# Notes on Heat Kernels on Infinite dimensional Torus.

Patrick Maheux
Département de Mathématiques.U.M.R 6628.MAPMO
Université d'Orléans B.P 6759
45 067 Orléans Cedex 2, France

February 16, 2008

(**VERSION 1**)

## Contents



# 1 Introduction

This set of notes deals with the different heat kernels we can built on the infinite dimensional torus $\mathbb{T}^\infty$. We are mainly interested by on and off diagonal estimates of different heat kernels on $\mathbb{T}^\infty$. On diagonal estimates (or asymtotics) means to study of the heat kernel $\mu_t(0)$ at the origin 0 (when it exists) for small and/or large $t > 0$.

Off-diagonal estimates means to study $\mu_t(x)$ with respect to $x \in \mathbb{T}^\infty$ and try to find gaussian bounds in terms of some metric $d$ related to the generator of the semigroup associated to the heat kernel. We are looking for are upper an lower bounds of the form $\mu_t(x) \sim C_t \exp(-\frac{d^2(x,0)}{ct})$ for some constant $c >$.

Note that we work in the setting of locally compact abelian groups and the results represented here are parts of the study of harmonic spaces. We shall not enter into the consideration of harmonic sheaves. For that purpose, we shall refer to [B1].

Our purpose is to present detailed proof of the results as self-contained as possible. These notes are essentially taken from the works of C.Berg and A.Bendikov (see [BG1, B2]). The existence of a (continuous) density on $\mathbb{T}^\infty$ for gaussian convolution semigroups was established independently by three authors: A. Bendikov, Ch. Berg and A. Siebert in 1974-75. (see [B3]).

We briefly recall some well-known facts we shall need about harmonic analysis on locally compact abelian groups and semigroup theory.

Let $G$ be a locally compact abelian group. We denote by 0 the neutral element. There exists a bi-invariant Borel positive measure (Haar measure) by the left and right actions of $G$. This measure is unique up to a constant (see [R]).

Let $(\mu_t)_{t>0}$ be a convolution semigroup of measures on $G$ i.e. a family of measures $\mu_t$ with $t > 0$ such that:

(1) $\mu_t(G) \leq 1, \quad \forall t > 0,$

(2) $\mu_t \star \mu_s = \mu_{t+s}, \quad \forall t, s > 0,$

(3) $\lim_{t\to 0^+} \mu_t = \delta_0$ vaguely, i.e. $\lim_{t\to 0^+} \mu_t(f) = f(0), \quad \forall f \in C_c(G).$

The convolution of two finite (positive) measures $\mu_1$ and $\mu_2$ on $G$ is defined for a Borel set $A$ of $G$ by:

$$\mu_1 \star \mu_2(A) = \int_G \mu_1(A - y)\, d\mu_2(y).$$

If $\mu_1$ and $\mu_2$ are absolutely continuous w.r.t. a fixed Haar measure $\nu$ of $G$ such that $d\mu_1(x) = f_1(x)\, d\nu(x)$ and $d\mu_2(x) = f_2(x)\, d\nu(x)$ then $\mu_1 \star \mu_2$ is absolutely continuous w.r.t. $\nu$ and $d(\mu_1 \star \mu_2)(x) = (f_1 \star f_2)(x)\, d\nu(x)$ where $f_1 \star f_2(x) = \int_G f_1(x-y) f_2(y)\, d\nu(y)$.

On $G$ is defined a Fourier transform through the dual group denoted by $\Gamma$. the dual groupe is also a locally compact abelian group so it posseses also a Haar measure.

Let $\mu$ a finite measure, for any $n \in \Gamma$,

$$\hat{\mu}(n) = \int_G (n,x)\, d\mu(x)$$

with $(n,x)$ the duality bracket between $\Gamma$ and $G$. It can be proved that

$$\hat{\mu_t}(n) = e^{-t\Psi(n)}$$

with $\Psi : \Gamma \to \mathbb{C}$ a continuous definite negative function (see p.52 of [BG1]) and [BF]). Conversely all continuous definite negative functions give rise to a convolution semigroup of measures.

To this family of measures, we associate a semigroup of operators $(P_t)_{t>0}$ on $C_0(G)$ defined by: for any $f \in C_0(G)$

$$P_t f(x) = \mu_t \star f(x) = \int_G f(x-y) d\mu_t(y).$$

We define the domain $\mathcal{D}$ of the generator in $C_0(G)$ of the semigroup $(P_t)$ by,

$$\mathcal{D} = \{ f \in C_0(G) \text{ s.t. } \lim_{t \to 0^+} \frac{P_t f - f}{t} \text{ exits in } C_0(G) \}$$

(The convergence is for the sup norm in $C_0(G)$).

The generator $\mathcal{L}$ is defined by

$$\mathcal{L} f = \lim_{t \to 0^+} \frac{P_t f - f}{t}$$

for any $f \in \mathcal{D}$.

We say that the semigroup or the generator is *local* if :for every $f \in C_0(G)$,

$$\text{support}(\mathcal{L} f) \subseteq \text{support}(f).$$

The following theorem is important to characterize the symmetric measures which have a continuous density w.r.t. the Haar measure. It will be apply in the sequel. A measure $\mu$ on $G$ is siad to be symmetric if $\mu(-A) = \mu(A)$ for any Borel set $A$ of $G$.

**Theorem 1.1** *Let $(\mu_t)$ be a symmetric convolution semigroup of measures on $G$ with associate negative definite function $\Psi$ on $\Gamma$. For each $t >$ the following conditions are equivalent :*

*(i)* $\mu_t$ *has a* continuous *density $g_t$ w.r.t. the Haar measure on $G$.*

*(ii)* $e^{-t\Psi} \in L^1(\Gamma)$.

*(iii)* *If (ii) holds true for any* $t > 0$ *then* $g_t \in \mathcal{D}$ *and*

$$\mathcal{L}g_t(x) = \frac{d}{dt}g_t(x), \quad t > 0, x \in G.$$

*Furthermore, the function* $g : ]0, +\infty[\times G \to \mathbb{R}$

For a proof see bg1 [BG1] p.53-54. The Cauchy problem

$$\mathcal{L}u(t,x) = \frac{d}{dt}u(t,x), \quad t > 0, x \in G.$$

$$u(0,x) = f(x), x \in G$$

has (formally) the solution $u(t,x) = P_t f(x), x \in G, t > 0$ when $f \in L^p(G)$ for $1 \leq p \leq +\infty$.

# 2 The one dimensional torus

In this section, we define the heat kernel on the one dimensional torus $\mathbb{T}$ and give some estimates on the heat kernel useful in the sequence.

## 2.1 Definition of the heat kernel on $\mathbb{T}$.

We first recall the explicit form of the heat kernel on the real line $\mathbb{R}$. We denote by $(p_t)$ the Gaussian kernel defining the heat semigroup $(\nu_t)$ on $\mathbb{R}$. Let $t > 0$,

$$p_t(y) = \frac{1}{(4\pi t)^{\frac{1}{2}}} \exp\left(-\frac{|y|^2}{4t}\right), \quad y \in \mathbb{R}. \tag{2.1}$$ heatR

where $|y|$ is the absolute value of $y$ (that is the euclidean distance from $y$ to 0). So

$$H_t f(x) = \nu_t \star f(x) = \int_{\mathbb{R}} p_t(x-y)\, f(y)\, dy$$

where $dy$ denotes the Lebesgue measure on $\mathbb{R}$ (The Haar measure on the locally compact abelian group $\mathbb{R}$). The Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined by

$$\hat{f}(y) = \int_{\mathbb{R}} f(x) \exp(-ix.y)\, dx.$$

So

$$\hat{p_t}(y) = \exp\left(-t|y|^2\right).$$

The function $H_t f$ gives the solution of the equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial^2 x}(t,x), \quad u(0,x) = f(x), \quad t > 0, x \in \mathbb{R}.$$

with $f \in L^p(\mathbb{R})$, $1 \le p \le +\infty$.

The convolution semigroup $(\nu_t)$ induced a semigroup of operators on $L^p(\mathbb{R})$ defined by $H_t f(x) = \nu_t \star f(x)$.

We are in position to define the heat kernel on the one dimensional torus.

We consider the one dimensional torus $\{z \in \mathbb{C}, |z| = 1\}$ denoted by $\mathbb{T}$ and represented by $[-\pi; \pi]$ with the application $\theta \in [-\pi; \pi] \to z = e^{i\theta}$ (In this process, we identify $\pi$ and $-\pi$). The torus can also be identified with the quotient $\mathbb{R}/2\pi\mathbb{Z}$. This representation explains why the heat kernel on $\mathbb{T}$ is given by

$$g_t(x) = 2\pi \sum_{k \in \mathbb{Z}} p_t(x + 2\pi k), \; x \in [-\pi; \pi].$$

which is a normalized $2\pi$-periodisation of $p_t$. Indeed, assume that $f \ge 0$ is $2\pi$-periodic an bounded,

$$\begin{aligned} \nu_t \star f(x) = \int_{\mathbb{R}} p_t(x-y)\, f(y)\, dy &= \sum_{k \in \mathbb{Z}} \int_{-\pi}^{\pi} p_t(x - z + 2k\pi) f(z - 2k\pi)\, dz \\ &= \int_{-\pi}^{\pi} \left[ 2\pi \sum_{k \in \mathbb{Z}} p_t(x - z + 2k\pi) \right] f(z)\, \frac{dz}{2\pi}. \end{aligned}$$

Now let $f$ be a bounded function on $\mathbb{T}$. This function can be extended on $\mathbb{R}$ as a $2\pi$-periodic bounded function on $\mathbb{R}$. So the preceding formula justify the definition of the heat semigroup on $\mathbb{T}$ denoted by $(\mu_t)$ and given by

$$\mu_t \star f(x) = \int_{\mathbb{T}} f(x-y)\, d\mu_t(y) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y) g_t(y)\, dy. \tag{2.2}$$

heatorus

where $\frac{1}{2\pi}\, dy$ is the normalised Haar (Lebesgue) measure on $\mathbb{T} \sim [-\pi; \pi]$. The function $f$ is assume to be in $L^\infty(\mathbb{T})$ with respect to the measure $\frac{1}{2\pi}\, dy$ but the semigroup can be extended to $L^p(\mathbb{T})$ for any $1 \le p \le +\infty$ by the formula (2.2).

Of course, $v(t, x) = \mu_t \star f(x), t > 0, x \in \mathbb{T}$ gives the solution of the Cauchy problem

$$\frac{\partial v}{\partial t}(t, x) = \frac{\partial^2 v}{\partial^2 x}(t, x), \quad v(0, x) = f(x), \quad t > 0, x \in \mathbb{T}.$$

with $f \in L^p(\mathbb{T})$, $1 \le p \le +\infty$.

The convolution semigroup $(\mu_t)$ induced a semigroup of operators on $L^p(\mathbb{T})$ defined by $P_t f(x) = \mu_t \star f(x)$.

Recall that the dual group of $\mathbb{T}$ is $\mathbb{Z}$ and the Fourier transfom is given by the sequence of Fourier coefficients $\hat{g}_t(n) = e^{-n^2 t}, n \in \mathbb{Z}$.

## 2.2 Gaussian estimates of the heat kernel on $\mathbb{T}$

We are interested by Gaussian estimates of the heat kernel $g_t$ similar to the exact formula (2.1). Note that (2.1) can be reformulated by

$$p_t(y) = p_t(0)\ \exp\left(-\frac{|y|^2}{4t}\right).$$

Note that $0 \in \mathbb{R}$ is the neutral element on the group $\mathbb{R}$ and $|y|$ is the euclidean distance on $\mathbb{R}$ from $y$ to 0. We identify $\mathbb{T}$ with $[-\pi, \pi]$ and we shall denote also by $0 \in \mathbb{T}$ the neutral element of the group $\mathbb{T}$. We introduce a metric $d$ on $\mathbb{T}$ as follows. Let $x \in [-\pi, \pi]$ and set $d(x, 0) = |x|$ where $|x|$ is the absolue value of $x$. Note that if we identify $\mathbb{T}$ with $[0, 2\pi]$ then $d(x, 0) = \inf\{x, 2\pi - x)$. We set $||x|| = d(x, 0)$. In fact, the distance $d$ is an intrinsic distance induced on the quotient $\mathbb{T} = \mathbb{T}/2\pi\mathbb{Z}$ by the euclidean distance on $\mathbb{R}$ by the formula: $d(x, y) = \inf\{|\theta + 2\pi k|, k \in \mathbb{Z}\}$ for any $\theta \in \mathbb{R}$ and $x, y \in \mathbb{T} \sim [-\pi, \pi]$ such that $e^{i(x-y)} = e^{i\theta}$.

estimoff **Theorem 2.1** *For any $t > 0$ and for any $x \in [-\pi; \pi]$,*

$$g_t(0)\exp\left(-\frac{d(x,0)^2}{4t}\right) \leq\ g_t(x)\ \leq 2\, g_t(0)\exp\left(-\frac{d(x,0)^2}{4t}\right). \tag{2.3}$$

bounds

**Proof**: By definition,

$$\begin{aligned} g_t(x) &= 2\pi \sum_{k\in\mathbb{Z}} p_t(x + 2\pi k) = \frac{2\pi}{\sqrt{4\pi t}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{(x+2\pi k)^2}{4t}\right) \\ &= \sqrt{\frac{\pi}{t}} \exp\left(\frac{-x^2}{4t}\right) \sum_{k\in\mathbb{Z}} \exp\left(\frac{-\pi k x}{t}\right) \exp\left(\frac{-\pi^2 k^2}{t}\right) \\ &= \sqrt{\frac{\pi}{t}} \exp\left(\frac{-x^2}{4t}\right) \left[1 + \sum_{k=1}^{\infty} \exp\left(\frac{-\pi^2 k^2}{t}\right) \left(\exp\left(\frac{-\pi k x}{t}\right) + \exp\left(\frac{+\pi k x}{t}\right)\right)\right] \end{aligned}$$

So we deduce the following exact formula:

$$g_t(x) = \sqrt{\frac{\pi}{t}} \exp\left(\frac{-x^2}{4t}\right) \left[1 + 2\sum_{k=1}^{\infty} \exp\left(\frac{-\pi^2 k^2}{t}\right) \cosh\left(\frac{\pi k x}{t}\right)\right] \tag{2.4}$$

exactform

(Which shows in particular that $g_t$ is an even function). The lower bound follows from $\cosh y \geq 1,\ y \in \mathbb{R}$. Indeed for any $t > 0$,

$$g_t(x) \geq \sqrt{\frac{\pi}{t}} \exp\left(\frac{-x^2}{4t}\right) \left[1 + 2\sum_{k=1}^{\infty} \exp\left(\frac{-\pi^2 k^2}{t}\right)\right] = g_t(0)\exp\left(\frac{-x^2}{4t}\right)$$

We now use the fact that $g_t$ is $2\pi$-periodic and that $x^2 = ||x||^2 = d(x, 0)^2$ for $x \in [-\pi, \pi]$.

For the upper bound, we need more work. We just need to give a uniform upper bound of

$$S(x) := 1 + 2\sum_{k=1}^{\infty} \exp\left(\frac{-\pi^2 k^2}{t}\right) \cosh\left(\frac{\pi k x}{t}\right)$$

since $g_t(x) = \sqrt{\dfrac{\pi}{t}} \exp\left(\dfrac{-x^2}{4t}\right) S(x)$.

By periodicity of $g_t$ and the fact that the function $g_t$ is even, we can assume $x \in [0, \pi]$, so for any $t > 0$, this allows us to bound as follows: for any $k \geq 1$,

$$2\cosh\left(\frac{\pi k x}{t}\right) \leq 2\cosh\left(\frac{\pi^2 k}{t}\right) \leq \exp\left(\frac{-\pi^2 k}{t}\right) + \exp\left(\frac{\pi^2 k}{t}\right) \leq 1 + \exp\left(\frac{\pi^2 k}{t}\right).$$

From which we deduce,

$$\begin{aligned} 2\exp\left(\frac{-\pi^2 k^2}{t}\right) \cosh\left(\frac{\pi k x}{t}\right) &\leq \exp\left(\frac{-\pi^2 k^2}{t}\right) + \exp\left(\frac{-\pi^2 k^2}{t} + \frac{\pi^2 k}{t}\right) \\ &= \exp\left(\frac{-\pi^2 k^2}{t}\right) + \exp\left(\frac{-\pi^2 k(k-1)}{t}\right). \end{aligned}$$

From the inequality just above,

$$S(x) \leq 1 + \sum_{k=1}^{\infty} \exp\left(\frac{-\pi^2 k^2}{t}\right) + \sum_{k=1}^{\infty} \exp\left(\frac{-\pi^2 k(k-1)}{t}\right)$$

Since $k(k-1) \geq (k-1)^2$,

$$\begin{aligned} S(x) &\leq \left(1 + \sum_{k=1}^{\infty} \exp\left(\frac{-\pi^2 k^2}{t}\right)\right) + \left(\sum_{k=1}^{\infty} \exp\left(\frac{-\pi^2 (k-1)^2}{t}\right)\right) = 2\left(1 + \sum_{k=1}^{\infty} \exp\left(\frac{-\pi^2 k^2}{t}\right)\right). \\ &= 1 + \left[1 + 2\sum_{k=1}^{\infty} \exp\left(\frac{-\pi^2 k^2}{t}\right)\right]. \end{aligned}$$

So we obtain,

$$\sqrt{\frac{\pi}{t}} S(x) \leq \sqrt{\frac{\pi}{t}} + g_t(0).$$

and

$$g_t(x) \leq \left[\sqrt{\frac{\pi}{t}} + g_t(0)\right] \exp\left(\frac{-x^2}{4t}\right). \tag{2.5}$$ better

Since $\sqrt{\frac{\pi}{t}} \leq g_t(0)$. we get the result. This completes the proof.

We note that we get a better upper bound with (2.5) but for our purpose (2.3) is enough.

We are now interested by the estimates of $g_t(0)$. Heuristically, the manifold $\mathbb{T}$ is locally as $\mathbb{R}$. So the estimates as $t$ goes to 0 should be like in $\mathbb{R}$ that is constant$\times\frac{1}{\sqrt{t}}$ as in $\mathbb{R}$ for $p_t(0)$. For large time $t$, $g_t(0)$ should be like the constant 1 since the manifold $\mathbb{T}$ is compact. So we are interested by the behavior of $g_t(0)-1$ when $t$ is large. This is related to ergodic property of the diffusion associated to the heat kernel. Indeed, heuristically, the heat diffusion should be spread in a uniform way on the compact set $[-\pi;\pi]$ with respect to the Haar measure on $\mathbb{T}$. Recall that the heat kernel $g_t$ has a mass one for any $t>0$.

estimon **Theorem 2.2** *For any $t>0$, we have*

$$\sqrt{\frac{\pi}{t}} \le g_t(0) \le 1+\sqrt{\frac{\pi}{t}} \tag{2.6}$$ gtloca

*and*

$$2e^{-t} \le g_t(0)-1 \le \frac{2e^{-t}}{1-e^{-t}}. \tag{2.7}$$ gtglob

*In particular,*

$$g_t(0) \sim \sqrt{\frac{\pi}{t}}, \quad t\to 0 \tag{2.8}$$ gtlocaest

*and*

$$g_t(0) \sim 2e^{-t}, \quad t\to+\infty. \tag{2.9}$$ gtglobest

Note that (2.6) and (2.7) are valid for any $t>0$ but are significant only when $t$ is small in (2.6) and for large $t$ is large in (2.7) as precised in (2.8) and (2.9). Note that our asymptotics are deduced from the upper-lower estimates.

**Proof**: By the exact formula

$$g_t(x) = \sqrt{\frac{\pi}{t}} \exp\left(\frac{-x^2}{4t}\right)\left[1+2\sum_{k=1}^{\infty}\exp\left(\frac{-\pi^2k^2}{t}\right)\cosh\left(\frac{\pi k x}{t}\right)\right],$$

we easily get

$$g_t(0 \ge \sqrt{\frac{\pi}{t}}.$$

For the upper bound of (2.6), we use the Fourier series to express $g_t(x)$. We have

$$g_t(x) = 1+2\sum_{k=1}^{\infty} e^{-tn^2}\cos(nx)$$

So

$$g_t(0) = 1+2\sum_{k=1}^{\infty} e^{-tn^2} = 1+2\varphi(t) \tag{2.10}$$ gtphi

with $\varphi(t)=\sum_{k=1}^{\infty} e^{-tn^2}$. We compare $\varphi$ with an integral

$$\varphi(t) \le \int_0^{\infty} e^{-tx^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{t}}$$

so

$$g_t(0) \le 1 + \sqrt{\frac{\pi}{t}}.$$

We have proved (2.6).

To prove the lower bound of (2.7), we use the simple fact $\varphi(t) \ge e^{-t}$, so

$$g_t(0) \ge 1 + 2e^{-t}$$

and for the upper bound,

$$\varphi(t) = \sum_{k=1}^{\infty} e^{-tn^2} \le \sum_{k=1}^{\infty} e^{-tn} = \frac{e^{-t}}{1-e^{-t}}.$$

So

$$g_t(0) \le 1 + 2\frac{e^{-t}}{1-e^{-t}}.$$

The estimations (2.8) and (2.9) are immediately deduced from (2.6) and (2.7). This completes the proof.

From Theorems 2.2 and 2.1, we deduce gaussian upper and lower bounds of the heat kernel $g_t$ on $\mathbb{T}$:

**Theorem 2.3** *For any $t > 0$, we have*

$$\sup\{\sqrt{\frac{\pi}{t}}, 1 + 2e^{-t}\} \exp\left(-\frac{||x||^2}{4t}\right) \le \; g_t(x)$$

$$\le 2\inf\{1 + \sqrt{\frac{\pi}{t}}, 1 + \frac{2e^{-t}}{1-e^{-t}}\} \exp\left(-\frac{||x||^2}{4t}\right). \tag{2.11}$$

*In particular, for any $0 < t < 1$,*

$$\sqrt{\frac{\pi}{t}} \exp\left(-\frac{||x||^2}{4t}\right) \le g_t(x) \le 2(1 + \sqrt{\frac{\pi}{t}}) \exp\left(-\frac{||x||^2}{4t}\right). \tag{2.12}$$

*For any $t > 1$,*

$$(1 + 2e^{-t}) \exp\left(-\frac{||x||^2}{4t}\right) \le g_t(x) \le 2(1 + \frac{2e^{-t}}{1-e^{-t}}) \exp\left(-\frac{||x||^2}{4t}\right). \tag{2.13}$$

*And*

$$\lim_{t\to+\infty} g_t(x) = 1$$

*(The convergence is unifom on $\mathbb{T}$).*

$$\lim_{t\to 0^+} g_t(x) = 0, \quad x \ne 0.$$

**Proof**: The first part is left to the reader. We only show the last statement. Let $x, y \in \mathbb{T}$ and $h_t(x,y) = g_t(x,y)$ so $h_t$ is symmetric i.e. $h_t(x,y) = h_t(y,x)$ and $h_t(x,x) = h_t(0,0) = g_t(0)$. By semigroup property and Cauchy-Schwarz inequality, we get

$$h_t(x,y) = \int_{\mathbb{T}} h_{t/2}(x,z)h_{t/2}(z,y)\,\frac{dz}{2\pi} \leq \left(\int_{\mathbb{T}} h^2_{t/2}(x,z)\,\frac{dz}{2\pi}\right)^{1/2} \left(\int_{\mathbb{T}} h^2_{t/2}(z,y)\,\frac{dz}{2\pi}\right)^{1/2}$$

$$= (h_t(x,x)h_t(y,y))^{1/2} = h_t(0,0) = g_t(0).$$

We take $y = 0$, so $g_t(x) = h_t(x,0) \leq g_t(0)$. In particular, $\sup_{x\in\mathbb{T}} g_t(x) = g_t(0)$. So we get, $g_t(x) - 1 \leq g_t(0) - 1 \leq \frac{2e^{-t}}{1-e^{-t}}$. By the gaussian lower bound, we have

$$(1 + 2e^{-t})\,\exp\left(-\frac{\pi^2}{4t}\right) - 1 \leq g_t(x) - 1.$$

So

$$sup_{x\in\mathbb{T}}|g_t(x) - 1| \leq \sup\{\frac{2e^{-t}}{1-e^{-t}}, |(1 + 2e^{-t})\,\exp\left(-\frac{\pi^2}{4t}\right) - 1|\}.$$

The uniform convergence of $g_t(x)$ to 1 as $t \to +\infty$ is then proved.

# 3 Heat kernel on finite dimensional torus

As we shall see, there exists essentially one heat kernel on the finite dimensional torus in the sense of the behavior at the origin of heats kernels is of the same type.

## 3.1 Definition of the heat kernel on $\mathbb{T}$.

Let $m \in \mathbb{N}$ and $\mathbb{T}^m = \mathbb{T} \times ... \times \mathbb{T}$ (m times) the $m$-dimensional torus. Let $\mathcal{A} = (a_1, ...a_m)$ some positive real numbers. For each $\mathcal{A}$, we define a heat kernel $\mu_t^{\mathcal{A}}$ as follows. For $x = (x_1, ..., x_m) \in \mathbb{T}^m$, we set $d\mu_t^{\mathcal{A}}$ the measure with density

$$\mu_t^{\mathcal{A}}(x) = \prod_{k=1}^{m} g_{a_k t}(x_k)$$

with respect to the Haar(-Lebesgue) measure $\displaystyle\prod_{k=1}^{m} \frac{dx_k}{2\pi}$.

We have $\hat{\mu}_t^{\mathcal{A}}(n) = e^{-tq(n)}$ where $q(n) = a_1\, n_1^2 + ... + a_m\, n_m^2$, $(n_1, ..., n_m) \in \mathbb{Z}^m$. Recall that the dual group of $\mathbb{T}^m$ is $\mathbb{Z}^m$. The function $q$ is the symbol of the generator of the convolution semigroup generated by $\mu_t$. So the generator is of local type (see [BF] p.?).

## 3.2 Gaussian estimates of the heat kernel

To formulate the gaussian estimates of heat kernels, we introduce the natural distance $d_{\mathcal{A}}$ associated to the diffusion generated by $\mu_t$: let $x = (x_1, ..., x_m) \in \mathbb{T}^m$, $y = (y_1, ..., y_m) \in \mathbb{T}^m$,

$$d^2_{\mathcal{A}}(x,y) = \frac{1}{a_1}||x_1 - y_1||^2 + ...\frac{1}{a_m}||x_m - y_m||^2$$

with $||x_k||$ the distance on $\mathbb{T}$ introduced in Theorem 2.1. From results obtain for $g_t$ on $\mathbb{T}$, we immediately deduce:

**Theorem 3.1** *For any $\mathcal{A}$ as above, for any $t > 0$ and any $x \in \mathbb{T}^m$:*

$$\left(\prod_{k=1}^m g_{a_k t}(0)\right) \exp\left(-\frac{d^2_{\mathcal{A}}(x,0)}{4t}\right) \leq \mu^{\mathcal{A}}_t(x) \leq 2^m \left(\prod_{k=1}^m g_{a_k t}(0)\right) \exp\left(-\frac{d^2_{\mathcal{A}}(x,0)}{4t}\right). \tag{3.14}$$

*For any $t > 0$ and any $x \in \mathbb{T}^m$,*

$$\left(\prod_{k=1}^m \sup\{\sqrt{\frac{\pi}{a_k t}}, 1 + 2e^{-a_k t}\}\right) \exp\left(-\frac{d^2_{\mathcal{A}}(x,0)}{4t}\right) \leq \mu^{\mathcal{A}}_t(x)$$

$$\leq 2^m \left(\prod_{k=1}^m \inf\{1 + \sqrt{\frac{\pi}{a_k t}}, 1 + \frac{2e^{-a_k t}}{1 - e^{-a_k t}}\}\right) \exp\left(-\frac{d^2_{\mathcal{A}}(x,0)}{4t}\right). \tag{3.15}$$

*For $0 < t < 1$,*

$$\left(\prod_{k=1}^m a_k\right)^{-1/2} \left(\frac{\pi}{t}\right)^{m/2} \exp\left(-\frac{d^2_{\mathcal{A}}(x,0)}{4t}\right) \leq \mu^{\mathcal{A}}_t(x)$$

$$\leq 2^m \left(\prod_{k=1}^m \left(1 + \sqrt{\frac{\pi}{a_k t}}\right)\right) \exp\left(-\frac{d^2_{\mathcal{A}}(x,0)}{4t}\right). \tag{3.16}$$

*For $t > 1$,*

$$\left(\prod_{k=1}^m (1 + 2e^{-a_k t})\right) \exp\left(-\frac{d^2_{\mathcal{A}}(x,0)}{4t}\right) \leq \mu^{\mathcal{A}}_t(x)$$

$$\leq 2^m \left(\prod_{k=1}^m (1 + \frac{2e^{-a_k t}}{1 - e^{-a_k t}})\right) \exp\left(-\frac{d^2_{\mathcal{A}}(x,0)}{4t}\right). \tag{3.17}$$

The inequalities (3.16) and (3.17) respectively gives the correct behavior (up to a multiplicative constant) of $\mu^{\mathcal{A}}_t(x)$ when $t$ is small and $t$ is large.

# 4 Heat kernel on infinite dimensional torus (Part 1)

As we shall see, there exists many heat kernels on the infinite dimensional torus in the sense of the behavior at the origin. First we face to the existence of such heat kernels. The generator of the semigroup generated by the convolution semigroup is easy to express on cylindrical functions contained in the domain.

Before going into the subject, we start a short (informal) discussion on quantities appearing above in the gaussian estimates of the heat kernel on $\mathbb{T}^m$.

Assume that we are given an infinite sequence $\mathcal{A} = (a_1, a_2, ..., a_n, ...)$ with $a_k > 0, \forall k \in \mathbb{N}$. We try to guess the behavior of the heat kernel $\mu_t^{\mathcal{A}}$ (that we assume to exist).

First, we note that if

$$\sum_{k=1}^{\infty} e^{-a_k t} < \infty \tag{4.18}$$

cond

then we can take the pointwise limit

$$\lim_m \left( \prod_{k=1}^m (1 + 2e^{-a_k t}) \right)$$

and get a lower bound in (3.17) above. We shall see that this quantity appears in our discussion below concerning the existence and the continuity of the density of the heat convolution semigroup associated to an infinite sequence $\mathcal{A} = (a_1, a_2, ..., a_m, ...)$. Note that, for the quantity appearing in (3.17), the limit

$$\lim_m 2^m \left( \prod_{k=1}^m (1 + \frac{2e^{-a_k t}}{1 - e^{-a_k t}}) \right)$$

is not finit under/or not under the condition (4.18). Indeed,

$$\infty = \lim_m 2^m \leq \lim_m 2^m \left( \prod_{k=1}^m (1 + \frac{2e^{-a_k t}}{1 - e^{-a_k t}}) \right).$$

Nevertheless,

$$\lim_m \left( \prod_{k=1}^m (1 + \frac{2e^{-a_k t}}{1 - e^{-a_k t}}) \right) < \infty$$

and this limit will be the appropriate quantity to bound $\mu_t^{\mathcal{A}}$ at the origin $0 \in \mathbb{T}^\infty$ (see (2.7)). Indeed, $\mu_t^{\mathcal{A}}(0) = \lim_m (\prod_{k=1}^m g_{a_k t}(0))$.

which converges under the condition (4.18) for large $t$ with $(a_k)$ bounded below by $\delta > 0$.

## 4.1 Definition of heat convolution semigroups on $\mathbb{T}^\infty$.

We define $\mathbb{T}^\infty$ as the product of countable many copies of the torus $\mathbb{T}$. It is a compact abelian group with respect to the ordinary product structure. We shall identify $\mathbb{T}$ with $[-\pi,\pi]$ so $\mathbb{T}^\infty$ with $[-\pi,\pi]^\infty$. We denote by $d\mu_k$ the normalized Haar measure on $\mathbb{T}$. The measure $d\mu_k(x_k)$ is identified to $\frac{dx_k}{2\pi}$ where $dx_k$ is the Lebesgue measure on $[-\pi,\pi]$. The normalized Haar measure on $\mathbb{T}^\infty$ is

$$d\mu(x) = \prod_{k=1}^{+\infty} d\mu_k(x_k), \quad x = (x_k)_{k=1}^{\infty}\mathbb{T}^\infty.$$

Such measure measure exists by the Caratheory extension theorem. The dual group of $\mathbb{T}^\infty$ is identified $\mathbb{Z}^{(\infty)}$ the subgroup of $\mathbb{Z}^\infty$ consisting of all sequences which are eventually zero i.e.

$$\mathbb{Z}^{(\infty)} = \{n = (n_k)_{k=1}^{\infty},\ \text{s.t}\ n_k \in \mathbb{Z},\ \exists\ N \in \mathbb{N} : \forall k, |k| \geq N, ; n_k = 0\}.$$

Namely, each $n = (n_k)_{k=1}^{\infty} \in \mathbb{Z}^{(\infty)}$ determines a character $e_n$ on $\mathbb{T}^\infty$,

$$e_n(x) = \exp\left(i(\sum_{k=1}^{\infty} n_k x_k)\right), \quad x = (x_k)_{k=1}^{\infty} \in \mathbb{T}^\infty.$$

Functions on $\mathbb{T}^\infty$ are identified with function on $\mathbb{R}^\infty$ which are periodic with $(2\pi\mathbb{T})^\infty$ as group of periodicity.

Let $\mathcal{A} = (a_k)_{k=1}^{\infty}$ with $a_k > 0$ for any $k \in \mathbb{N}$. For each $t > 0$ we define the product measure

$$\mu_t^{\mathcal{A}} = \bigotimes_{k=1}^{\infty} \mu_{a_k t}$$

where $\mu_t$ is the measure on $\mathbb{T}$ with density $g_t(x)$ with respect to the normalized Haar measure on $\mathbb{T}$.

It is easy to see that $\mu_t^{\mathcal{A}}$ defines a symmetric convolution semigroup on $\mathbb{T}^\infty$. Since $\mathbb{T}^\infty$ is a locally compact abelian group there exists a Fourier transform. (see Rudin?) This transform is explicit and use the dual group to be expressed:

$$\hat{f}(n) = \int_{\mathbb{T}^\infty} f(x) e_n(x)\, d\mu(x), \quad n = (n_k)_{k=1}^{\infty} \in \mathbb{Z}^{(\infty)}.$$

So taking the Fourier transform of $\mu_t^{\mathcal{A}}$ satisfies :

$$\hat{\mu}_t^{\mathcal{A}}(n) = e^{-\psi(n)} \quad \mathit{with} \quad \psi(n) = \sum_{k=1}^{\infty} a_k n_k^2, \quad n = (n_k)_{k=1}^{\infty} \in\in \mathbb{Z}^{(\infty)}.$$

By the form of $\psi$, the semigroup $(\mu_t^{\mathcal{A}})_{t>0}$ is of local type (see [BF] p....).

The aim of the next section is to study the existence of a (continuous) density of $\mu_t^{\mathcal{A}}$ with respect to the Haar measure on $\mathbb{T}^\infty$. We shall see that some condition on the sequence $\mathcal{A}$ has to be fulfill for the existence of a (continuous) density.

## 4.2 Existence of density (heat kernels) of $\mu_t^{\mathcal{A}}$.

We now formulate a remarkable theorem due to C. Berg (Thm. 4.3 and Thm 4.6 [BG1]). This theorem characterizes all the situations where a density exits or not and, in particular, a *continuous* density exits or not. This is an application of Kakutani theorem about existence of (continuous) density for an infinite product of probability measures with respect to a a fixed measure on the space under consideration (see Appendix A).

bergth **Theorem 4.1 (C.Berg)** *Let* $t_0 = \inf\{t > 0 : \sum_{k=1}^{\infty} e^{-2ta_k} < \infty\} \in [0, +\infty]$.

1. *For any* $0 < t < t_0$, *the measure* $\mu_t^{\mathcal{A}}$ *is singular with respect to the Haar measure on* $\mathbb{T}^\infty$.
2. *For any* $t_0 < t < \infty$, *the measure* $\mu_t^{\mathcal{A}}$ *is absolutely continuous with respect to the Haar measure on* $\mathbb{T}^\infty$ *but has no continuous density if* $t_0 < t < 2t_0$.
3. *If* $2t_0 < t$ *the measure* $\mu_t^{\mathcal{A}}$ *has a continuous density with respect to the Haar measure on* $\mathbb{T}^\infty$.

We shall call $t_0$ the critical time for the convolution semigroup **Proof**: It is a direct proof of Kakutani's theorem (see Appendix A for details). We set $\mu_k = g_{a_k t}\nu_k$ with $\nu_k$ the Haar measure of $\mathbb{T}$. So

$$H(\mu, \nu) = \prod_{k=1}^{\infty} \int_{\mathbb{T}} \sqrt{g_{a_k t}(x_k)}\, d\nu_k(x_k)$$

We set $\rho(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sqrt{g_t(\theta)}\, d\theta$. So $H(\mu, \nu) = \prod_{k=1}^{\infty} \rho(a_k t)$.

By Kakutani's theorem the measure $\mu_t^{\mathcal{A}}$ posseses a density with respect to the Haar measure on $\mathbb{T}^\infty$ if an only if $H(\mu, \nu) > 0$ i.e.

$$\ln H(\mu, \nu) > -\infty.$$

The condition is equivalent to

$$\sum_{k=1}^{\infty} \ln \rho(a_k t) > -\infty.$$

i.e.

$$\sum_{k=1}^{\infty} \ln \frac{1}{\rho(a_k t)} < +\infty. \tag{4.19}$$ criter

In the case $\ln H(\mu, \nu) = -\infty$, $\mu$ is singular with respect to $\nu$ the Haar measure on $\mathbb{T}^\infty$.

So, we need some estimates of $\rho(t)$. We have (Prop. 2.8 of [BG1] p.59):

$$(a) \quad 0 < \rho(t) < 1, \quad \forall t > 0, (b) \quad \lim_{t \to 0^+} \rho(t) = 0, (c) \quad \rho(t) \sim 1 - \frac{1}{4} e^{-2t}, t \to +\infty.$$

The first result is obtained by Cauchy-Schwarz, the second is deduced from the gaussain upper bound. The last one is more difficult but crucial to get the criterium expressed by $t_0$ (we refer to [BG1] p.60) .

Before proving theses results, we show how to conclude. In particular, this condition implies that $\lim_k \rho(a_k t) = 1$ (The general term of the series tends to 0). By (a) and (b) above, this limit can be obtained only by the fact that $\lim_k a_k t = +\infty$ (for this fixed $t > 0$) i.e $\lim_k a_k = +\infty$. So our condition (4.19) is equivalent to

$$\sum_{k=1}^{\infty} \ln\left(1 - (1 - \rho(a_k t))\right) > -\infty.$$

Now, since $\lim_k a_k = +\infty$ thus, by using (c), $\lim_k (1 - \rho(a_k t)) = 0$ for any $t > 0$. So the condition is equivalent to

$$\sum_{k=1}^{\infty} -(1 - \rho(a_k t)) > -\infty.$$

that is

$$\sum_{k=1}^{\infty} (1 - \rho(a_k t)) < +\infty.$$

More explicitely

$$\sum_{k=1}^{\infty} e^{-2a_k t} < +\infty (*)$$

This proves the first and second statement of Berg's result. Let's fixe $t > 0$. Indeed (to sum up the situation), we have $\mu_t^{\mathcal{A}}$ is singular with respect to the Haar measure in the other case that is if

$$\sum_{k=1}^{\infty} e^{-2a_k t} < +\infty$$

and $\mu_t^{\mathcal{A}}$ is absolutely continuous with respect to the Haar measure if

$$\sum_{k=1}^{\infty} e^{-2a_k t} = +\infty$$

Let $t_0$ as defined in the theorem. Then $t_0$ is a critical value in the following sense: for any $t > t_0$, we know that $\mu_t^{\mathcal{A}}$ is absolutely continuous with respect to the Haar measure and for any $t < t_0$, we have that $\mu_t^{\mathcal{A}}$ is singular with respect to the Haar measure. In the general case we don't know what happens if $t = t_0$ (see Berg's paper for a discussion of one example such that...).

We now prove 3). (see Thm 4.6 of Berg). We use a general theorem on l.c.a (locally compact abelian) groups saying that a symmetric convolution semigroup $(\mu_t)$ has continuous density with respect to the Haar measure iff $\hat{\mu}_t \in L^1(\Gamma)$ with $\Gamma$

the dual group of $G$ endowed with its Haar measure. We apply this with $G = \mathbb{T}^\infty$ and $\Gamma = \mathbb{Z}^{(\infty)}$ (with counting measure).

In this proof, we assume that $t > t_0$ to get the existence of a density (heat kernel) for the measure $\mu_t^{\mathcal{A}}$.

*First step*: we prove

$$||\hat{\mu}_t^{\mathcal{A}}||_{L^1(\mathbb{Z}^{(\infty)})} := \sum_{n=(n_1,n_2,...,)\in\mathbb{Z}^{(\infty)}} |\hat{\mu}_t^{\mathcal{A}}|(n) = \prod_{k=1}^{\infty} g_{a_k t}(0).$$

Indeed, let $\tilde{\mathbb{Z}}^p = \{n = (n_1, n_2, ..., n_p, 0, 0, ...), n_k \in \mathbb{Z}, k = 1...p\}$. So $\mathbb{Z}^{(\infty)} = \cup_{p=1}^{\infty} \tilde{\mathbb{Z}}^p$. By dominated convergence,

$$||\hat{\mu}_t^{\mathcal{A}}||_{L^1(\mathbb{Z}^{(\infty)})} = \lim_{p\to+\infty} \sum_{n\in\tilde{\mathbb{Z}}^p} |\hat{\mu}_t^{\mathcal{A}}|(n)$$

For any $n \in \tilde{\mathbb{Z}}^p$, since $\mu_t^{\mathcal{A}} = \bigotimes_{k=1}^{\infty} \mu_{a_k t}$ with $\hat{\mu}_{a_k t}(n_k) = \hat{g}_{a_k t}(n_k)$,

$$\hat{\mu}_t^{\mathcal{A}}(n) = \int_{\mathbb{T}^\infty} \hat{\mu}_t^{\mathcal{A}}(x) e^{-ix.n}\, d\nu(x) = \prod_{k=1}^{p} \hat{g}_{a_k t}(n_k) = \prod_{k=1}^{p} e^{-a_k n_k^2 t} = |\prod_{k=1}^{p} e^{-a_k n_k^2 \frac{t}{2}}|^2 = |(\prod_{k=1}^{p} g_{a_k \frac{t}{2}})\hat{}\,|^2(n).$$

Thus, by Parseval equality

$$\sum_{n\in\tilde{\mathbb{Z}}^p} |\hat{\mu}_t^{\mathcal{A}}|(n) = ||(\prod_{k=1}^{p} g_{a_k\frac{t}{2}})\hat{}\,||^2_{L^2(\mathbb{Z}^p)} = ||(\prod_{k=1}^{p} g_{a_k\frac{t}{2}})\,||^2_{L^2(\mathbb{T}^p)} = \prod_{k=1}^{p} ||g_{a_k\frac{t}{2}}||^2_{L^2(\mathbb{T})} = \prod_{k=1}^{p} g_{a_k t}(0).$$

Indeed, $g_s$ defines a symmetric convolution semigroup so

$$||g_s||^2_{L^2(\mathbb{T})} = \int_{\mathbb{T}} g_s(0-x) g_s(x)\, d\nu_1(x) = g_{2s}(0).$$

(take $2s = a_k t$). We conclude

$$||\hat{\mu}_t^{\mathcal{A}}||_{L^1(\mathbb{Z}^{(\infty)})} = \lim_{p\to+\infty} \prod_{k=1}^{p} g_{a_k t}(0) = \prod_{k=1}^{\infty} g_{a_k t}(0).$$

*Second step*: The condition of existence of a *continuous* density is characterized by

$$||\hat{\mu}_t^{\mathcal{A}}||_{L^1(\mathbb{Z}^{(\infty)})} < \infty$$

i.e.

$$\lim_{p\to+\infty} \prod_{k=1}^{p} g_{a_k t}(0) < \infty.$$

Taking the log, it is equivalent to

$$\sum_{k=1}^{\infty} \log g_{a_k t}(0) < \infty.$$

At this point, let's again discuss the behavior of $a_k$'s.

The convergence implies the necessary condition $\lim_{k\to+\infty} g_{a_k t}(0) = 1$. We know that $g_t(0) = 1 + 2\varphi(t)$ with $\varphi(t) = \sum_{n=1}^{\infty} e^{-tn^2}$. So, it implies

$$\lim_{k\to+\infty} \varphi_{a_k t}(0) = 0.$$

This implies that $\lim_{k\to+\infty} a_k = +\infty$. Indeed, recall that (see Thm estimon 2.2), we have proved:

$$\varphi(t) \sim \frac{1}{2}\sqrt{\frac{\pi}{t}}, \quad t \to 0, \qquad \varphi(t) \sim e^{-t}, \quad t \to +\infty.$$

we also know that $0 < \varphi_{a_k t}(0)$ for all $k$'s and that $t \to \varphi(t)$ is non-increasing. So necessarily, for this fixed $t > 0$, $\lim_{k\to+\infty} a_k t = +\infty$ that is $\lim_{k\to+\infty} a_k = +\infty$. (This condition is already contained in the discussion of the existence of the density of $\mu_t^{\mathcal{A}}$ above).

We come back to our criterium. The criterium is equivalent to

$$\sum_{k=1}^{\infty} \log\left(1 + 2\varphi(a_k t)\right) < \infty.$$

i.e.

$$\sum_{k=1}^{\infty} \varphi(a_k t) < +\infty$$

since $\lim_k \varphi(a_k t) = 0$. From the facts that $a_k$ tends to $+\infty$ and by $\varphi(t) \sim e^{-t}, \quad t \to +\infty$, we conclude

$$\sum_{k=1}^{\infty} e^{-a_k t} < +\infty.$$

This condition is satisfied if $\frac{t}{2} > t_0$ that is $t > 2t_0$.

The proof is now completed.

**Exercice 4.2** *: Let $\nu$ be the normalized Haar measure on $\mathbb{T}^{\infty}$.*

1. *Let $\alpha > 0$ and $a_k = \frac{1}{2\alpha}\ln(k+1), k \geq 1$. Show that $t_0 = \alpha$ is the critical time and that $\mu_{t_0}^{\mathcal{A}}$ and $\nu$ are mutually singular.*

2. *Let $\alpha > 0$ and $a_k = \frac{1}{2\alpha}\ln(k+1) + \frac{1}{\alpha}\ln\ln(k+2), k \geq 1$. Show that $t_0 = \alpha$ is the critical time. Show that $\mu_{t_0}^{\mathcal{A}}$ is absolutely continuous with respect to $\nu$.*

3. *Find a larger family satisfying the following conditions: $t_0 = \alpha$ is the critical time and $\mu_{t_0}^{\mathcal{A}}$ is absolutely continuous with respect to $\nu$*

4. *Let* $a_k = 1, k \geq 1$. *Show that* $t_0 = +\infty$. *Find a larger class satisfying this condition.*

5. *Let* $\beta > 0$ *and* $a_k = k^\beta, k \geq 1$. *Show that* $t_0 = 0$. *(?)*

6. *For the examples above, what can be said about the existence of the density and its continuity of* $\mu_t^{\mathcal{A}}$*?*

**Exercice 4.3** *Show that for any* $t_0 \in [0, +\infty]$, *there exists a sequence* $\mathcal{A} = (a_k)_{k\geq 1}$ *such that the critical time for the convolution semigroup is exactly* $t_0$.

**Appendix A : Kakutani's theorem.**

Theses notions are taken from the excellent book dp [DP] p.26-30.

Kakutani's theorem is a dichtomy theorem about existence of density in the case of infinite product of probability measures with respect to a reference probability measure which is also a product of probability measures .

We recall some well-known definitions and results. Let $\mu$ and $\nu$ two (positive) measures on $(\Omega, \mathcal{F})$ with $\mathcal{F}$ a $\sigma-$ algebra on the set $\Omega$. We recall that a measure $\mu$ is absolutely continuous with respect to $\nu$ (denoted by $\mu << \nu$) if

$$\forall A \in \mathcal{F},\ (\nu(A) = 0) \Rightarrow (\mu(A) = 0).$$

By Radon-Nikodym theorem, there exists a function $f = \frac{\partial\mu}{\partial\nu} \in L^1(\nu)$ such that $\mu(A) = \int_A f(x)\, d\nu(x)$. If $\mu$ and $\nu$ are probability measures then $f$ is called a density i.e. $f \geq 0$ and $\int_\Omega f(x)\, d\nu(x) = 1$. We say that $\mu$ and $\nu$ are equivalent if $\mu$ is absolutely continuous with respect to $\nu$ and $\nu$ is absolutely continuous with respect to $\mu$. We say that $\mu$ is singular with respect to $\nu$ if there exists two disjoint sets $A, B \in \mathcal{F}$ such that $\Omega = A \cup B$ and $\mu(A) = 0$ and $\nu(B) = 0$ (for positive measures $\mu$ and $\nu$). We say that $\mu$ and $\nu$ are mutually singular if $\mu$ is singular with respect to $\nu$ and $\nu$ is singular with respect to $\mu$.

We define the Hellinger integral as follows:

$$H(\mu, \nu) = \int_\Omega \sqrt{\frac{\partial\mu}{\partial\zeta}\frac{\partial\nu}{\partial\zeta}}\, d\zeta$$

where $\zeta = (\mu + \nu)/2$. Note that $\mu$ and $\nu$ are absolutely continuous with respect to $\zeta$ so $\sqrt{\frac{\partial\mu}{\partial\zeta}\frac{\partial\nu}{\partial\zeta}}$ is defined. The Hellinger integral gives a criterium about existence of a density of product of probabilty measures. Kakutani's theorem assets that there are two possible cases: 1) these measures are mutually absolutely continuous or 2) theses measures are mutually singular.

kutanith **Theorem .4** *Let $(\mu_k)_{k\geq 1}$ and $(\nu_k)_{k\geq 1}$ be two sequences of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\mu_k$ and $\nu_k$ are equivalent for any $k \geq 1$. Let $\mu = \bigotimes_{k=1}^{\infty} \mu_k$ and $\nu = \bigotimes_{k=1}^{\infty} \nu_k$ the two associated product probability measures on $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R}...$ (countable product of $\mathbb{R}$). We have*

$$H(\mu, \nu) = \prod_{k=1}^{\infty} H(\mu_k, \nu_k).$$

*Then we have two possibilities:*

1. *If $H(\mu, \nu) > 0$ then $\mu$ and $\nu$ are equivalent and the density of $\mu$ with respect to $\nu$ is given by*

$$\frac{\partial \mu}{\partial \nu}(x) = \lim_{n \to +\infty} \prod_{k=1}^{n} \frac{\partial \mu_k}{\partial \nu_k}(x_k), \quad x = (x_k)_{k=1}^{\infty} \in \mathbb{R}^\infty.$$

2. *If $H(\mu, \nu) = 0$ then $\mu$ and $\nu$ are mutually singular.*

For a proof see dp [DP] p. 29.

In the case of measures $\alpha$ on $\mathbb{T} \sim [-\pi, \pi]$, we extend these measures on $\mathbb{R}$ (denoted by $\tilde{\alpha}$) in a natural way: let $A \in \mathcal{B}(\mathbb{R})$, we set $\tilde{\alpha} = \alpha(A \cap [-\pi, \pi])$.

Recall that Kolmogorov's extension theorem gives the existence and uniqueness of the measures $\mu$ and $\nu$ of the theorem above. We recall that theorem which can be found in many text books (ref?).

Let $(\Omega_n, \mathcal{F}_n, P_n)$ be countable family of probability spaces. Let $\Omega = \Omega_1 \times \Omega_2 \times \Omega_3 ... \Omega_n ...$ be the cartesian product of $\Omega_n$ and $\mathcal{F}$ be the $\sigma-$algebra generated by $C = A_1 \times A_2 \times ... A_n$ with $A_k \in \mathcal{F}_k$ and $n \in \mathbb{N}$. Then there exists a unique probability measure $P$ on $\mathcal{F}$ such that

$$P(C) = P_1(A_1) \times P_2(A_2) \times ... P_n(A_n).$$

This theorem can be applied to countable many copies of the Haar measure on the torus $\mathbb{T}$ to prove the existence of the Haar measure on $\mathbb{T}^\infty$ and to prove the existence of the measure $\mu_t^{\mathcal{A}} = \otimes \mu_{a_k t}$ on $\mathbb{T}^\infty$.

**Solutions of exercices**:

*Aknowlegement: I warmly thank A.Bendikov for mentioning that Th. has been proved independently by Bendikov, Berg, ?.*

# References


b1 [B1] Bendikov A.D *Potential Theory on Infinite Dimensional Abelian Groups.* Walter De Gruyter & Co., Berlin-New York, 1995.

b2 [B2] Bendikov.A.D *Symmetric stable semigroups on the infinite dimensional torus.* Expo.Math.13 (1995), 39-80.

b3 [B3] Bendikov.A.D *Personal communication.*

bg1 [BG1] Berg.Christian, *Potential theory on the infinite dimensional torus.* Invent. Math. 32 (1976), no. 1, 49–100.

bf [BF] Berg Ch., Forst G. *Potential theory on locally compact Abelian groups.* Erg. der Math. und ihrer Grenzgeb., Band 87, Springer-Verlag 1975.

dp [DP] Da Prato, Giuseppe, *An introduction to infinite-dimensional analysis* . Revised and extended from the 2001 original by Da Prato. Universitext. Springer-Verlag, Berlin, 2006. x+209 pp.

h [H] Heyer, Herbert, *Probability measures on locally compact groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 94. Springer-Verlag, Berlin-New York, 1977.

r [R] Rudin, Walter, *Fourier analysis on groups.* Reprint of the 1962 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley and Sons, Inc., New York, 1990.

s [S] Skorokhod,A.V: *Basic principles and applications of probability theory.* Springer