for the convenience of the reader (a straight forward proof -without use of these theorems- is possible but much longer).

Theorem 2.5 [BH02] *Let m be a probability measure in $[0, 1]^D$ equipped with the filtration of ℓ -adic cubes, $\ell \in \mathbb{N}$. Then*

$$\dim_*(m) \leq h_*(m) .$$

Moreover, the following properties are equivalent :

1. $\dim_*(m) = h_*(m)$
2. $\dim_*(m) = \dim^*(m) = h_*(m)$
3. There exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that for m -almost every $x \in [0, 1]^D$,

$$\lim_{k \rightarrow +\infty} \frac{\log m(I_{n_k}(x))}{-n_k \log \ell} = \dim_*(m) .$$

Theorem 2.6 [BH02] *We also have*

$$h^*(m) \leq \text{Dim}^*(m),$$

and the following properties are equivalent :

1. $\text{Dim}^*(m) = h^*(m)$
2. $\text{Dim}_*(m) = \text{Dim}^*(m) = h^*(m)$
3. There exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that for m -almost every $x \in [0, 1]^D$,

$$\lim_{k \rightarrow +\infty} \frac{\log m(I_{n_k}(x))}{-n_k \log \ell} = \text{Dim}^*(m) .$$

3 Proofs of the theorems

To prove theorem 1.2 we will use the following strong law of large numbers (cf. [HH80]).

Theorem 3.1 (Law of Large Numbers) *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of uniformly bounded in \mathcal{L}^2 real random variables on a probability space $(\mathbb{X}, \mathcal{B}, P)$ and let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be an increasing sequence of σ -subalgebras of \mathbb{B} such that X_n is measurable with respect to \mathcal{F}_n for all $n \in \mathbb{N}$. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (X_k - \mathbb{E}(X_k | \mathcal{F}_{k-1})) = 0 \text{ } P\text{-almost surely} \quad (9)$$

Remark that the assumptions on the random variables are not optimal but it will be sufficient for our goal. The space here is \mathbb{K} , the filtration will be the dyadic one and μ will take the place of the probability measure P .

Proof of Theorem 1.2. Consider the random variables X_n , $n \in \mathbb{N}$, defined on \mathbb{K} , given by

$$X_n(x) = \log \frac{\mu(I_n(x))}{\mu(I_{n-1}(x))},$$

where, for $x \in \mathbb{K}$, we have noted $I_n(x)$ the unique element of \mathcal{F}_n containing x . The previous lemma implies that for all positive p 's

$$\lim_{n \rightarrow \infty} \frac{1}{(n+1)} \sum_{j=1}^n \left(\frac{1}{p} \sum_{k=1}^p [X_{jp+k} - \mathbb{E}(X_{jp+k} | \mathcal{F}_{jp})] \right) = 0, \quad \mu\text{-almost surely.} \quad (10)$$

On the other hand, on each $I \in \mathcal{F}_n$, the conditional expectation $\frac{1}{p} \sum_{k=1}^p \mathbb{E}(X_{np+k} | \mathcal{F}_{np})$ is given by

$$\frac{1}{p} \sum_{k=1}^p \mathbb{E}(X_{np+k} | \mathcal{F}_{np}) = \frac{1}{p} \sum_{K \in \mathcal{F}_p} \log \left(\frac{\mu(IK)}{\mu(I)} \right) \frac{\mu(IK)}{\mu(I)}. \quad (11)$$

By proposition 2.4, for every $\epsilon > 0$ there exists $p \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and all I in \mathcal{F}_n

$$\left| \frac{1}{p} \sum_{K \in \mathcal{F}_p} \log \left(\frac{\mu(IK)}{\mu(I)} \right) \frac{\mu(IK)}{\mu(I)} - c_n \right| < \epsilon, \quad (12)$$

where c_n is a constant depending only on n and on the chosen p but not on the cylinder I of \mathcal{F}_n .

It is also easy to see that the variable $(X_n)_{n \in \mathbb{N}}$ are uniformly bounded in $\mathcal{L}^2(\mu)$. We deduce, using the relations (10) and (11), that for every $\epsilon > 0$ there exists $p \in \mathbb{N}$ and a sequence $(c_n)_{n \in \mathbb{N}}$ of real numbers such that

$$\begin{aligned} -\epsilon &< \liminf_{n \rightarrow \infty} \frac{1}{(n+1)} \sum_{j=1}^n \left(\frac{1}{p} \sum_{k=1}^p X_{jp+k} - c_j \right) \leq \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{(n+1)} \sum_{j=1}^n \left(\frac{1}{p} \sum_{k=1}^p X_{jp+k} - c_j \right) < \epsilon, \end{aligned} \quad (13)$$

μ -almost everywhere on \mathbb{K} . This relation implies that

$$\liminf_{n \rightarrow \infty} \frac{-1}{(n+1)} \sum_{j=1}^n c_j - \epsilon < \liminf_{n \rightarrow \infty} \frac{-1}{p} \frac{1}{(n+1)} \sum_{j=1}^n \sum_{k=1}^p X_{jp+k} < \liminf_{n \rightarrow \infty} \frac{-1}{(n+1)} \sum_{j=1}^n c_j + \epsilon \quad (14)$$

and

$$\limsup_{n \rightarrow \infty} \frac{-1}{(n+1)} \sum_{j=1}^n c_j - \epsilon < \limsup_{n \rightarrow \infty} \frac{-1}{p} \frac{1}{(n+1)} \sum_{j=1}^n \sum_{k=1}^p X_{jp+k} < \limsup_{n \rightarrow \infty} \frac{-1}{(n+1)} \sum_{j=1}^n c_j + \epsilon \quad (15)$$

μ -almost everywhere on \mathbb{K} . If we note

$$\underline{c} = \liminf_{n \rightarrow \infty} \frac{-1}{(n+1) \log 2} \sum_{j=1}^n c_j \text{ and } \bar{c} = \limsup_{n \rightarrow \infty} \frac{-1}{(n+1) \log 2} \sum_{j=1}^n c_j,$$

we deduce from (14) and (15) that $\dim_* \mu = \underline{c}$ and $\text{Dim}_* \mu = \bar{c}$.

Furthermore, the inequalities (13) imply that for every positive ϵ there is a strictly increasing sequence of natural numbers $(n_l)_{l \in \mathbb{N}}$ verifying

$$-\epsilon < \liminf_{l \rightarrow \infty} \frac{-1}{(n_l+1)} \sum_{j=1}^{n_l} \left(\frac{1}{p} \sum_{k=1}^p X_{jp+k} \right) - \underline{c} \leq \limsup_{l \rightarrow \infty} \frac{-1}{(n_l+1)} \sum_{j=1}^{n_l} \left(\frac{1}{p} \sum_{k=1}^p X_{jp+k} \right) - \underline{c} < \epsilon.$$

One easily proves (using, for instance, Cantor's diagonal argument) that there exists a strictly increasing sequence of natural numbers $(n_l)_{l \in \mathbb{N}}$ such that

$$\lim_{l \rightarrow \infty} \frac{-1}{n_l \log 2} \log \mu(I_{n_l}(x)) = \dim_*(\mu),$$

for μ -almost all $x \in \mathbb{K}$.

Similarly, there exists a strictly increasing sequence of natural numbers $(\hat{n}_l)_{l \in \mathbb{N}}$ such that

$$\lim_{l \rightarrow \infty} \frac{-1}{\hat{n}_l \log 2} \log \mu(I_{\hat{n}_l}(x)) = \text{Dim}_*(\mu),$$

for μ -almost all $x \in \mathbb{K}$. We use theorems 2.5 and 2.6 to finish the proof. •

To prove theorem 1.3 we will use proposition 2.4 and lemma 3.1.

Proof of theorem 1.3 Take $\epsilon > 0$ and let $(p_n, q_n)_{n \in \mathbb{N}}$ and $(p'_n, q'_n)_{n \in \mathbb{N}}$ be two sequences of weights satisfying $0 < p_n, q_n, p'_n, q'_n < 1$ for all $n \in \mathbb{N}$ and

$$\|(p_n, q_n)_{n \in \mathbb{N}} - (p'_n, q'_n)_{n \in \mathbb{N}}\|_\infty < \zeta.$$

We note μ and μ' the measures corresponding to these two sequences of weights. We will show that

$$|\dim_*(\mu) - \dim_*(\mu')| < \epsilon,$$

if ζ is small enough.

It follows from proposition 2.4 that there exist a natural number p large enough and two sequences of real numbers $(c_n)_{n \in \mathbb{N}}, (c'_n)_{n \in \mathbb{N}}$ such that the following relations hold:

$$\left| \frac{1}{p} \sum_{K \in \mathcal{F}_p} \log \left(\frac{\mu(IK)}{\mu(I)} \right) \frac{\mu(IK)}{\mu(I)} - c_n \right| < \frac{\epsilon}{4}$$

and

$$\left| \frac{1}{p} \sum_{K \in \mathcal{F}_p} \log \left(\frac{\mu'(IK)}{\mu'(I)} \right) \frac{\mu'(IK)}{\mu'(I)} - c'_n \right| < \frac{\epsilon}{4}$$

for all cylinders $I \in \mathcal{F}_{np}$ and all $n \in \mathbb{N}$. Since p is a fixed finite number it suffices to take ζ small in order to have

$$\left| \frac{1}{p} \sum_{K \in \mathcal{F}_p} \log \left(\frac{\mu(IK)}{\mu(I)} \right) \frac{\mu(IK)}{\mu(I)} - \frac{1}{p} \sum_{K \in \mathcal{F}_p} \log \left(\frac{\mu'(IK)}{\mu'(I)} \right) \frac{\mu'(IK)}{\mu'(I)} \right| < \frac{\epsilon}{2},$$

for all $I \in \mathcal{F}_{np}$ and all $n \in \mathbb{N}$. Hence,

$$-\epsilon < \liminf_{n \rightarrow \infty} \frac{1}{(n+1)} \sum_{j=1}^n |c_j - c'_j| \leq \limsup_{n \rightarrow \infty} \frac{1}{(n+1)} \sum_{j=1}^n |c_j - c'_j| < \epsilon.$$

we deduce from (14) and (15) that $|\dim_*(\mu) - \dim_*(\mu')| < \epsilon$ and $|\text{Dim}_*(\mu) - \text{Dim}_*(\mu')| < \epsilon$, which completes the proof. •

Theorem 1.3 have a limited validity as we show in the following section.

4 A counterexample

For every $\epsilon > 0$ we construct two dyadic doubling measures μ and ν on \mathbb{K} such that if $X_n(x) = \log \frac{\mu(I_n(x))}{\mu(I_{n-1}(x))}$ and $Y_n(x) = \log \frac{\nu(I_n(x))}{\nu(I_{n-1}(x))}$, for $n \in \mathbb{N}$ then

$$\sup_{n \in \mathbb{N}} \left\| X_n - Y_n \right\|_{L^\infty} < \epsilon \quad (16)$$

and $|\dim_*(\mu) - \dim_*(\nu)| > \frac{1}{4}$. A first example was proposed to us by professor Alano Ancona; the proof provided here is of a similar nature.

The construction is carried out in two stages. We start by finding two Bernoulli measures satisfying 16 and afterwards we will modify them by a recurrent process to get the corresponding dimensions very different.

For $I \in \mathcal{F}_n$ we note \hat{I} the unique cylinder of the $(n-1)$ th generation \mathcal{F}_{n-1} contenant I . Relation (16) can now be reformulated in the following way

$$\left| \frac{\mu(I)}{\mu(\hat{I})} : \frac{\nu(I)}{\nu(\hat{I})} - 1 \right| < \epsilon, \quad \text{for all cylinders } I \text{ of } \bigcup_{n \in \mathbb{N}} \mathcal{F}_n \quad (17)$$

4.0.1 The starting point

Take $\epsilon > 0$ and λ_0 the Lebesgue (uniform) measure (of dimension 1) on \mathbb{K} .

Consider the Bernoulli measure ρ_0 of weight variable $\frac{1}{2} - \epsilon$, i.e. such that for $I \in \mathcal{F}_n$, $n \in \mathbb{N}$,

$$\rho_0(II_0) = \left(\frac{1}{2} - \epsilon\right)\rho(I), \quad \rho_0(II_1) = \left(\frac{1}{2} + \epsilon\right)\rho(I). \quad (18)$$

Put $\mu_0 = \lambda_0$ and $\nu_0 = \rho_0$. By construction the measures λ_0 and ρ_0 verify condition (16), are exact and doubling on the dyadics. Moreover, we have

$$\dim \rho_0 = h_*(\rho) = -\frac{(\frac{1}{2} - \epsilon)}{\log 2} \log \left(\frac{1}{2} - \epsilon \right) - \frac{(\frac{1}{2} + \epsilon)}{\log 2} \log \left(\frac{1}{2} + \epsilon \right)$$

It is clear that λ_0 and ρ_0 are singular. Furthermore by the Shannon-MacMillan formula (cf for instance [Zin97]),

$$\lim_{n \rightarrow \infty} \frac{\log \rho_0(I_n(x))}{n} = h_*(\rho_0) \quad \rho_0\text{-almost everywhere on } \mathbb{K}.$$

Hence, we can find $n_1 \in \mathbb{N}$ and a partition $\{F_0, F_1\}$ of \mathcal{F}_{n_1} verifying :

1. $F_0 \cup F_1 = \mathcal{F}_{n_1}$
2. $\left| \frac{\log \rho_0(I)}{n} + h_*(\rho_0) \right| < \epsilon$ for all $I \in F_1$
3. $\left| \frac{\log \lambda_0(I)}{n} + \log 2 \right| < \epsilon$ for all $I \in F_0$
4. $\sum_{I \in F_1} \rho_0(I) > 1 - \epsilon$
5. $\sum_{I \in F_0} \lambda_0(I) > 1 - \epsilon$

Let us also define the Bernoulli measures λ_1 and ρ_1 on \mathbb{K} in the following way

$$\begin{aligned} \rho_1(I_0) &= \delta & \text{and} & & \rho_1(I_1) &= 1 - \delta \\ \lambda_1(I_0) &= \delta(1 - \epsilon) & \text{and} & & \lambda_1(I_1) &= 1 - \delta(1 - \epsilon) \end{aligned} \tag{19}$$

where $\delta > 0$ will be fixed later.

4.0.2 Going on with the construction

For $I_{i_1 \dots i_n} \subset I \in F_1$ we put

$$\begin{aligned} \mu_1(I_{i_1 \dots i_n}) &= \mu_0(I_{i_1 \dots i_{n_1}}) \lambda_1(I_{i_{n_1} \dots i_n}) \\ \nu_1(I_{i_1 \dots i_n}) &= \nu_0(I_{i_1 \dots i_{n_1}}) \rho_1(I_{i_{n_1} \dots i_n}) \end{aligned} \tag{20}$$

and for $I_{i_1 \dots i_n} \subset I \in F_0$

$$\mu_1(I_{i_1 \dots i_n}) = \mu_0(I_{i_1 \dots i_n}) \quad , \quad \nu_1(I_{i_1 \dots i_n}) = \nu_0(I_{i_1 \dots i_n}) \tag{21}$$

Remark that for $I = I_{i_1 \dots i_n}$ with $n \leq n_1$ we leave $\mu_1(I) = \mu_0(I)$ and $\nu_1(I) = \nu_0(I)$.

The restrictions of the measures μ_1 and ν_1 in the cylinders of $\mathcal{F}_{n_1} = F_0 \cup F_1$ are Bernoulli measures of different dimensions, so they are singular between them. Therefore, we can find $n_2 \in \mathbb{N}$ and a partition $\{F_{00}, F_{01}, F_{10}, F_{11}\}$ of \mathcal{F}_{n_2} such that

1. $I \in F_{j0} \cup F_{j1}$ if and only if there is $J \in F_j$ such that $I \subset J$, $j \in \{0, 1\}$.
2. $\left| \frac{\log \mu_1(I)}{n_2} + \log 2 \right| < \epsilon^2$ for all $I \in F_{00}$.
3. $\left| \frac{\log \nu_1(I)}{n_2} + h_*(\rho_1) \right| < \epsilon^2$ for all $I \in F_{11}$.
4. $\sum_{\substack{J \in F_{00} \\ J \subset I}} \mu_1(J) > (1 - \epsilon^2)\mu_1(I)$ et $\sum_{\substack{J \in F_{01} \\ J \subset I}} \nu_1(J) > (1 - \epsilon^2)\nu_1(I)$ pour $I \in F_0$
5. $\sum_{\substack{J \in F_{10} \\ J \subset I}} \mu_1(J) > (1 - \epsilon^2)\mu_1(I)$ and $\sum_{\substack{J \in F_{11} \\ J \subset I}} \nu_1(J) > (1 - \epsilon^2)\nu_1(I)$ pour $I \in F_1$

If $I \in F_{00} \cup F_{10}$ and $J \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$, we put

$$\mu_2(IJ) = \mu_1(I)\lambda_0(J) \quad , \quad \nu_2(IJ) = \nu_1(I)\rho_0(J).$$

If $I \in F_{01} \cup F_{11}$ and $J \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ we put

$$\mu_2(IJ) = \mu_1(I)\lambda_1(J) \quad , \quad \nu_2(IJ) = \nu_1(I)\rho_1(J).$$

Finally, for $I \in \mathcal{F}_n$, with $n \leq n_2$ we keep the same mass distribution $\mu_2(I) = \mu_1(I)$ et $\nu_2(I) = \nu_1(I)$.

Suppose the measures μ_k, ν_k and the partition $\{F_{i_1 \dots i_k}, i_1, \dots, i_k \in \{0, 1\}\}$ de \mathcal{F}_{n_k} are constructed. As in the two first stages, the restrictions of the measures μ_k and ν_k on such cylinder of \mathcal{F}_{n_k} are supposed to be Bernoulli measures: whether λ_0 and ρ_0 whether λ_1 and ρ_1 , respectively.

The measures μ_k and ν_k are singular between them. Hence, there is $n_{k+1} > n_k$ and a partition $\{F_{i_1 \dots i_{k+1}}, i_1, \dots, i_{k+1} \in \{0, 1\}\}$ of $\mathcal{F}_{n_{k+1}}$ satisfying

1. $I \in F_{i_1 \dots i_k 0} \cup F_{i_1 \dots i_k 1}$ if and only if there is $J \in F_{i_1 \dots i_k}$ such that $I \subset J$, with $i_1, \dots, i_k \in \{0, 1\}$
2. $\left| \frac{\log \mu_k(I)}{n_{k+1}} + \log 2 \right| < \epsilon^{k+1}$ for all $I \in F_{i_1 \dots i_{k-1} 00}$.
3. $\left| \frac{\log \nu_k(I)}{n_2} + h_*(\rho_1) \right| < \epsilon^{k+1}$ for all $I \in F_{i_1 \dots i_{k-1} 11}$.
4. $\sum_{\substack{J \in F_{i_1 \dots i_{k-1} 00} \\ J \subset I}} \mu_k(J) > (1 - \epsilon^{k+1})\mu_k(I)$ and $\sum_{\substack{J \in F_{i_1 \dots i_{k-1} 01} \\ J \subset I}} \nu_k(J) > (1 - \epsilon^{k+1})\nu_k(I)$,
for all cylinders $I \in F_{i_1 \dots i_{k-1} 0}$.
5. $\sum_{\substack{J \in F_{i_1 \dots i_{k-1} 10} \\ J \subset I}} \mu_k(J) > (1 - \epsilon^{k+1})\mu_k(I)$ et $\sum_{\substack{J \in F_{i_1 \dots i_{k-1} 11} \\ J \subset I}} \nu_k(J) > (1 - \epsilon^{k+1})\nu_k(I)$,
for all cylinders $I \in F_{i_1 \dots i_{k-1} 1}$.

If $I \in F_{i_1 \dots i_k 0}$, $i_1, \dots, i_k \in \{0, 1\}$, then for all $J \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ we put

$$\mu_{k+1}(IJ) = \mu_k(I)\lambda_0(J) \quad \text{and} \quad \nu_{k+1}(IJ) = \nu_k(I)\rho_0(J).$$

If $I \in F_{i_1 \dots i_k 1}$, $i_1, \dots, i_k \in \{0, 1\}$, then for all $J \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ we put

$$\mu_{k+1}(IJ) = \mu_k(I)\lambda_1(J) \quad \text{and} \quad \nu_{k+1}(IJ) = \nu_k(I)\rho_1(J).$$

4.0.3 Properties of the measures defined

It is clear that the sequences $(\mu_n)_{n \in \mathbb{N}}$ and $(\nu_n)_{n \in \mathbb{N}}$ converge towards two probability measures μ and ν respectively. By the construction μ and ν are doubling on the dyadics, exacts and satisfy (16).

On the other hand, clearly $\dim_* \mu = 1$ and it is not difficult to see that $\dim_* \nu \leq \frac{1}{2}$, if δ is small enough, since $\liminf_{n \rightarrow \infty} \frac{-\log \nu(I_n(x))}{n \log 2} = \frac{h_*(\rho_1)}{\log 2}$, ν -almost everywhere. Evenmore, the measures μ and ν satisfy the conclusion of theorem 1.2. The counterexample is complete.

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