# Nash-type Inequalities and decay of semigroups.

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Workshop in Nanterre . June 4-5-6, 2007.

# SETTING

- $(X, \mu)$ **measure space with** $\mu$ **a** $\sigma$**-finite measure.**

- $(T_t)_{t>0}$**: a symmetric submarkovian semi-group:**

  - $(T_t f, h) = (f, T_t h), \quad \forall f, g \in L^2$

  - $\forall f \in L^2, \quad 0 \leq f \leq 1 \Rightarrow 0 \leq T_t f \leq 1$

- $(T_t)_{t>0}$ **acts as a contraction on** $L^p$**.**

- $\mathcal{L}$: **infinitesimal generator associated to the semigroup** $T_t = e^{-t\mathcal{L}}$

$$\frac{\partial T_t}{\partial t} = -AT_t f$$

- $\mathcal{E}$: **Dirichlet form associated to** $(T_t)$**:**

$$\mathcal{E}(f) = (\mathcal{L}f, f), \qquad f \in \mathcal{D}(\mathcal{L})$$

**Important property**

$$\forall f \in \mathcal{D}(\mathcal{E}) \Rightarrow h = (f \vee 0) \wedge 1 \in \mathcal{D}(\mathcal{E})$$

$$\text{and} \qquad \mathcal{E}(h) \leq \mathcal{E}(f)$$

# MOTIVATION

**Let $\lambda > 0$.**

- **Spectral decay :**

$$(SG) \quad ||T_t f||_2^2 \leq e^{-2\lambda t}||f||_2^2$$

- **Poincaré inequality :**

$$(P) \quad \lambda||f||_2^2 \leq (\mathcal{L}f, f).$$

- **(SG) can be written as**

$$\mathcal{G}\left(||T_t f||_2^2\right) \leq e^{-2t}\mathcal{G}\left(||f||_2^2\right), \qquad ||f||_q \leq 1.$$

**with** $\mathcal{G}(x) = x^{1/\lambda}$**.**

**So the spectral dimension can be seen as a function namely** $\mathcal{G}$ **.**

## Main goal

**Study the Exponential Functional Decay**
**( $\mathcal{G}$-decay for short)**

**Let** $1 \leq q < p < \infty$ **and** $\mathcal{G}$ **increasing**

$$(EFD)_{p,q} \quad \mathcal{G}\left(||T_t f||_p^p\right) \leq e^{-pt}\mathcal{G}\left(||f||_p^p\right), \quad ||f||_q \leq 1$$

**Under some conditions, we deduce:**

- **a Generalized Spectral Decay:**

$$(GSG) \qquad ||T_t f||_p^p \leq \phi(t)||f||_p^p, \qquad \forall t > 0$$

- **Ultracontractivity:**

$$(ULT) \qquad ||T_t f||_2^2 \leq \psi(t)||f||_1^2, \qquad \forall t > 0$$

## Nash-type inequality

**Consider SuperPoincaré inequality**

$$(SP) \quad ||f||_2^2 \le s(\mathcal{L}f, f) + \beta(s)||f||_1, \qquad \forall s > 0$$

**What is a Nash-type inequality ?**

$$(NTI) \quad \Theta(||f||_2^2) \le (\mathcal{L}f, f), \qquad ||f||_1 \le 1$$

**( $\Theta$ increasing)**

**What relation between Nash-type inequality (NTI) and SuperPoincaré ? (in $L^2$)**

**Answer** **(SP) implies (NTI) and** $\Theta(x) = \sup_{t>0}(tx - t\beta(1/t))$ **(and conversely...).**

**Here we study a more general situation**

$(NTI)_{p,q}$ $\quad \Theta(||f||_p^p) \le (\mathcal{L}f, f^{p-1}), \quad ||f||_q \le 1.$

**We assume $\Theta$ of the form $\Theta(x) = x\,\mathcal{N}(\ln x)$ with $\mathcal{N}$ increasing continuous (which often appears...). So**

$$(NTI)_{p,q} \quad ||f||_p^p\,\mathcal{N}\left(\log ||f||_p^p\right) \leq (\mathcal{L}f, f^{p-1})$$

**with $||f||_q \leq 1$.**

**On the other side, we have**

$$(EFD)_{p,q} \quad \mathcal{G}\left(||T_t f||_p^p\right) \leq e^{-pt}\mathcal{G}\left(||f||_p^p\right), \qquad ||f||_q \leq 1$$

- **This is equivalent to**

$$t \rightarrow e^{pt}\mathcal{G}\left(||T_t f||_p^p\right)$$

**non-increasing.**

- **Why $\Theta$ of this special form ? (**$p = 2, q = 1$)

Assume $\quad ||T_t f||_2 \leq e^{M(t)}||f||_1, \quad \forall t > 0$

**from Davies-Simon Thm**

$$\int f^2 \ln\left(\frac{f^2}{||f||_2^2}\right) d\mu \leq 2t\mathcal{E}(f) + 2M(t)||f||_2^2$$

**Let** $0 \leq f, ||f||_1 = 1$ **i.e** $f\,d\mu$ **is a probability. Then**

$$||f||_2^2 \ln ||f||_2^2 \, d\mu \leq 2t\mathcal{E}(f) + 2M(t)||f||_2^2$$

**So**

$$||f||_2^2 \, \mathcal{N}\left(\log ||f||_2^2\right) \leq \mathcal{E}(f)$$

**with**

$$\mathcal{N}(\ln x) = \sup_{t>0} \left(\frac{1}{2t} \ln x - \frac{1}{t} M(t)\right)$$

## Main result

**Theorem**

**Let** $\mathcal{L}$ **be the generator of a symmetric submarkovian semigroup** $(T_t)$**. Let** $1 \leq q < p < \infty$ **and** $\mathcal{D}$ **be the domain of** $\mathcal{L}$**. The two following statements (1) and (2) are equivalent:**

1. **There exists $\mathcal{N} : \mathbb{R} \longrightarrow (0, +\infty)$ a continuous non-decreasing function such that for all $f \in \mathcal{D}$ with $||f||_q \leq 1$,**

$$||f||_p^p \mathcal{N}\left(\ln ||f||_p^p\right) \leq (\mathcal{L}f, f^{p-1}). \qquad (1)$$

2. **There exits $\mathcal{G} \in C^1((0,\infty),(0,\infty))$ an increasing function such that for all $t > 0$ and for all $f \in \mathcal{D}$ with $||f||_q \leq 1$,**

$$\mathcal{G}(\,||T_t f||_p^p\,) \leq e^{-pt}\,\mathcal{G}(\,||f||_p^p\,). \qquad (2)$$

**Moreover**

- **(1) implies (2) with**

$$\mathcal{G} = \exp o \mathcal{F} o \ln$$

**with the derivative of** $\mathcal{F}$ **satisfiying** $\mathcal{F}' = 1/\mathcal{N}$

- **(2) implies (1) with**

$$\mathcal{N}(y) = \frac{\mathcal{G}(e^y)}{e^y\, \mathcal{G}'(e^y)},\ y \in \mathbb{R}$$

## Corollary 1 (spectral decay)

- **Assume that the semigroup $(T_t)$ satisfies a $\mathcal{G}$-decay for some $1 \le q < p < \infty$.**

- **If the measure $\mu$ is finite, then**

$$||T_t f||_p^p \le \psi(t)\, ||f||_p^p$$

**with** $\psi(t) = \mu(X)^\gamma\, \mathcal{G}^{-1}\left(e^{-pt}\, \mathcal{G}(\mu(X)^{-\gamma})\right)$ **and** $\gamma = \frac{p}{q} - 1$.

**Corollary 2 (Ultracontractivity)**

- **Assume that the semigroup** $(T_t)$ **satisfies a** $\mathcal{G}$**-decay for some** $1 \leq q < p < \infty$**.**

- **If** $\mathcal{G}$ **is bounded by** $k$**, then**

$$||T_t f||_p \leq \phi(t)\, ||f||_q$$

**with**

$$\phi(t) = \left[\mathcal{G}^{-1}\left(k\, e^{-pt}\right)\right]^{1/p}$$

# How to get Nash-type inequalities ?

- **Directly form the example under study.**

- **From other functional inequalities:**
- **Ultracontractivity**

$$||T_t f||_\infty \le U(t)||f||_1, \quad \forall t > 0$$

**(Through Davies-Simon log-Sobolev inequality for instance or Coulhon's Thm)**

- **From Orlicz-Sobolev inequality...**

- **From a Nash-type inequality itself : "from $L^2$ to $L^p$" ($p > 2$) ($f \to f^{p/2}$)**

- **From one operator to another:**
- **Comparison of Dirichlet forms.**

- **From $\mathcal{L}$ to its fractional powers (A.Bendikov & P.M):**

$$||f||_2^2 \, B(||f||_2^2) \le (\mathcal{L}f, f), \quad ||f||_1 \le 1$$

**Then $0 < \alpha < 1$, There exists $c_1, c_2$ s.t.**

$$c_1 ||f||_2^2 \, B^{\alpha}(c_2 \, ||f||_2^2) \le (\mathcal{L}^{\alpha} f, f), \quad ||f||_1 \le 1$$

## Example 1

- **Usual Laplacian on $\mathbb{R}^n$: Nash inequality**

$$c_0||f|_2^{2+4/n} \leq (\mathcal{L}f, f), \qquad ||f|_1 \leq 1$$

**also for fractional operator** $0 < \alpha \leq 1$**,**

$$c||f|_2^{2+4\alpha/n} \leq (\mathcal{L}^\alpha f, f), \qquad ||f|_1 \leq 1$$

i.e.$(\alpha = 1)$ $\qquad \mathcal{N}(y) = c_0 \exp\left(\frac{2}{n} y\right), \qquad y \in \mathbb{R}$

**So**

$$\mathcal{G}(x) = \exp\left(-\frac{n}{2c_0}x^{-\frac{2}{n}}\right), \qquad x > 0$$

$\mathcal{G}$**-decay can be written as : for any** $f \in L^1 \cap L^2$

$$||T_t f||_2 \le H_t(f)\,||f||_2$$

$$H_t(f) = \left(1 + c\,t\,\left(\frac{||f||_2}{||f||_1}\right)^{4/n}\right)^{-n/4}$$

**implies**

$$||T_t f||_2 \leq ||f||_2 \quad , \quad ||T_t f||_2 \leq \frac{c}{t^{n/4}} ||f||_1$$

$\mathcal{G}$**-decay can be seen as " interpolated**

**inequalities " between $L^2$-contraction and**

**ultracontractivity of the semigroup.**

## Example 2

- **Let** $\mathcal{N}(y) = k\, y_+^{1+1/\gamma}$**. We have**

$$\mathcal{G}(x) = \exp\left(-\frac{\gamma}{k}\,[\ln x]^{-1/\gamma}\right),\ x > 1$$

**This is the case when**

$$||T_t f||_2 \le\ c\, e^{c_1/t^\gamma} ||f||_1, \quad \forall t > 0$$

# Example 3

- **Let** $\mathcal{N}(y) = y_+(\ln y_+)_+^{1/\gamma},\ \gamma > 0$**.**

**If** $\gamma \neq 1$**,**

$$\mathcal{G}(x) = \exp\left(\frac{\gamma}{\gamma - 1}(\ln \ln x)^{1-1/\gamma}\right) \qquad x > e$$

**If** $\gamma = 1$**,**

$$\mathcal{G}(x) = \ln(\ln x)) \qquad x > e$$

$\mathcal{G}$ **is bounded if and only if** $\gamma < 1$**.**

## Example 4 (Weak Gross inequality)

- **Let** $\mathcal{N}(y) = c\,y,\ c > 0$**.**

$$c||f||_2^2 \ln ||f||_2^2 \leq (\mathcal{L}f, f), \qquad ||f||_1 \leq 1.$$

$$\mathcal{G}(x) = (\ln x)^{1/c}\ , \qquad\qquad x > 1$$

$$\mathcal{G} - decay \qquad ||T_t f||_2^2 \leq ||f||_2^{2\alpha(t)}.$$

**with** $\alpha(t) = e^{-2ct}$

- **Weak SuperPoincaré (weak Nash-T.I)**

$$||f||_2^2 \leq 2t\,\mathcal{E}(f) + ||f||_2^{2\alpha(t)}, \qquad \forall t > 0$$

**with** $\alpha(t) = e^{-2t}$ **(for instance)**

**is equivalent to**

$$||f||_2^2 \ln ||f||_2^2 \leq (\mathcal{L}f, f), \qquad ||f||_1 \leq 1.$$

## Nash revisiting Gross, Nelson, ...:

**(proof may be already known )**

$$(Hyper) \quad ||T_t f||_2 \le ||f||_{p(t)}, \quad p(t) = 1 + e^{-2t},\ \forall t > 0$$

$$\Rightarrow (WNTI) \quad ||f||_2^2 \le 2t\, \mathcal{E}(f) + ||f||_{p(t)}^2$$

**Weak form of interpolated inequality (between Gauss and Poincaré):** $1 < p < 2$

$$\Rightarrow ||f||_2^2 - ||f||_p^2 \leq \ln\left(\frac{1}{p-1}\right)\mathcal{E}(f)$$

$\ln\left(\frac{1}{p-1}\right) = \ln\left(1 + \frac{2-p}{p-1}\right) \sim (2-p)$ **as** $p \to 2^-$**.**

$$\Rightarrow (Gross) \qquad \int f^2 \ln\left(\frac{f^2}{||f^2||_2^2}\right) \leq \mathcal{E}(f)$$

## Example 5 (symmetric Γ-semigroup)

$$\text{Let} \quad \mathcal{E}(f) = \int_{\mathbb{R}} \ln(1 + 4\pi^2|x|^2)|\hat{f}(x)|^2\, dx$$

$dx$: **Lebesgue measure (not finite!)**

$$\text{Let} \quad \mathcal{E}(f) = (\ln(I + \Delta)f, f)$$

**for any** $0 < \varepsilon < 1$ **and any** $||f||_1 \leq 1$**:**

$$(1-\varepsilon)\,||f||_2^2 \ln\left(1 + \pi^2\varepsilon^2||f||_2^4\right) \leq (\ln(I+\Delta)f, f)$$

$$\mathcal{N}(\ln x) = (1-\varepsilon)\ \ln\left(1+\pi^2\varepsilon^2 x^2\right)$$

$$||f||_2^2 \ln\left(||f||_2^2\right) \leq (\ln(I+\Delta)f, f) \qquad (\varepsilon = 1/2)$$

**but** $d\mu(x) = dx$ **is not finite!**

**We have a so-called defective Gross inequality.** ♣