Oblique repulsion in the nonnegative quadrant

Dominique Lépingle
Université d’Orléans, FDP-MAPMO

January 24, 2014

Abstract

We consider the differential system
\[\begin{align*}
\dot{x} &= \frac{\alpha}{x} + \frac{\beta}{y} \\
\dot{y} &= \frac{\gamma}{x} + \frac{\delta}{y}
\end{align*}\]
in the nonnegative quadrant. Here \(\alpha\) and \(\delta\) are positive, \(\beta\) and \(\gamma\) are real constants. Under some condition on the constants there exists a unique global solution. The main difficulty is to prove uniqueness when starting at the corner of the quadrant.

1 Introduction.

We are interested in the question of existence and uniqueness of the solution \(u(.)(t) = (x(.)(t), y(.)(t))\) to the following integral system
\[\begin{align*}
x(t) &= x + \alpha \int_0^t \frac{ds}{x(s)} + \beta \int_0^t \frac{ds}{y(s)} \\
y(t) &= y + \gamma \int_0^t \frac{ds}{x(s)} + \delta \int_0^t \frac{ds}{y(s)}
\end{align*}\]
where \(x(.)(t)\) and \(y(.)(t)\) are continuous functions from \([0, \infty)\) to \([0, \infty)\) with the conditions
\[\begin{align*}
\int_0^t \mathbb{1}_{\{x(s) = 0\}} ds &= 0 \\
\int_0^t \mathbb{1}_{\{y(s) = 0\}} ds &= 0 \\
\int_0^t \mathbb{1}_{\{x(s) > 0\}} \frac{ds}{x(s)} &= \infty \\
\int_0^t \mathbb{1}_{\{y(s) > 0\}} \frac{ds}{y(s)} &= \infty
\end{align*}\]
for any \(t \geq 0\). Here \(\alpha\), \(\beta\), \(\gamma\) and \(\delta\) are four real constants with \(\alpha > 0\) and \(\delta > 0\).

The system has a single singularity at each side of the nonnegative quadrant \(S = \{(x, y) : x \geq 0, y \geq 0\}\) and a double singularity at the corner \(0 = (0, 0)\). We write \(S^\circ := S \setminus \{0\}\) for the punctured quadrant.

We will note \(\dot{x}(t)\) the derivative \(dx(t)/dt\). So the integral system (1) may be written as an initial-value problem
\[\begin{align*}
\dot{x} &= \frac{\alpha}{x} + \frac{\beta}{y} \\
\dot{y} &= \frac{\gamma}{x} + \frac{\delta}{y}
\end{align*}\]
with the initial condition \((x(0), y(0)) \in S\).

We first remark that if \(\beta < 0, \gamma < 0\) and \(\alpha \delta < \beta \gamma\), there exist \(\lambda > 0\) and \(\mu > 0\) such that \(\lambda \alpha + \mu \gamma < 0\) and \(\lambda \beta + \mu \delta < 0\). Thus \(z(t) := \lambda x(t) + \mu y(t)\) is decreasing, \(\min (x(t), y(t)) \to 0\) and \(\dot{z}(t) \to -\infty\) as \(t \to t_f\) where \(t_f < \infty\) and there is no solution. If \(\beta < 0, \gamma < 0\) and \(\alpha \delta = \beta \gamma\), then \(v(t) := \alpha y(t) - \gamma x(t)\) remains equal to \(\alpha y - \gamma x\) and there is a unique solution \((x(t), y(t))\) that converges to \((\frac{\gamma x - \alpha y}{\beta + \gamma}, \frac{\beta y - \delta x}{\beta + \gamma})\) except if \((x, y) = 0\) in which case there is no solution.

From now on we will make the following hypothesis:
\[\begin{align*}
(H) \quad & \max (\beta, \gamma) \geq 0 \quad \text{or} \quad \beta \gamma < \alpha \delta.
\end{align*}\]
This is equivalent to the existence of $\lambda \geq 0$ and $\mu \geq 0$ such that $\lambda \alpha + \mu \gamma > 0$ and $\lambda \beta + \mu \delta > 0$. This last formulation amounts to saying that the matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is an $S$-matrix in the terminology of [1]. In the sequel, we fix a pair $(\lambda, \mu)$ with $\lambda > 0$, $\mu > 0$, such that $\lambda \alpha + \mu \gamma > 0$ and $\lambda \beta + \mu \delta > 0$.

The aim of this note is to prove the following result.

**Theorem 1** Under condition $(H)$, there exists a unique solution to (1) for any starting point $(x, y) \in S$.

2 Some preliminary lemmata.

We begin with a comparison lemma.

**Lemma 2** Let $x_1$ and $x_2$ be nonnegative continuous functions on $[0, \infty)$ which are solutions to the system

$$\begin{align*}
    x_1(t) &= v_1(t) + \alpha \int_0^t \frac{ds}{x_1(s)} \\
    x_2(t) &= v_2(t) + \alpha \int_0^t \frac{ds}{x_2(s)}
\end{align*}$$

where $\alpha > 0$, $v_1$ and $v_2$ are continuous functions such that $0 \leq v_1(0) \leq v_2(0)$, and $v_2 - v_1$ is nondecreasing. Then $x_1 \leq x_2$ on $[0, \infty)$.

Proof. Assume there exists $t > 0$ such that $x_1(t) > x_2(t)$. Set

$$\tau := \max \{s \leq t : x_1(s) \leq x_2(s)\}.$$ 

Then

$$x_2(t) - x_1(t) = x_2(\tau) - x_1(\tau) + (v_2(t) - v_1(t)) - (v_2(\tau) - v_1(\tau)) + \alpha \int_\tau^t \left(\frac{1}{x_2(s)} - \frac{1}{x_1(s)}\right)ds \geq 0,$$

a contradiction. ■

**Lemma 3** Let the system

$$\begin{align*}
    \dot{x} &= \frac{\alpha}{x} + \phi(x, z) \\
    \dot{z} &= \psi(x, z)
\end{align*}$$

with $x(0) = x_0 \geq 0$, $z(0) = z_0 \in \mathbb{R}$, $\alpha > 0$, $\phi$ and $\psi$ two Lipschitz functions on $\mathbb{R}_+ \times \mathbb{R}$, and $|\phi| \leq c$ for some $c < \infty$. Then there exists a unique solution to (4). Moreover, for this solution, $x(t) > 0$ for any $t > 0$.

Proof. Assume first $x_0 > 0$. Then the system (4) is Lipschitz on $[\min \{x_0, \frac{\alpha}{x}\}, \infty) \times \mathbb{R}$ and the solution does not step out of this domain, so there is a unique global solution. When $x_0 = 0$, we let $w_0(t) = 0$, $z_0(t) = z_0$ and for $n \geq 1$

$$\begin{align*}
    w_n(t) &= 2\alpha t + 2 \int_0^t \sqrt{w_{n-1}(s)} \phi(\sqrt{w_{n-1}(s)}, z_{n-1}(s))ds \\
    z_n(t) &= z_0 + \int_0^t \psi(\sqrt{w_{n-1}(s)}, z_{n-1}(s))ds.
\end{align*}$$

2
Let $M > 0$ and assume $|w_{n-1}(t)| \leq M$ on some interval $[0, T]$. Then, for $0 \leq t \leq T$, 

$$|w_n(t)| \leq T(2\alpha + 2c\sqrt{M})$$

and this is again $\leq M$ for $T$ small enough. We also have $|z_n(t)| \leq M'$ for any $n \geq 0$ for $T$ small enough. Equicontinuity of $(w_n, z_n)_{n \geq 0}$ is easily verified and from the Arzela-Ascoli theorem it follows there exists a subsequence $(w_{n_k}, z_{n_k})$ converging on $[0, T]$ to a solution $(w, z)$ of the system

$$\begin{align*}
\dot{w} &= 2\alpha + 2\sqrt{w}\phi(\sqrt{w}, z) \\
\dot{z} &= \psi(\sqrt{w}, z)
\end{align*}$$

with the initial conditions $w(0) = 0$, $z(0) = z_0$. For small $T$, $\dot{w} > 0$ on $[0, T]$. Set now $x(t) = \sqrt{w(t)}$. Then $(x, z)$ is a solution to (4) on $[0, T]$ with $x(T) > 0$. We may extend the solution to $[0, \infty)$ by using the above result with $x_0 > 0$.

We now prove uniqueness. Let $(x, z)$ and $(x', z')$ be two solutions of (4). Then

$$\frac{(x(t) - x'(t))^2 + (z(t) - z'(t))^2}{2} = 2\alpha \int_0^t (x(s) - x'(s))(\frac{1}{x(s)} - \frac{1}{x(s)}) ds + 2 \int_0^t (x(s) - x'(s))(\phi(x(s), z(s)) - \phi(x'(s), z'(s))) ds$$

$$+ 2 \int_0^t (z(s) - z'(s))(\psi(x(s), z(s)) - \psi(x'(s), z'(s))) ds$$

$$\leq 4L \int_0^t ((x(s) - x'(s))^2 + (z(s) - z'(s))^2) ds$$

where $L$ is the Lipschitz constant of $\phi$ and $\psi$. Uniqueness follows from Gronwall’s inequality. ■

**Lemma 4** Let $u(.) = (u(.), y(.))$ be a solution to (1) and let $\nu = (\lambda, \mu)$. Then the function $z(t) := \nu.u(t) = \lambda x(t) + \mu y(t)$ is increasing on $[0, \infty)$ and we have $u(t) \in S^0$ for any $t > 0$.

Proof. Recall that condition (H) is in force. We easily check that $\dot{z}(t)$ is positive. ■

### 3 Existence. Case $x = 0, y = 0.$

There is an explicit solution to (1) when the starting point is the corner.

**Proposition 5** There is a solution to (1) with initial condition $0$ given by

$$\begin{align*}
x(t) &= c\sqrt{t} \\
y(t) &= d\sqrt{t}
\end{align*}$$

where

$$\begin{align*}
c &= (2\alpha + \frac{\beta}{2}(\beta - \gamma + \sqrt{(\beta - \gamma)^2 + 4\alpha \delta}))^{1/2} \\
d &= (2\delta + \frac{\gamma}{2}\beta - \beta + \sqrt{(\beta - \gamma)^2 + 4\alpha \delta}))^{1/2}.
\end{align*}$$

Proof. Writing down $x(t) = c\sqrt{t}$ and $y(t) = d\sqrt{t}$ we have to solve the system

$$\begin{align*}
\frac{c}{x^2} &= \frac{\alpha}{c} + \frac{\beta}{x} \\
\frac{c}{y^2} &= \frac{\gamma}{2c} + \frac{\delta}{y}.
\end{align*}$$

We first compute

$$\frac{d}{c} = \frac{\gamma - \beta + \sqrt{(\beta - \gamma)^2 + 4\alpha \delta}}{2\alpha}$$
and then obtain (7) provided that
\[
C = 2\alpha + \beta (\beta - \gamma + \sqrt{(\beta - \gamma)^2 + 4\alpha\delta})
\]
\[
D = 2\delta + \frac{\gamma}{\alpha} (\gamma - \beta + \sqrt{(\beta - \gamma)^2 + 4\alpha\delta})
\]
are positive. If \(\beta \geq 0\), \(C\) is clearly positive. This is also true if \(\beta < 0\) and \(\beta\gamma < \alpha\delta\) since \(C\) may be written
\[
C = \frac{4\alpha(\alpha\delta - \beta\gamma)}{2\alpha\delta - \beta\gamma + \beta^2 - \beta\sqrt{4(\alpha\delta - \beta\gamma) + (\beta + \gamma)^2}}.
\]
The proof for \(D\) is similar. ■

Uniqueness in this case is more involved and will be treated in the last section. We only remark for the moment that the system (3) with \(\alpha = \delta = 0\), \(\beta > 0\), \(\gamma > 0\) and initial value \(0\) has a one-parameter family of solutions.

4 Angular behavior.

We are now in a position to study the behavior of \(y(t)/x(t)\). For any \(u = (x, y) \in S^0\) we set
\[
\theta(u) = \arctan \frac{y}{x}.
\]
We also set
\[
u_* = (x_*, y_*) := \left(\frac{c}{\lambda c + \mu d}, \frac{d}{\lambda c + \mu d}\right).
\]

Proposition 6 Let \(u(.)\) be a solution to (1) starting at \(u = (x, y) \in S^0\). Then for any \(t > 0\)

1. \[
\frac{d\theta(u(t))}{dt} > 0 \quad \text{and} \quad \theta(u(t)) < \theta(u_*) \quad \text{if} \quad \theta(u) < \theta(u_*)
\]
2. \[
\frac{d\theta(u(t))}{dt} = 0 \quad \text{and} \quad \theta(u(t)) = \theta(u_*) \quad \text{if} \quad \theta(u) = \theta(u_*)
\]
3. \[
\frac{d\theta(u(t))}{dt} < 0 \quad \text{and} \quad \theta(u(t)) > \theta(u_*) \quad \text{if} \quad \theta(u) > \theta(u_*)
\]

Proof. From Lemma 4 we know that \(u(t) \in S^0\) for any \(t \geq 0\).

1. We compute
\[
\frac{d\theta(u(t))}{dt} = \frac{1}{x^2(t) + y^2(t)} \left( \frac{d}{c} - \frac{y(t)}{x(t)} \right) \left[ \alpha + \frac{x(t)(\beta - \gamma + \sqrt{(\beta - \gamma)^2 + 4\alpha\delta})}{2y(t)} \right]
\]
and the conclusion follows.

2. Let \(a, b \in S^0\) with \(0 \leq \theta(a) < \theta(u_*)\), \(\theta(u_*) < \theta(b) \leq \frac{\pi}{2}\) and let \(l > 0\). We set
\[
A = \{v \in S^0 : \theta(a) \leq \theta(v) \leq \theta(u_*)\}
\]
\[
B = \{v \in S^0 : \theta(b) \geq \theta(v) \geq \theta(u_*)\}.
\]
It follows from above that any solution starting from $A$ stays in $A$, and the same is true for $B$. If $u \in A$,

$$x(t) \geq -\frac{\mu}{\lambda}y(t) + (x + \frac{\mu}{\lambda}y) \geq -\frac{\mu d}{\lambda c}x(t) + x + \frac{\mu}{\lambda}y$$

and therefore

$$x(t) \geq c \frac{\lambda x + \mu y}{\lambda c + \mu d}.$$  

If $u \in B$,

$$x(t) \geq \frac{x}{y}y(t) \geq \frac{x}{y}(\frac{\lambda}{\mu}x(t) + \frac{\lambda x + \mu y}{\mu})$$

and therefore

$$x(t) \geq x.$$  

Same estimations for $y(t).$  

**Corollary 7** Let $u(.)$ be a solution to (1). Then

$$\lim_{t \to \infty} \theta(u(t)) = \theta(u_*),$$ i.e. 

$$\lim_{t \to \infty} \frac{y(t)}{x(t)} = \frac{d}{c}.$$  

Proof. If $u = (x, y) \in S^0$, this is an easy consequence of (10). If $u = (0, 0)$ we may apply Lemma 4 and then (10).  

**5 Existence and uniqueness. Case $x > 0, y > 0$.**  

**Proposition 8** There exists a unique solution $u(.)$ to (1) starting at $u = (x, y)$ with $x > 0, y > 0$. It satisfies $x(t) > 0, y(t) > 0$ for any $t \geq 0$.

Proof. We now assume $\theta(a) > 0$ and $\theta(b) < \frac{\pi}{2}$ in (11). Let $l > 0$ and $\nu = (\lambda, \mu)$. We set $L := \{v \in S^0 : \nu \cdot v \geq l\}$. From Lemma 4 and Proposition 6 we know that any solution starting from $A \cap L$ stays in $A \cap L$, and the same is true for $B \cap L$. As the system is Lipschitz in $A \cap L$ and in $B \cap L$, there is a unique global solution to (1) in both cases.  

**6 Existence and uniqueness. Case $x = 0, y > 0$.**  

**Proposition 9** There exists a unique solution $u(.)$ to (1) starting at $u = (x, y)$ with $x = 0, y > 0$. It satisfies $x(t) > 0, y(t) > 0$ for any $t > 0$.

Proof. Let $\varepsilon \in (0, y\frac{\mu d}{\lambda c + \mu d})$. We define on $\mathbb{R}_+ \times \mathbb{R}$

$$\psi_\varepsilon(x, z) := \frac{1}{\max(\gamma x + z, \alpha \varepsilon)}.$$  

We apply Lemma 3 to obtain a unique solution $x_\varepsilon(.)$, $z_\varepsilon(.)$ to

$$x_\varepsilon(t) = \alpha \int_0^t \frac{ds}{x_\varepsilon(s)} + \alpha^m \beta \int_0^t \psi_\varepsilon(x_\varepsilon(s), z_\varepsilon(s))ds$$

$$z_\varepsilon(t) = \alpha y + \alpha(\alpha \delta - \beta \gamma) \int_0^t \psi_\varepsilon(x_\varepsilon(s), z_\varepsilon(s))ds.$$  

(12)
Let
\[ y_\varepsilon(t) = \frac{1}{\alpha} (\gamma x_\varepsilon(t) + z_\varepsilon(t)) \]
\[ \tau_y(\varepsilon) = \inf \{ t > 0 : y_\varepsilon(t) < \varepsilon \} \].

On the interval \([0, \tau_y(\varepsilon)]\), \((x_\varepsilon(.), y_\varepsilon(.))\) is the unique solution to (1). From (9) we know that \(y_\varepsilon(t) > \varepsilon\) on this interval. Thus \(\tau_y(\varepsilon) = \infty\) and \((x(.), y(.)) := (x_\varepsilon(.), y_\varepsilon(.))\) is the unique global solution to (1).

\[ \blacksquare \]

7 Path behavior.

Let us note \(u(t, u_0)\) the solution to (1) starting at \(u_0 \in S^\circ\). Using Gronwall’s inequality as in the proof of uniqueness, it is easily seen that for any \(t > 0\) the solution \(u(t, u_0)\) continuously depends on the initial condition \(u_0\). It has the Scaling Property:

\[ (SC) \quad u(r^2 t, u_0) = ru(t, \frac{u_0}{r}) \]

for any \(r > 0, t \geq 0, u_0 \in S^\circ\). Using Lemma 4 we also note that any solution \(u(\cdot)\) to (1) has the Semi-group Property:

\[ (SG) \quad u(s + t) = u(t, u(s)) \]

for any \(s > 0\) and \(t \geq 0\). With Proposition 8 and Proposition 9 this entails that \(x(t) > 0\) and \(y(t) > 0\) for any \(t > 0\). We now set for any \(r > 0\):

\[ L_r := \{ u = (x, y) : x > 0, y > 0, \nu. u = r, x \geq 0, y \geq 0, \nu. u = r \} \]

\[ L_r := \{ u = (x, y) : x > 0, y > 0, \nu. u = r, x \geq 0, y \geq 0, \nu. u = r \} \]

**Lemma 10** Let \(u(\cdot)\) be a solution to (1) starting at \(u_0 \in S\) with \(\nu. u_0 \leq r\). We set

\[ \tau(r) := \inf \{ t \geq 0 : \nu. u(t) = r \} \]

Then

\[ \tau(r) \leq \frac{r^2}{2(\lambda(\lambda\alpha + \mu\gamma) + \mu(\lambda\beta + \mu\delta))}. \]

**Proof.** Set

\[ z(t) := \nu. u(t). \]

As

\[ z(t) = \nu. u_0 + [\lambda(\lambda\alpha + \mu\gamma) + \mu(\lambda\beta + \mu\delta)] \int_0^t \frac{ds}{z(s)} + \int_0^t f(s) ds \]

with

\[ f(s) = \mu(\lambda\alpha + \mu\gamma) \frac{y(s)}{x(s)} + \lambda(\lambda\beta + \mu\delta) \frac{x(s)}{y(s)} > 0 \]

for \(s > 0\), it follows from Lemma 2 that \(z(t) \geq w(t)\) where

\[ w(t) = \nu. u_0 + [\lambda(\lambda\alpha + \mu\gamma) + \mu(\lambda\beta + \mu\delta)] \int_0^t \frac{ds}{w(s)}. \]

and then

\[ z^2(t) \geq w^2(t) = 2[\lambda(\lambda\alpha + \mu\gamma) + \mu(\lambda\beta + \mu\delta)] t + (\nu. u_0)^2. \]
The conclusion follows.

We now define $q : L_1 \rightarrow L_1$ by

$$q(u_1) = \frac{1}{2} u(\tau(2), u_1)$$

where

$$\tau(2) = \inf \{ t \geq 0 : \nu.u(t, u_1) = 2 \}$$

is finite from the above Lemma. Let now $r > 0$ and $u \in \overline{L}_r$. From (SC), the geometric paths in $S$ of $u(., u)$ and $ru(., \frac{u}{r})$ are identical. Therefore

$$u(\tau(2r), u) = ru(\tau(2), \frac{u}{r})$$

where in this equality $\tau(2r)$ is relative to $u(., u)$ and $\tau(2)$ is relative to $u(., \frac{u}{r})$. Thus

$$q\left(\frac{u}{r}\right) = \frac{1}{2r} u(\tau(2r), u).$$

Iterating and using (SG), we get for any $n \geq 1$

$$q^n\left(\frac{u}{r}\right) = \frac{1}{2^n r} u(\tau(2^n r), u).$$

**Proposition 11** There exists $k \in (0, 1)$ such that for any $u_1 \in \overline{L}_1$

$$|q(u_1) - u_*| \leq k |u_1 - u_*|.$$

Proof. From Proposition 6 we know that $q$ has a unique invariant point $u_*$. We consider the solution $u(t, u_1) = (x(t), y(t))$ on the time interval $[0, \tau(2)]$, where $\tau(2)$ was defined in (13). We first assume that

$$\frac{y_1}{x_1} < \frac{y_*}{x_*} = \frac{d}{c}.$$

We note for further use that

$$x_* < x_1 \leq \frac{1}{\lambda}$$

$$0 \leq y_1 < y_* < \frac{1}{\mu}.$$

We set

$$u_2 = (x_2, y_2) := u_1 + \frac{(\alpha y_1 + \beta x_1, \gamma y_1 + \delta x_1)}{\lambda(\alpha y_1 + \beta x_1) + \mu(\gamma y_1 + \delta x_1)}.$$

Then, $2u_*, u_* + u_1$ and $u_2 \in L_2$. Setting for $z \in [0, \infty]$

$$h(z) = \frac{\gamma z + \delta}{\alpha z + \beta}$$

we compute

$$\frac{dh}{dz}(z) = \frac{\beta \gamma - \alpha \delta}{(\alpha z + \beta)^2}.$$ (15)

From Proposition 6 we know that for any $t \in [0, \tau(2)]$

$$\frac{y(t)}{x(t)} < \frac{d}{c}.$$ (16)

7
When \( \alpha \delta > \beta \gamma \), it follows from (15) and (16) that
\[
\frac{\dot{y}(t)}{\dot{x}(t)} = h\left(\frac{y(t)}{x(t)}\right) > h\left(\frac{\mu}{c}\right) = \frac{d}{c}
\]
and then \( 2q(u_1) \) belongs to the open interval \((2u_*, u_1 + u_1)\) on \(L_2\). Therefore,

\[
|2q(u_1) - 2u_*| < |u_1 - u_*|.
\]

When \( \alpha \delta = \beta \gamma \), the path of the solution is a straight half-line with slope \( \frac{d}{c} \) and

\[
|2q(u_1) - 2u_*| = |u_1 - u_*|.
\]

When \( \alpha \delta < \beta \gamma \), \( \frac{\dot{y}(t)}{\dot{x}(t)} \) is increasing on \([0, \tau(2)]\) and then

\[
\frac{\gamma y_1 + \delta x_1}{\alpha y_1 + \beta x_1} \leq \frac{\dot{y}(t)}{\dot{x}(t)} < \frac{d}{c}.
\]

As a result, \( 2q(u_1) \) belongs to the open interval \((u_* + u_1, u_1)\) on \(L_2\). Moreover, using the relation \( \lambda x_1 + \mu y_1 = 1 \) twice, we get

\[
2x_1 - x(\tau(2)) > 2x_1 - x_2 = x_1 - \frac{\alpha y_1 + \beta x_1}{\lambda(y_1 + \beta x_1) + \mu(y_1 + \beta x_1)} = \frac{\alpha \lambda y_1 + \beta \gamma x_1 + \gamma \mu y_1 y_1 + \delta \mu y_1^2 - \alpha y_1 - \beta x_1}{\lambda(\alpha y_1 + \beta x_1) + \mu(\gamma y_1 + \delta x_1)} = \frac{\gamma y_1 + \delta x_1}{\alpha y_1 + \beta x_1} (x_1 \frac{d}{c} - y_1) \left( y_1 + \frac{\beta - \gamma + \sqrt{(\beta - \gamma)^2 + 4\alpha \delta}}{2\alpha} x_1 \right).
\]

In the same way,

\[
y(\tau(2)) - 2y_1 > y_2 - 2y_1 = \frac{\alpha(\lambda c + \mu d)}{\lambda(\alpha y_1 + \beta x_1) + \mu(\gamma y_1 + \delta x_1)} (y_1 - y_1) \left( 1 + \frac{\beta - \gamma + \sqrt{(\beta - \gamma)^2 + 4\alpha \delta}}{2\alpha} x_1 \right).
\]

Setting

\[
k_1 = \frac{\lambda \mu(\beta - \gamma + \sqrt{(\beta - \gamma)^2 + 4\alpha \delta})}{4\lambda(\lambda \alpha \mu + \mu(\lambda \beta + \mu \gamma))} > 0
\]

we obtain

\[
|2q(u_1) - 2u_*| > k_1 |u_1 - u_*|
\]

and then

\[
|q(u_1) - u_*| = |u_1 - u_*| - |q(u_1) - u_1| < (1 - k_1) |u_1 - u_*|.
\]

If now \( \frac{y_1}{x_1} > \frac{d}{c} \), in the same way there exists \( k_2 > 0 \) such that

\[
|q(u_1) - u_*| < (1 - k_2) |u_1 - u_*|
\]

We may take \( k = 1 - \min(k_1, k_2) \).
8 Uniqueness. Case $x = 0, y = 0$.

Existence was proven in Section 3. We may now conclude the proof of Theorem 1.

**Proposition 12** The solution given by (6) is the unique solution to (1) starting at 0.

Proof. Let $u(.)$ be a solution to (1) starting at 0. For any $n \geq 1$ and $s > 0$,

$$\mathbf{u}(\tau(s)) = \mathbf{u}(\tau(s), \mathbf{u}(s2^{-n}))$$

where $\tau(s)$ in the l.h.s. is relative to $\mathbf{u}(.)$ and $\tau(s)$ in the r.h.s. is relative to $\mathbf{u}(., \mathbf{u}(s2^{-n}))$. We may apply (14) with $r = s2^{-n}$ and $\mathbf{u} = \mathbf{u}(s2^{-n})$. We obtain

$$\frac{\mathbf{u}(\tau(s))}{s} = q^n(\frac{\mathbf{u}(s2^{-n})}{s2^{-n}}).$$

From Proposition 11 (or directly from (10) it follows that the r.h.s. converges to $\mathbf{u}_*$ as $n \to \infty$. Thus for any $s > 0$

$$\frac{\mathbf{u}(\tau(s))}{s} = \mathbf{u}_*$$

and this implies

$$\frac{y(\tau(s))}{x(\tau(s))} = \frac{d}{c}.$$ 

From Lemma 10 we know that $\tau$ is one-to-one from $[0, \infty)$ to $[0, \infty)$, and thus for any $t > 0$

$$\frac{y(t)}{x(t)} = \frac{d}{c}.$$ 

Going back to the system (1) we conclude that

$$x(t) = c\sqrt{t} \\ y(t) = d\sqrt{t}.$$

References