Nash-type Inequalities on Locally Compact Abelian Groups

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I. SETTING

- Let $\mathbf{G}$ be a Locally Compact Abelian Group (LCA group)

- $\mathbf{\Gamma}$ its dual group

(Fourier analysis tools available)

- Let $(\mu_{\mathbf{t}})_{\mathbf{t>0}}$ be a symmetric convolution semigroup of measures on G

$$\mu_t \star \mu_s = \mu_{t+s}, \qquad \lim_{t \to 0} \mu_t = \delta_0$$

- ψ : its **Lévy exponent**
(real continuous non-negative definite function) defined by

$$\widehat{\mu}_t = e^{-t\psi},$$

$$\psi : \Gamma \longrightarrow \mathbb{R}$$

$$\theta \longrightarrow \psi(\theta)$$

- $T_t f(x) = \mu_t \star f(x), x \in G$: one parameter semi-group associated to (μ_t)

- $\mathcal{L}_\psi$ the infinitesimal generator associated to the semi-group

II. QUESTION

What kind of functional inequality is satisfied by the **quadratic form**

$$(\mathcal{L}_\psi u, u) = \int_\Gamma \psi(\theta) |\widehat{u}(\theta)|^2 \, d\theta \quad ???$$

($d\theta$: the Haar measure on Γ)

EXAMPLE : $G = \mathbb{R}^n$,

$\Psi(\theta) = |\theta|^2$ ($\mathcal{L} = \Delta$ Laplacian)

i.e. $(\mu)_{t>0}$: **Heat** convolution semigroup.

III. MOTIVATION

Nash inequality on $\mathbb{R}^d$:

$$c\,||u||_2^{2+4/d} \leq (\Delta u, u), \quad ||u||_1 \leq 1$$

implies ("equivalent")

Sobolev inequality :

$$||u||^2_{\frac{2d}{d-2}} \leq (\Delta u, u)$$

It is also equivalent to the upper bound estimate for the heat kernel:

$$||T_t||_{1\to+\infty} = h_t(0) \le \frac{c}{t^{n/2}}, \quad t > 0$$

(This holds true in a very abstract setting)

IV. What is a Nash-type inequality ? (generalized Nash inequality)

For some non-decreasing function

$$\Lambda : [0, +\infty) \longrightarrow [0, +\infty)$$

$$\Lambda(\, ||u||_2^2 \,) \le (\mathcal{L}_\psi u, u), \quad ||u||_1 \le 1$$

• Example : Euclidean setting with the Laplacian:

$$\Lambda(x) = x^{1+2/d}$$

V. Why are we so interested by Nash T.I ?

Assume that

$$\Lambda(\,||u||_2^2\,) \le (\mathcal{L}u, u), \quad ||u||_1 \le 1$$

is equivalent to (for some function Λ and m)

$$||T_t u||_\infty \le m(t)\,||u||_1, \qquad t > 0.$$

i.e

$$\sup_x h_t(x,x) \le m(t), \qquad t > 0.$$

1) From Nash T.I. we deduce heat kernel bounds

2) If A is a generator (a simple one) and and B is another operator (a complicated one) s.t.

$$(Au, u) \leq (Bu, u)$$

If N.T.I holds for A then it holds also

immediately for B !!!

and the heat kernel for B satisfies the same upper bound as the theat kernel for A !!!

VI. First result: Theorem 1 (Nash approach)

- Assume that ψ is real Lévy exponent
- We set the "volume" function

$$V(t) = d\theta\{\theta \in \Gamma : \psi(\theta) \leq t\}, \quad t > 0$$

- We denote by

$$\Lambda(s) = \sup_{t>0}\{st - tV(t)\}$$

Then (Nash type-inequality) :

$$\Lambda(||u||_2^2) \leq (\mathcal{L}_\psi u, u), \quad u \in \mathcal{D}(\mathcal{L}_\psi) \ , ||u||_1 \leq 1 \tag{1}$$

VII. Non-radial examples (product of α-stable semigroups)

- Let $G = \mathbb{R}^d$,
$d = d_1 + d_2 + ... + d_k, \qquad (d_j \geq 1, j = 1...k)$.

- Let $a_1, ..., a_k > 0$ and $\alpha_j \in (0, 1]$.

- $\psi(x) = a_1|x_1|^{2\alpha_1} + a_2|x_2|^{2\alpha_2} + ... + a_k|x_k|^{2\alpha_k}$

$x = (x_1, ..., x_k) \in \mathbb{R}^d$ and $x_j \in \mathbb{R}^{d_j}$

$$\mathcal{L}_\psi = a_1\Delta_1^{\alpha_1} + a_2\Delta_2^{\alpha_2} + ... + a_k\Delta_k^{\alpha_k},$$

Let $\nu = 2(\frac{d_1}{\alpha_1} + ... + \frac{d_k}{\alpha_k})$.
(this number may not be an integer!)

Then there exists a constant $c_1 > 0$, for all $f \in \mathcal{D}$ with $||u||_1 \leq 1$:

$$c_1 ||u||_2^{2+4/\nu} \leq (\mathcal{L}_\psi u, u)$$

Then the semigroup (T_t) generated by ψ is ultracontractive and satisfies:

$$||T_t u||_\infty \leq c t^{-\nu/2} ||u||_1, \quad t > 0.$$

i.e.

$$h_t(0) \leq c t^{-\nu/2}, \quad t > 0.$$

When $\nu > 2$,

this implies the **Sobolev inequality**:

$$||u||^2_{\frac{2\nu}{\nu-2}} \leq (\mathcal{L}_\psi u, u)$$

VIII. Theorem on $\mathbb{R}^d$ for radial symbol

- Let ψ be radial real-valued Lévy exponent

$$\tilde{\psi}(|\theta|) = \psi(\theta)$$

- Assume $r \to \tilde{\psi}(r)$ is non-decreasing

Then (Nash-type inequality)

$$c||u||_2^2 \; \tilde{\psi}\left(c||u||_2^{2/d}\right) \le (\mathcal{L}_\psi u, u) \qquad (2)$$

for all $||u||_1 \le 1$.

IX. Corollary: Sobolev-Orlicz-type inequality

- Let ψ be radial real-valued Lévy exponent on $G = \mathbb{R}^d$
- Assume $r \to \tilde{\psi}(r)$ is non-decreasing.

Then

$$c \int_{\mathbb{R}^d} u^2 \, \tilde{\psi} \left(c' \frac{u^{2/d}}{||u||_2^{2/d}} \right) dm \leq (\mathcal{L}_\psi u, u) \qquad (3)$$

X. Examples on $\mathbb{R}^d$

1. Γ^*-semigroup: $\mathcal{L} = \log(1+\Delta)$

$g(x) = \log(1+x)$ (Bernstein function)

$go\tilde{\psi}(r) = \log(1+r^2)$ with $\psi(r) = r^2$

Then (N.T.I)

$$c\,||u||_2^2 \ln\left(1 + c||u||_2^{4/d}\right) \leq (\ln(1+\Delta)u, u) \tag{4}$$

Then **Sobolev-Orlicz inequality !**

$$c \int_{\mathbb{R}^d} u^2 \log\left(1 + c' \frac{u^{4/d}}{||u||_2^{4/d}}\right) dm \leq (\log(1+\Delta)u, u) \tag{5}$$

This implies **Gross-type inequality !**

$$c'' \int_{\mathbb{R}^d} u^2 \log\left(c' \frac{u^2}{||u||_2^2}\right) dm \leq (\log(1+\Delta)u, u) \tag{6}$$

2. Poisson semigroup with jump $\lambda > 0$

$g(x) = 1 - e^{\lambda x}$ (Bernstein function)

$go\tilde{\psi}(r) = 1 - e^{-\lambda r^2}$ with $\psi(r) = r^2$

$$c||u||_2^2\left(1 - e^{-\lambda c||u||_2^{4/d}}\right) \leq ((I - T_\lambda)u, u) \quad (7)$$

$\mathcal{L} = (I - T_\lambda)$ is the (bounded) generator of the Poisson semigroup given by $P_t = e^{-t(I - T_\lambda)}$

XI. PROBLEM:

- $G = \mathbb{R}$, $\quad \psi_x(\theta) = 1 - \cos(x.\theta)$, $\quad x \in \mathbb{R}$

Then $\quad V_x(t) = +\infty \ (\forall x \in \mathbb{R})$!

- **Importance of ψ_x**: Lévy-Khinchine representation formula (symmetric case)

$$\psi(\theta) = \int_{\mathbb{R}^n - \{0\}} \psi_x(\theta) \, d\nu(x)$$

- But if $G = \mathbb{Z}^d$ then $\Gamma = \mathbb{T}^d$ and the volume V_ψ is always finite for any ψ!

XII. Examples on $G = \mathbb{T}^d$

Let $\psi(n) = |n|^{2\alpha}$ with $\alpha \in (0,1]$ and $n = (n_1, ..., n_d) \in \Gamma = \mathbb{Z}^d$ with $|n|$ the euclidean norm of n.

$$(\mathcal{L}_\psi u, u) = \sum_{n \in \mathbb{Z}^d} |n|^{2\alpha} |\hat{u}(n)|^2 = (\Delta^\alpha u, u)$$

The Haar measure on $\mathbb{Z}^d$ is the counting measure

We denote by $\underline{u} = \int_{\mathbb{T}^d} u(x)\, dx$ the mean value of u.

Theorem on $\mathbb{T}^d$

There exists a constant $c > 0$ such that: For all $u \in \mathcal{D}$, with $\underline{u} = 0$,

$$||u||_2^{2+\frac{4\alpha}{d}} \leq (\mathcal{L}_\psi u, u), \qquad ||u||_1 \leq 1$$

Or equivalently: for all $u \in \mathcal{D}$, $||u - \underline{u}||_1 \leq 1$,

$$||u - \underline{u}||_2^{2+\frac{4\alpha}{d}} \leq (\mathcal{L}_\psi u, u)$$

XIII. Examples on $G = \mathbb{Z}^d$

Theorem on $\mathbb{Z}$ ($d = 1$)

- Assume $u \in L^1(\mathbb{Z})$ with $||u||_1 \leq 1$
- Then for all $\rho \in (0, 1/2)$,

$$c_\rho ||u||_2^2 \left(1 - \cos(\rho ||u||_2^2)\right) \leq (\mathcal{L}_\psi u, u)$$

with $c_\rho = (1 - 2\rho)^2$.

Argument : $\psi(\theta) = 1 - \cos\theta, \theta \in \mathbb{T} = [-\pi, +\pi)$ is radial and $\tilde{\psi}(r) = 1 - \cos r$ is non decreasing on $[0, \pi[$

XIV. Partial result on $\mathbb{R}$ with $\psi(\theta) = 1 - \cos(\theta x)$

Theorem on $\mathbb{R}$

- Assume $u \in L^1(\mathbb{R})$ satisfies the condition for all $x \in \mathbb{R}$,

$$\sum_{n \in \mathbb{Z}} |\widehat{u}(x + 2n\pi)|^2 \leq 1 \qquad (8)$$

- Then for all $\rho \in (0, 1/2)$ with $c_\rho = (1-2\rho)^2$,

$$c_\rho ||u||_2^2 \left(1 - \cos(\rho ||u||_2^2)\right) \leq (\mathcal{L}_\psi u, u)$$

work still in progress...