A General Version of the Triple $\Pi$ Operator

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Recent developments of sensors and computers have raised the problem of handling huge amounts of complex data that users try to synthesize for decision making. Aggregation operators, such as those appearing in fuzzy sets theory, are useful tools for this synthesis but in their present formulation, these operators only deal with a finite set of arguments. In this paper, we introduce $G_3\Pi$, an extension of both Yager–Rybalov Triple $\Pi$ and Mean Triple $\Pi$ operators to general measure spaces that can deal with temporal or spatiotemporal intensive data streams. Known properties and inequalities are extended in this more general setting. The notion of moving $G_3\Pi$ is also introduced and it can be applied to a solar radiation data stream. This may lead to further works on data fusion and on similar extensions of some other operators. © 2013 Wiley Periodicals, Inc.

1. INTRODUCTION

Recent technological advances have popularized the use of outstanding performance sensors and measuring instruments, which provide many detailed and varied information. This concerns quite all domains including industry (microelectronics, food industry), safe (airport, air quality, department store), and even everyday life (transportation, housing). While the problem was, in the past, the lack of sensors to collect reliable and useful data, leading to the question of how to get information, nowadays the question is rather “what to do with so much information and how to make a decision?”. The saying “too much information kills information” has never been so true.

A recurrent problem concerns the design of tools and techniques that synthesize information for decision making. One way to address this problem was the use of aggregation (or fusion) operators mainly developed in the framework of fuzzy sets theory to perform information fusion (or merging). Information is often considered

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as a degree of membership to a class or to a state and the operators are only dealt with a finite number of arguments. Motivated by present problems of handling huge datasets, our purpose in the present work consists of extending these operators to infinite sets of arguments.

The choice of a suited operator depends on the context and on the fusion itself, recalling the following definition (Bloch and Hunter\(^1\)): fusion consists in joining together, or aggregating, the information coming from various sources, and to exploit this new resulting information in various applications like decision, numerical estimation, etc . . . .

This definition first emphasizes the combination of the information and then the accent is put on the aim of the fusion. More formally,

**Definition 1.1.** (Ref. 2) Let \( I = [0, 1] \) and \( J = \cup_{n=1,2,...} I^n \), an aggregation operator (AO) \( f \) is a function \( f : J \rightarrow I \) increasing on \( I^n, \forall n \).

AOs can be divided into four groups according to their behavior w.r.t. min (minimum) and max (maximum) (Ref. 3) (see also Ref. 4):

- **Conjunctive** operators which verify \( f \leq \min \). Triangular norms (\( t \)-norms) and copulas\(^5\) belong to this class.
- **Disjunctive** operators which verify \( f \geq \max \). Triangular conorms (\( t \)-conorms) belong to this group.
- **Average** (or mean) operators which verify \( \min \leq f \leq \max \). For example, OWA\(^6,7\) belongs to this class.
- **Hybrid** operators. This group represents all other operators which are not comparable with min or max. For example, mixed connectives,\(^8,9\) uninorms,\(^10-12\) nullnorms,\(^13\) and symmetric sums\(^14\) belong to this group.

An important aspect of aggregation is the type of data that we are dealing with. Several types of data were proposed by Bloch et al.\(^1\) (see also Ref. 2):

- **Observations.** From more or less particular point of views, they describe the world through numerical data provided by sensors.
- **Knowledge.** It describes how the world is in general through data reported by human observations rather than by sensors.
- **Preferences.** Information that describes how we would like the world to be. Of course, this information is reported by human observations.
- **Regulations.** Generic information presented by laws or rules.

AOs are generally used to synthesize information at a given time and not over a time interval. As far as we know, even some integral type operators, like Choquet and Sugeno integrals, are usually applied to localized point features and not to data ranges. Note, however, that some AOs dealing with time series are used for smoothing\(^15\) and forecasting.\(^16,17\)

The \( G3\Pi \) AO that we define is a common reformulation of both totally reinforced Yager–Rybalov Triple \( \Pi \) AO, denoted \( 3\Pi \), and mean reinforced Mean Triple \( \Pi \) one, denoted \( M3\Pi \). The idea is to define an AO not only on \( J \) but on any general measure space that includes the case of time series and spatiotemporal series of membership degrees.
The paper is organised as follows: Section 2 recalls the main properties of $3\Pi$ and $M3\Pi$ AO$s. Section 3 concerns our main results, that is, $G3\Pi$, reinforcement, and moving $G3\Pi$. In Section 4, we present some applications and perspectives and we conclude in Section 5.

2. 3\Pi AND M3\Pi OPERATORS

2.1. 3\Pi and full reinforcement

Before giving the definition of $3\Pi$, recall that

- $3\Pi$ is an uninorm,$^{12}$
- $3\Pi$ can be viewed as a generalization of a symmetric sum.$^{18}$

**Definition 2.1[11].** A uninorm is a mapping $U : J \rightarrow I$ such that for all $x, y, z, s, t \in I^n$, $n = 1, 2, \ldots$, the following holds:

1. Commutativity: $U(x, y) = U(y, x)$
2. Monotonicity: $U(x, y) \geq U(s, t)$ if $x \geq s, y \geq t$
3. Associativity: $U(x, U(y, z)) = U(U(x, y), z)$
4. Neutral element: there exists some fixed element $g \in [0, 1]$ such that $U(x, g) = U(x)$.

The main property of an uninorm is to be totally reinforced (the definition of reinforcement is given below).

**Definition 2.2[14].** A symmetric sum $S$ is a continuous self-dual AO, where self-duality means that

$$S(x_1, x_2, \ldots, x_n) = 1 - S(1 - x_1, 1 - x_2, \ldots, 1 - x_n)$$

for $x_i \in [0, 1], i = 1, \ldots, n$.

As pointed in [19], Silvert showed that any symmetric sum of two arguments can be written in the form:

$$S(x, y) = \frac{G(x, y)}{G(x, y) + G(1 - x, 1 - y)}$$

where $G$ is a continuous, increasing, and positive function satisfying $G(0, 0) = 0$.

A symmetric sum $S$ applied to two fuzzy sets has, for fuzzy complement, the value of $S$ applied to the fuzzy complements so that $S$ takes in account both a fuzzy subset and its fuzzy complement, unlike classical aggregation operators.

Let us now recall the definition of total (or full) reinforcement (Yager–Rybalov$^{12}$), a main property of uninorms. Suppose that for a given class, its object has strong membership degrees for all the considered features. In human reasoning, an aggregation of such degrees will be higher than each degree taken separately.$^{12, 20}$ In this kind of reasoning, the membership degrees which are strong are mutually reinforced. This behavior is called positive (upward) reinforcement. Similarly, if
an object has small membership degrees, then the aggregation will be weaker than the weakest membership degree. This is called negative (downward) reinforcement. Total reinforcement is a property that traduces certain aspects of human reasoning. Using an operator having such a property can thus be interesting when dealing with a system close to this type of reasoning. More formally,

**Definition 2.3[12].** An AO $f$ is positively (resp. negatively) reinforced if

$$f(x_1, \ldots, x_n) \geq \max_i(x_i)$$

when all the $x_i$’s are affirmative (i.e., $\geq 0.5$)

$$\text{(resp. } f(x_1, \ldots, x_n) \leq \min_i(x_i) \text{)}$$

when all the $x_i$’s are nonaffirmative (i.e., $\leq 0.5$).

An operator which has both properties is called completely reinforced (or fully reinforced).

Note that $t$-norms are negatively (but not positively) reinforced while $t$-conorms are positively but not negatively reinforced. Yager and Rybalov\cite{12} have given a counterexample of combination of $t$-norms and $t$-conorms (as the connective mixed ones) that is not completely reinforced. A mean AO, say $M$, is neither positively nor negatively reinforced, since by definition, we have $\min_i(x_i) \leq M(x_1, \ldots, x_n) \leq \max_i(x_i)$.

As said above, as far as we know, the operators which are completely reinforced are uninorms.\cite{12} The uninorm $3\Pi$ defined by Yager and Rybalov is fully reinforced and is defined as follows:

$$3\Pi(x_1, \ldots, x_n) = \frac{\Pi_{j=1}^n x_j}{\Pi_{j=1}^n x_j + \Pi_{j=1}^n (1-x_j)}.$$  \hspace{1cm} (5)

Note that $g = 0.5$ is a neutral element for $3\Pi$.

$3\Pi$ as an uninorm, and more generally uninorms, have been widely studied (a list of studies is given in \cite{21}). Several variants of $3\Pi$ have been proposed, see, for example, \cite{21–23}, but most of these works concern the uninorm aspect and very few concern the symmetric sum aspect. A first study\cite{24} on this last aspect has led to the definition of a new AO called Mean $3\Pi$ and denoted $M3\Pi$, the idea being to get a mean operator having the symmetric sum and the reinforcement properties.

Why a mean operator? Actually in many practical cases, a lot of sensors, with various physical characteristics and configurations, analyze the same phenomenon. So they can be considered as the pieces of a same source and in these cases, average or mean operators are well suited.\cite{1,2} This motivated the idea of finding a $3\Pi$ close to average. In order to develop $M3\Pi$, the function $G$ in equation (2), which is the product of the arguments in $3\Pi$, was taken in $M3\Pi$ as the product of the $n$th-roots of the arguments, as indicated in the next section.
2.2. Mean 3\Pi and Mean Reinforcement

Let us define

$$3\Pi_G(x_1, \ldots, x_n) = \frac{\prod_{j=1}^{n} G(x_j)}{\prod_{j=1}^{n} G(x_j) + \prod_{j=1}^{n} G(1-x_j)}$$  \hspace{1cm} (6)$$

where $G(x)$ is a generating function which is nonnegative and increasing.\(^{14,18}\) To get an idempotency property, we have considered the function $G(x) = x^{1/n}$ where $n$ is the number of arguments. Thus a new AO can be defined as follows\(^{24}\):

$$M3\Pi(x_1, \ldots, x_n) = \frac{\prod_{j=1}^{n} (x_j)^{(1/n)}}{\prod_{j=1}^{n} (x_j)^{(1/n)} + \prod_{j=1}^{n} (1-x_j)^{(1/n)}}$$  \hspace{1cm} (7)$$

\begin{align*}
&= \frac{1}{1 + \prod_{j=1}^{n} \left[ \frac{1-x_j}{x_j} \right]^{1/n}}, & x_j \in (0, 1].
\end{align*}$$

Recall the following properties:

If $x_j \geq 0.5$, $\forall j \in 1, \ldots, n$, then

$$M3\Pi(x_1, \ldots, x_n) \geq \frac{1}{n} \sum_{j=1}^{n} x_j.$$  \hspace{1cm} (8)$$

If $x_j \leq 0.5$, $\forall j \in 1, \ldots, n$, then

$$M3\Pi(x_1, \ldots, x_n) \leq \frac{1}{n} \sum_{j=1}^{n} x_j.$$  \hspace{1cm} (9)$$

This property, presented and proved in Ref. 24 by using a convexity argument, is called mean reinforcement in analogy with 3\Pi total reinforcement.

Mean reinforcement traduces the fact that when the degrees agree, $M3\Pi$ discriminates the classes better than the arithmetic mean. The first applications of $M3\Pi$ were elaborated in bioprocessing\(^{24}\) and more recently in image analysis\(^{25}\).

Keeping in mind the symmetric sum aspect, we now address the question of extending $M3\Pi$ to sequences (discrete case) and time series (continuous case) of membership degrees.

3. GENERALIZED 3\Pi

Our idea consists of modifying the expression of 3\Pi so that the symmetry property and similar inequalities still hold. Our general formulation covers several cases and extends both 3\Pi and $M3\Pi$. 

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3.1. General Setting

Let \((V, \mathcal{V}, \mu)\) be a measure space, where \(V\) denotes a set of indices, \(\mathcal{V}\) a \(\sigma\)-algebra on \(V\), and \(\mu\) a nonnegative measure defined on \(\mathcal{V}\). Let \(f\) be a measurable mapping from \(V\) to the interval \([0, 1]\).

In certain situations, \(f(v)\) can represent an opinion, a preference, or a membership degree on a scale from 0 to 1. The function \(f\) is defined on a set of contributors \(V\) and \(\mu\) represents a weight on the contributors. The following examples cover many real cases:

- **The finite case**
  \(V = \{1, \ldots, n\}\) and \(f(v), v \in V\) represents \(n\) membership degrees provided by \(n\) sources.

- **The discrete case**
  \(V = \{1, 2, \ldots\}\) and \(f(v), v \in V\) represents an infinite but countable set of membership degrees.

- **The temporal case**
  \(V = [0, T]\) is a time interval and \(f(t)\) represents a membership degree at time \(t \in [0, T]\).

- **The spatiotemporal case**
  \(V = S \times [0, T]\) where \(S\) is a set of locations and \(f(s, t)\) represents a membership degree at location \(s\) and time \(t\).

As usual, in the two first cases, \(\mathcal{V}\) will be the power set of \(V\), while it can be any Borel \(\sigma\)-algebra in the two last cases.

3.2. G3Π Aggregating Operator

Let \(V\), \(\mu\), and \(f\) be the same as in the preceding section, with \(0 \leq f(v) \leq 1, \forall v \in V\).

First, \(\log(f(v))\) is defined for any \(v\) such that \(0 < f(v) \leq 1\). However, extending the log function on \([0, 1]\) by putting \(\log 0 = -\infty\), it is seen that \(\log(f(v))\) is defined \(\forall v \in V\) and that \(\log(f)\) is a measurable function whose range is \([-\infty, 0]\). Therefore, since \(\mu\) is a nonnegative measure, \(\int_{\mathcal{V}} \log(f) d\mu\) is well defined and belongs to \([-\infty, 0]\) (see, for example, 26, p. 19) so that \(e^{\int_{\mathcal{V}} \log(f) d\mu}\) is well defined and belongs to \([0, 1]\), with the convention \(e^{-\infty} = 0\). The same holds for \(e^{\int_{\mathcal{V}} \log(1-f) d\mu}\), since \(0 \leq 1 - f \leq 1\).

**Definition 3.1**. The generalized 3Π operator, \(G3\Pi\), is defined as

\[
G3\Pi(f) = \frac{e^{\int_{\mathcal{V}} \log(f) d\mu}}{e^{\int_{\mathcal{V}} \log(f) d\mu} + e^{\int_{\mathcal{V}} \log(1-f) d\mu}}
\]

for any measurable function \(f : V \rightarrow [0, 1]\) such that either \(e^{\int_{\mathcal{V}} \log(f) d\mu} > 0\) or \(e^{\int_{\mathcal{V}} \log(1-f) d\mu} > 0\).

For the relationship between \(G3\Pi\) and 3Π, see section 3.3. Note that the two integrals in \(G3\Pi\) denominator can simultaneously be 0, as seen in the trivial
example \( V = \{1, 2\}, \mu = \mu(2) = \frac{1}{2}, f(1) = 1, f(2) = 0, \) since \( \int_V \log f \, d\mu = \int_V \log(1-f) \, d\mu = -\infty. \)

Also, if \( \int_V \log(f) \, d\mu = -\infty \) (and \( -\infty < \int_V \log(1-f) \, d\mu \)) then clearly, \( G3\Pi(f) = 0. \) On the other hand, if \( -\infty < \int_V \log(f) \, d\mu, \) then, necessarily, \( f > 0 \mu\text{-a.e.} \) and \( e^{\int_V \log(f) \, d\mu} > 0, \) and dividing both \( G3\Pi \) numerator and denominator by \( e^{\int_V \log(1-f) \, d\mu}, \) we get

\[
G3\Pi(f) = \frac{1}{1 + e^{\int_V \log(1-f) \, d\mu}}
\]

Conversely, (11) clearly implies 10 if \( -\infty < \int_V \log(f) \, d\mu. \)

So, we arrive at the following equivalent definition:

**Definition 3.2.** The Generalized 3\( \Pi \) operator, \( G3\Pi, \) is defined as

\[
G3\Pi(f) = \begin{cases} 
0 & \text{if } \int_V \log(f) \, d\mu = -\infty \\
\frac{1}{1 + e^{\int_V \log(1-f) \, d\mu}} & \text{if } -\infty < \int_V \log(f) \, d\mu
\end{cases}
\]

for any measurable function \( f : V \to [0, 1] \) such that either \( e^{\int_V \log(f) \, d\mu} > 0 \) or \( e^{\int_V \log(1-f) \, d\mu} > 0. \)

### 3.3. Deriving 3\( \Pi \) and \( M3\Pi \)

Yager and Rybalov 3\( \Pi \) operator can be derived from \( G3\Pi \) by just considering the finite set case \( V = \{1, \ldots, n\} \) with \( \mu = 1 \) and putting \( x_v = f(v). \) Indeed, \( \int_V \log(f) \, d\mu = \sum_{v=1}^n \log(x_v) = \log(\Pi_{v=1}^n x_v) \) so that \( e^{\int_V \log(f) \, d\mu} = e^{\log(\Pi_{v=1}^n x_v)} = \Pi_{v=1}^n x_v. \) Similarly, \( e^{\int_V \log(1-f) \, d\mu} = \Pi_{v=1}^n (1-x_v), \) so that

\[
G3\Pi = \frac{\Pi_{v=1}^n x_v}{\Pi_{v=1}^n x_v + \Pi_{v=1}^n (1-x_v)} = 3\Pi
\]

More generally, Triple \( \Pi \) operator with generating function \( G \) can be derived from \( G3\Pi \) by just taking \( \mu = 1 \) and \( f(v) = G(v). \)

\( M3\Pi \) operator can be derived from \( G3\Pi \) by just considering the finite set case \( V = \{1, \ldots, n\} \) with \( \mu = \frac{1}{n} \) and putting \( x_v = f(v). \) Indeed \( \int_V \log(f) \, d\mu = \frac{1}{n} \sum_{v=1}^n \log(x_v) = \log(\Pi_{v=1}^n \frac{1}{n} x_v) \) and \( e^{\int_V \log(f) \, d\mu} = \Pi_{v=1}^n \frac{1}{n} x_v. \) Similarly, \( e^{\int_V \log(1-f) \, d\mu} = \Pi_{v=1}^n (1-x_v)^\frac{1}{n}, \) so that

\[
G3\Pi = \frac{\Pi_{v=1}^n \frac{1}{n} x_v}{\Pi_{v=1}^n \frac{1}{n} x_v + \Pi_{v=1}^n (1-x_v)^\frac{1}{n}} = M3\Pi
\]
3.4. Monotonicity

**Proposition 3.3.** If \( f \leq g \), then \( G3\Pi(f) \leq G3\Pi(g) \)

*Proof.* If \( \int_V \log(f) d\mu = -\infty \), then \( G3\Pi(f) = 0 \) and the conclusion trivially holds. If \( -\infty < \int_V \log(f) d\mu \), then, by (11), we have

\[
G3\Pi(f) = \frac{1}{1 + e^{\int_V \log(\frac{1}{f} - 1) d\mu}}.
\]

Since \( f > 0 \) \( \mu \)-a.e. (otherwise \( \int_V \log(f) d\mu = -\infty \)) and \( f \leq g \), we also have \( g > 0 \) \( \mu \)-a.e. so that \( \frac{1}{g} \) is defined a.e. and

\[
\frac{1}{f} - 1 \geq \frac{1}{g} - 1 \text{ a.e.}
\]

This clearly yields \( G3\Pi(f) \leq G3\Pi(g) \).

\[\blacksquare\]

3.5. Full Reinforcement

**Proposition 3.4.** Let \( m = \inf_{v \in V} f(v) \) (resp. \( M = \sup_{v \in V} f(v) \)).

- If \( \mu(f = m) \geq 1 \) and \( 0 \leq f \leq \frac{1}{2} \), then \( G3\Pi(f) \leq m \).
- If \( \mu(f = M) \geq 1 \) and \( \frac{1}{2} \leq f \leq 1 \), then \( G3\Pi(f) \geq M \).

*Proof.* If \( \int_V \log(f) d\mu = -\infty \), then \( G3\Pi(f) = 0 \) and we trivially have \( G3\Pi(f) \leq m \). If \( -\infty < \int_V \log(f) d\mu \), then \( f > 0 \) a.e. and

\[
\int_V \log\left(\frac{1}{f} - 1\right) d\mu = \int_{\{f = m\}} \log\left(\frac{1}{f} - 1\right) d\mu + \int_{\{f > m\}} \log\left(\frac{1}{f} - 1\right) d\mu
\]

\[
\geq \int_{\{f = m\}} \log\left(\frac{1}{f} - 1\right) d\mu = \log\left(\frac{1}{m} - 1\right) \mu(f = m)
\]

\[
\geq \log\left(\frac{1}{m} - 1\right)
\]

the first inequality being due to \( \log(\frac{1}{f} - 1) \geq \log\left(\frac{1}{\frac{1}{2}} - 1\right) = \log(1) = 0 \) and the second one due to \( \mu(f = m) \geq 1 \).

This implies that

\[
G3\Pi(f) = \frac{1}{1 + e^{\int_V \log(\frac{1}{f} - 1) d\mu}} \leq \frac{1}{1 + e^{\log\left(\frac{1}{m} - 1\right)}} = m.
\]

The proof for \( M \) is similar since \( \log\left(\frac{1}{f} - 1\right) \geq 0 \) if \( \frac{1}{2} \leq f \leq 1 \).

\[\blacksquare\]
Remark. The conditions of Proposition 2 are satisfied in the finite case $V = \{1, \ldots, n\}$ for $\mu\{v\} = 1$, that is in the case of Yager and Rybalov Triple $\Pi$. Indeed, $\inf(f)$ and $\sup(f)$ are reached at least for one $v \in V$ which is of measure one.

### 3.6. Mean Reinforcement

The following inequality is the main property of $M3\Pi$. It shows that this mean operator provides a better discrimination than the classical arithmetic mean.

**Proposition 3.5.** Assume that $\mu(V) = 1$, then for any measurable $f : V \rightarrow [0, \frac{1}{2}]$, we have

$$G3\Pi(f) \leq \int_V f d\mu$$

while if $f : V \rightarrow [\frac{1}{2}, 1]$, then the converse inequality holds.

**Proof.** If $\int_V \log(f) d\mu = -\infty$, then $e^{\int_V \log(f) d\mu} = 0$ and inequality (14) is trivially verified. Thus, we may and do assume that $-\infty < \int_V \log(f) d\mu \leq \log(\frac{1}{2})$, so that $0 < e^{\int_V \log(f) d\mu} < \frac{1}{2} < +\infty$. This assumption also implies that $\int_V f d\mu > 0$ because $\int_V f d\mu = 0$ implies $f = 0$ a.e., $\int_V \log(f) d\mu = -\infty$ and $e^{\int_V \log(f) d\mu} = 0$. So, under this assumption, both members of (14) are (strictly) positive, and using (11), inequality (14) is then equivalent to the following ones

$$\frac{1}{\int_V f d\mu} \leq 1 + e^{\int_V \log(\frac{1}{f} - 1) d\mu}$$

$$\frac{1}{\int_V f d\mu} - 1 \leq e^{\int_V \log(\frac{1}{f} - 1) d\mu}$$

$$\log \left( \frac{1}{\int_V f d\mu} - 1 \right) \leq \int_V \log \left( \frac{1}{f} - 1 \right) d\mu$$

$$\varphi \left( \int_V f d\mu \right) \leq \int_V \varphi(f) d\mu$$

where $\varphi(x) = \log(\frac{1}{x} - 1)$. But this last inequality is true as an application of the classical Jensen’s inequality (see, for example, 26, p. 62). Indeed, we have $0 \leq f \leq \frac{1}{2}$ and the function $\varphi$ is convex on the interval $[0, \frac{1}{2}]$ since $\varphi'(x) = -\frac{1}{1-x} - \frac{1}{x} = \frac{1}{x^2-x}$ and $\varphi'(x) = \frac{1-2x}{(x^2-x)^2} > 0$ for $0 < x \leq \frac{1}{2}$. ■

#### 3.6.1. Finite Set Case

Let $V = \{1, \ldots, n\}$ and let $\mu$ be the uniform probability measure, that is, $\mu\{i\} = \frac{1}{n}$ for $i = 1, \ldots, n$. If $f(i)$ is denoted by $x_i$, then we have
\[ \int_V f \, d\mu = \frac{1}{n} \sum_{i=1}^{n} x_i, \]

\[ \int_V \log(f) \, d\mu = \frac{1}{n} \sum_{i=1}^{n} \log(x_i) = \log \left( (x_1 \ldots x_n)^{\frac{1}{n}} \right) \]

and

\[ e^{\int_V \log(f) \, d\mu} = (x_1 \ldots x_n)^{\frac{1}{n}} \]

so that in this finite set case, inequality (14) can be written as

\[ \frac{(x_1 \ldots x_n)^{\frac{1}{n}}}{(x_1 \ldots x_n)^{\frac{1}{n}} + ((1 - x_1) \ldots (1 - x_n))^{\frac{1}{n}}} \leq \frac{1}{n} \sum_{i=1}^{n} x_i \]

whenever \( 0 \leq x_i \leq \frac{1}{2} \) for all \( i \), the converse inequality being true whenever \( \frac{1}{2} \leq x_i \leq 1 \) for all \( i \).

Inequality (15) was proved in Ref. 24 in the finite case with an application in classification of bioprocesses.

### 3.6.2. Discrete Case

More generally, let \( V = \{0, 1, \ldots, n, \ldots\} \) and let \( \mu \) be a probability distribution on \( V \). If \( f(i) \) is still denoted by \( x_i \), then we have

\[ \int_V f \, d\mu = \sum_{i=0}^{+\infty} x_i \mu\{i\}, \]

\[ \int_V \log(f) \, d\mu = \sum_{i=0}^{+\infty} \log(x_i) \mu\{i\} = \log \left( \prod_{i=0}^{+\infty} x_i^{\mu\{i\}} \right) \]

and therefore

\[ e^{\int_V \log(f) \, d\mu} = \prod_{i=0}^{+\infty} x_i^{\mu\{i\}} \]

so that in this discrete case, inequality (14) can be written as

\[ \frac{\prod_{i=0}^{+\infty} x_i^{\mu\{i\}}}{\prod_{i=0}^{+\infty} x_i^{\mu\{i\}} + \prod_{i=0}^{+\infty} (1 - x_i)^{\mu\{i\}}} \leq \sum_{i=0}^{+\infty} x_i \mu\{i\}. \]

For example, if \( \mu \) is a geometric(\( p \)) distribution with parameter \( 0 \leq p \leq 1 \), that is, \( \mu\{i\} = (1 - p) p^i \), we get
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\[
\frac{\prod_{i=0}^{+\infty} x_i^{(1-p)p^i}}{\prod_{i=0}^{+\infty} x_i^{(1-p)p^i} + \prod_{i=0}^{+\infty} (1-x_i)^{(1-p)p^i}} \leq (1 - p) \sum_{i=0}^{+\infty} x_i p^i
\] (17)

while if \(\mu\) is a Poisson\((\lambda)\) distribution with parameter \(\lambda > 0\), that is, \(\mu\{i\} = e^{-\lambda} \frac{\lambda^i}{i!}\), we get

\[
\frac{\prod_{i=0}^{+\infty} x_i^{e^{-\lambda} \frac{\lambda^i}{i!}}}{\prod_{i=0}^{+\infty} x_i^{e^{-\lambda} \frac{\lambda^i}{i!}} + \prod_{i=0}^{+\infty} (1-x_i)^{e^{-\lambda} \frac{\lambda^i}{i!}}} \leq e^{-\lambda} \sum_{i=0}^{+\infty} \frac{\lambda^i}{i!}.
\] (18)

Obviously, (18) is a particular case of (16) when taking \(\mu\{i\} = \frac{1}{n}\) if \(i = 1, \ldots, n\) and \(\mu\{i\} = 0\) otherwise.

This inequality is surely promising as it directly corresponds to the use of an AO on a discrete time series. Certainly, the choice of the probability distribution should be discussed by practicians to get an optimal decision from this discrete information.

3.6.3. Temporal Case, Moving G3\Pi

Let \(V = \mathbb{R}\) be the real line and \(\mu = g dt\) be a probability distribution on \(V\) with density \(g\) w.r.t. the Lebesgue measure \(dt\). Then inequality (14) can be written as

\[
\frac{e^{\int_a^b \log(f(t))g(t)dt}}{e^{\int_a^b \log(f(t))g(t)dt} + e^{\int_a^b \log(1-f(t))g(t)dt}} \leq \int_{\mathbb{R}} f(t)g(t)dt.
\] (19)

For example, if \(\mu\) is the uniform distribution on a finite interval \([a, b], a < b\) in \(\mathbb{R}\), then inequality (19) yields

\[
\frac{e^{\int_a^b \log(f(t))dt}}{e^{\int_a^b \log(f(t))dt} + e^{\int_a^b \log(1-f(t))dt}} \leq \frac{1}{b-a} \int_a^b f(t)dt
\] (20)

while if \(\mu\) is the exponential distribution on \([0, +\infty)\) with parameter \(\theta > 0\), we get

\[
\frac{e^{\theta \int_0^{+\infty} \log(f(t))e^{-\theta t}dt}}{e^{\theta \int_0^{+\infty} \log(f(t))e^{-\theta t}dt} + e^{\theta \int_0^{+\infty} \log(1-f(t))e^{-\theta t}dt}} \leq \theta \int_0^{+\infty} f(t)e^{-\theta t}dt.
\] (21)

A similar inequality holds for any standard distribution, such as a Normal \((\mu, \sigma^2)\) one or an exponential type one, the choice of the distribution having to be discussed by practicians. Two illustrations of these inequalities are given in the following Figures, for uniformly simulated data (Figure 1) and for a Brownian-type simulation (Figure 2).

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Figure 1. Illustration of $G_3\Pi$ mean reinforcement property. Curve of membership degree to a class, the degrees scaling from 0 (no membership to a class) to 1 (membership to a class). A data stream was uniformly simulated in $[0, 0.5]$ during the time interval $[0, 500]$ and in $[0.5, 1]$ during the time interval $[501, 1000]$. As announced, $G_3\Pi$ is lower (resp. greater) than the mean on $[0, 500]$ (resp. on $[501, 1000]$).

Figure 2. Illustration of $G_3\Pi$ mean reinforcement property with a membership degree data stream simulated by a Brownian Bridge-type stochastic process instead of a uniform one as in Figure 1, with smaller differences between $G_3\Pi$ and the mean.

In practical situations, $f$ is sampled at time $t_1, t_2, \ldots, t_n, \ldots$ to get the time series $f(t_1), f(t_2), \ldots, f(t_n), \ldots$. As moving averages (MA) and its variants, such as exponential MA (EMA), Kaufman adaptive MA (KAMA), Fractal adaptative MA (FAMA) and so on, are very useful in the study of time series trending, a natural question is whether such notions can be extended to the $G_3\Pi$ case.
This can be done by just using equation (11). Indeed, as moving average at time $t$ is a weighted arithmetic mean of $f$ on an interval $[t-h, t]$, in the case of a look-back lag $h > 0$, we can formally define the moving $G3\Pi$ as follows

**Definition 3.6.** Moving $G3\Pi$, denoted $MG3\Pi$, w.r.t. a weight function $g$ and a look-back lag $h > 0$, is defined as

$$
MG3\Pi(f)(t) = \begin{cases} 
0 & \text{if } \int_{t-h}^{t} \log(f(s))g(s)ds = -\infty \\
\frac{1}{1 + e^{\int_{t-h}^{t} \log(\frac{1}{f} - 1)g(s)ds}} & \text{otherwise.}
\end{cases}
$$

In other words, moving $G3\Pi$ just consists of first computing standard MA and its variants from the time series $\log(\frac{1}{f} - 1)$ and then applying the function $\frac{1}{1+\exp}$ to these MA. An illustrative example with KAMA is given in the Applications section 4.

### 3.6.4. Spatiotemporal case

Consider a spatiotemporal model introduced in Ref 27. Let $S = \mathbb{R}^2$ represent a set of locations and $T = [0, +\infty[$ represent time. Let $V = S \times T$ and $\mu \propto \exp(-a||s||^2) \exp(-bt^2)dsdt$, where $a > 0$, $b > 0$ are parameters. Let $f(s, t)$ represent a preference at location $s$ and time $t$. Then, $C$ being a normalizing constant, let

$$
u = \int_V \log(1 - f) d\mu$$

and similarly, let $w = \int_V f d\mu$.

If $f(s, t) \leq \frac{1}{2}$ then inequality (14) yields

$$
\frac{e^{\mu}}{e^{\mu} + e^{\nu}} \leq w
$$

### 3.6.5. Positive Functions

Now, only assume that $f > 0$. Then, as we have $0 < \frac{f}{2(1+f)} \leq \frac{1}{2}$, inequality (14) implies the following:

**Proposition 3.7.** Let $f : V \to (0, +\infty)$ be a positive measurable function defined on any probability space, then

$$
\frac{e^{\int_V \log(\frac{f}{2(1+f)})} d\mu}{e^{\int_V \log(\frac{f}{2(1+f)})} d\mu + e^{\int_V \log(\frac{f}{2(1+f)})} d\mu} \leq \int_V \frac{f}{2(1+f)} d\mu
$$
By the way, in the above Proposition, \( \frac{f}{2(1+f)} \) can be replaced by \( \phi(f) \) for any measurable function \( \phi \) mapping \( \mathbb{R}_+ \) on the interval \([0, \frac{1}{2}]\). A simple example for a bounded function \( f \) consists of dividing \( f \) by \( 2\sup(f) \).

This last case enlarges the possibility of using inequality (14) and can be a starting point to further works in the field of aggregating time series.

4. APPLICATIONS AND PERSPECTIVES

The finite set case is the most studied case for AOs and especially for \( 3\Pi_1 \) and \( M3\Pi_1 \). Two applications with real world data (classification of physiological states of microorganisms in a bioprocess and plant-image analysis) are detailed in Refs. 24 and 25, respectively.

An application in the temporal case can be developed in the solar energy domain. Indeed, the so-called extraterrestrial solar radiation (W/m²) at a given geographical position \( p \) of the earth is given for each day by a deterministic function of time, say \( e_p \). However, the solar radiation that can be measured at \( p \), also called global solar radiation, say \( g_p \), depends on the geographical environment and the local meteorological conditions. The ratio \( \frac{g_p(t)}{e_p(t)} \) is known as the clearness index at time \( t \), say \( K_p(t) \) or more simply \( K(t) \), which belongs to \([0, 1]\). The more clear the sky is at time \( t \), the more index \( K(t) \) is close to 1, so that \( K \) appears as a time series of degrees of membership to a certain class of clearness. KAMA\textsuperscript{28} can be computed for \( K \) and compared to moving \( M3\Pi_1 \), which is obtained by computing KAMA for \( \log(\frac{1}{K} - 1) \) and taking \( \frac{1}{1+\exp} \) of these moving averages as indicated in Section 3.6.3.

It may be interesting to apply AO \( G3\Pi \) in the following context. Suppose that the curve in Figure 3 represents a data stream giving, at each time, the membership to a danger state class of a given system. As all the values are higher than 0.5 in the interval \([26, 50]\), \( G3\Pi \) reinforces the classical mean and better emphasizes...
the danger state of the system. It seems that the potential applications of these inequalities remain to be discovered. We suggest two possible options of using $G\Pi_1$, depending on the priority purpose of the application. If local analysis (of signals) is primordial, then the following approach can be applied (Figure 4):

1. Step 1: At each time $t$, use $3\Pi$ to aggregate the values at $t$ of several variables, thus defining a new aggregate stream.
2. Step 2: Use $G\Pi_1$ to aggregate the aggregate stream on an interval.

If global analysis (of signal) is primordial, then the method can be the following (Figure 5):

1. Step 1: For each variable, compute $G\Pi_1$ on a time interval. The number of resulting values equals the number of variables.
2. Step 2: Aggregate all these values using $3\Pi$.

The choice of the approach depends on the application framework. Whatever the approach be, both aggregate time series. Obviously, for experimental applications, the discrete case (Section 3.3) seems more useful. A discretized version of the continuous case is possible. Choosing a suited weighting function $g$ is an important task. Possible applications could be analysis of medical and physiological signals, functional data, etc.

5. CONCLUSIONS

We have introduced $G\Pi_1$ AO that generalizes both $3\Pi$ and $M\Pi_1$ and aggregates information on any measured space. This point seems to be new and could...
be studied for other AO. A main inequality shows that $G_3\Pi$ is more discriminating than arithmetical mean. Aggregating information from time series on a given interval provides a better global vision and step back rather than a localized analysis. This could improve big data management and also, decision making. In the short term, further works could concern some applications of this generalization to some other real time series or spatiotemporal series. In the long term, one could envisage multiscale aggregation and also some applications to prediction.

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