An Example of Ultracontractive Lévy semigroup

Patrick Maheux (Mapmo.Orléans)

Joint work with Alexander Bendikov (Wroclaw.Pl).

(Toulouse. November 1st, 2008)

Contents:

I. Motivation

Let (T_t) be a symmetric submarkovian

semigroup $L^2(X, \mu)$ with μ

σ-finite measure on X

$0 \leq f \leq 1 \Longrightarrow 0 \leq T_t f \leq 1$ for all $t > 0$.

T_t : contraction semigroup on L^p,
$1 \leq p \leq \infty$.

$(-A, \mathcal{D})$ be its generator.

$$-Af = \lim_{t \to 0} \frac{T_t f - f}{t} \quad (\text{in } L^2)$$

$$T_t = e^{-tA}$$

For $n > 2$, the following conditions are equivalent

$$(S) \quad \| f \|^2_{2n/n-2} \leq c(Af, f), \quad \forall f \in \mathcal{D}$$

$$(N) \quad \| f \|_2^{2+4/n} \leq c_1 (Af, f) \| f \|_1^{4/n}, \quad \forall f \in \mathcal{D}$$

$$(U) \quad \| T_t f \|_\infty \leq c_2 \, t^{-n/2} \| f \|_1, \quad \forall t > 0.$$

(Varopoulos, Carlen-Kusuoka-Stroock)

Let M be a non-increasing non-negative function

defined on $(0, +\infty)$ with $M(0^+) = +\infty$

$$N(x) = \sup_{t>0} \left(xt - tM(1/t) \right).$$

When M is C^1, we set

$$Q(x) = -M' \, o \, M^{-1}(x) \quad if \quad x > M(\infty)$$

and $Q(x) = 0$ otherwise.

Thm: (Coulhon.T) Let $-A$ be the infinitesimal generator of a sub-Markovian semigroup on $L^2(X,\mu)$. Let M, N, Q as above.

(1)

$$(GU) \quad If\ ||T_t f||_\infty \le e^{M(t)}||f||_2,\ t > 0$$

Then

$$(NTI_1) \quad ||f||_2^2\, N\left(\log ||f||_2^2\right) \le (Af, f),\ \ ||f||_1 \le 1,$$

(2)

If

$$(NTI_2)||f||_2^2\, Q\left(\log ||f||_2^2\right) \leq (Af, f), \quad ||f||_1 \leq 1,$$

Then,

$$||T_t f||_\infty \leq e^{M(t)} ||f||_2, \; t > 0$$

Examples:

(1) Assume that $M(t) = c\,\log_+(\frac{1}{t})$. Then

$$N(x) = ce^{-1}\,\exp\left(\frac{x}{c}\right) = c'\,Q(x),\ x >> 1$$

(2) Assume that $M(t) = \frac{c}{t^\alpha}$ with $\alpha > 0$.

$$N(x) = c_3\,x^{1+1/\alpha} = c_4\,Q(x),\ x > 0,$$

(3) Assume that $M(t) = \exp(\frac{1}{t^\alpha})$ with $\alpha > 0$. Then

$$N(x) \sim x\,(\log x)^{1/\alpha}, \quad \text{as } x \to +\infty.$$

$$Q(x) = \alpha\, x\, [\log(x)]^{1+\frac{1}{\alpha}}, \quad \forall x > 1.$$

Observation: We see that in the first two examples that N and Q are comparable,

whereas in Example 3. the ratio Q/N goes to infinity at infinity.

Thm: (Bendikov,Coulhon,Saloff-Coste):
M convex non-increasing function s.t.

$$-tM'(t) \leq bM(t),\ t \to 0$$

Then the following properties are $\sim$

(1) $\exists\ c_1, c_2 > 0$ s.t.

$$\forall t > 0, \quad ||T_t||_{2,\infty} \leq e^{c_1 M(c_2 t)}$$

(2) $\exists\ c_3, c_4 > 0$ s.t. $\forall f \in \mathcal{D}, \quad ||f||_1 \leq 1,$

$$c_3 ||f||_2^2\, N\left(c_4 \log ||f||_2^2\right) \leq (Af, f)$$

Remark: Function $t \to M(t)$ as at most polynomial explosion so the heat kernel has at most exponential explosion at $t = 0$.

In this case, Nash-type inequality gives an exact description of the behavior of the function $t \to ||T_t||_{2,\infty}$.

The situation changes in the double exponential case that is

$$\log\log ||T_t f||_{2,\infty} \sim \frac{1}{t^\alpha} \text{ as } t \to 0.$$

By Coulhon's Thm. : (NTI): for all f s.t. $||f||_1 \leq 1$,

$$||f||_2^2 \left(\log ||f||_2^2\right) \left[\log(\log ||f||_2^2)\right]_+^{1/\alpha} \leq (Af, f)$$

Question:

Does this Nash-type inequality imply estimate of the function $t \to ||e^{-tA}||_{2,\infty}$ for small t?

No for $\alpha \geq 1$:

Counter- example of Davies-Simon:

There exists A s.t. $||f||_1 \leq 1$

$$||f||_2^2 \left(\log ||f||_2^2\right) \left[\log(\log ||f||_2^2)\right]_+ \leq (Af, f)$$

$$(\alpha = 1)$$

but $||e^{-tA}||_{2,\infty} = \infty$ at least for small $t > 0$

Yes for $0 < \alpha < 1$ but with some loss.

$$||e^{-tA}||_{2,\infty} \leq \exp\left(\exp\left(\frac{1}{t^{\alpha'}}\right)\right), \quad t > 0.$$

$$\alpha' = \frac{\alpha}{1-\alpha} > \alpha$$

Aim of the paper:

In the setting of Lévy semigroup (on $\mathbb{R}$),

we are interested in relationships between

$$\log \log ||T_t f||_{2,\infty} \sim \frac{1}{t^{\alpha}} \text{ as } t \to 0.$$

and

$$||f||_2^2 \left(\log ||f||_2^2\right) \left[\log(\log ||f||_2^2)\right]^{1/\alpha} \leq (Af, f)$$

with $||f||_1 \leq 1$.

II. Symmetric Lévy generators on $\mathbb{R}$

$$Af(x) = -\frac{1}{2}\int_{-\infty}^{\infty} [f(x+y) - 2f(x) + f(x-y)]\, d\Pi(y),$$

where the Lévy measure Π is a symmetric measure such that

$$\int_{\mathbb{R}^*} (1 \wedge y^2)\, d\Pi(y) < \infty.$$

A is the generator of a

translation invariant Markov semigroup
$T_t = e^{-tA}$.

Let μ_t be the measure such that

$$T_t f = \mu_t * f.$$

Fourier transform

$$(T_t f)\hat{}\,(y) = e^{-t\psi(y)}\hat{f}(y)$$

where

$$\psi(y) = 2\int_0^\infty [1 - \cos(xy)]\, d\Pi(x)$$

Lévy symbol (or exponent)

$$(Af, f) = \int_{\mathbb{R}} \psi(x)\, |\widehat{f}(x)|^2\, dx$$

- The function ψ is a continuous negative definite function

$$\widehat{\mu}_t(x) = e^{-t\psi(x)}, \quad x \in \mathbb{R}$$

- The asymptotic behavior of $\Pi(x)$, $x > 0$,

as $x \longrightarrow 0$ reflects into the growth of ψ at

infinity, and ultimately in the behavior of

$$\mu_t(0) = 2\int_0^\infty e^{-t\psi(x)}\,dx \quad t \text{ small}$$

Examples:

$$(1) \qquad \alpha \in (0, 1)$$

$$d\Pi(x) = c_\alpha |x|^{-1-2\alpha} dx, \; .$$

$$\psi(y) = |y|^{2\alpha}$$

$$A = (\Delta)^\alpha$$

T_t is the symmetric α-stable semigroup.

(2)

$$d\Pi(x) = |x|^{-1} e^{-|x|} dx.$$

$$\psi(y) = 2 \log(1 + |y|^2)$$

$$\forall u \in C_0^\infty,$$

$$Au = 2 \log(1 + \Delta) u$$

T_t is the symmetric Γ-semigroup.

(Berg-Forst)

μ_t is absolutely continuous and

has a continuous density

(with respect to Lebesgue measure)

iff

its Fourier transform $\widehat{\mu}_t$ is in L^1.

if $\widehat{\mu}_t \in L^1$,

$$||e^{-(t/2)A}||_{2,\infty}^2 = \mu_t(0) = 2\int_0^\infty e^{-t\psi(x)}\,dx$$

$$= 2\int_0^\infty e^{-ts}\,dV(s)$$

where (Volume function)

$$V(s) = |\{t > 0 : \psi(t) \leq s\}|,$$

($t \rightarrow \mu_t(x)$ denotes the density of μ_t.)

III. General results

ψ a be a real (even) negative definite function on $\Gamma = \mathbb{R}^d$.

ψ is radial, we define $\tilde{\psi}$ by $\tilde{\psi}(r) = \psi(\theta)$

for any $r = |\theta| > 0$, $\theta \in \mathbb{R}^d$

ψ is said radially non-decreasing

if ψ is radial and if $r \longrightarrow \tilde{\psi}(r)$ is non-increasing.

Thm:[In a forthcoming paper B-M]

On $\mathbb{R}^d$, let ψ be a Lévy symbol

radially non-decreasing.

Then $\forall \rho > 0$, we have

$$(1 - w_d\,\rho^d)\;||f||_2^2\;\tilde{\psi}\left(\rho||f||_2^{2/d}\right) \le (Af, f) \quad (1)$$

with $||f||_1 \le 1$.

Thm: (On LCA groups) [In a forthcoming paper B-M]

Let ψ is real and non- negative definite function on Γ (dual group of G).

We set $V(t) = d\theta\{\theta \in \Gamma : \psi(\theta) \leq t\}$, $t > 0$.

Assume $V(t) < \infty, \ \forall t > 0$.

Let $\Lambda(s) = \sup_{t>0}\{st - tV(t)\}$.

$\Rightarrow (NTI): \quad \Lambda(||f||_2^2) \leq (Af, f), \quad ||f||_1 \leq 1$

On $\mathbb{R}$, we associate to $x \to \psi(x), x > 0$

$$\psi^*(x) = \frac{1}{x}\int_0^x \psi(\tau)\, d\tau, \quad x > 0.$$

We extend $\psi^*(x)$ to be even.

Thm

(1) For any $x \in \mathbb{R}$, $\frac{1}{3}\psi(x) \leq \psi^*(x)$

(2) ψ^* is again a continuous

even negative definite function having representation

$$\psi^*(x) = 2\int_0^\infty (1 - \cos x\xi)\Pi^*(\xi)\, d\xi$$

with $\Pi^*(\xi)$ being continuous non-increasing function.

(3) Under one of the following two conditions ψ^* is comparable to ψ:

a) ψ is a non-decreasing function at infinity

b) ψ has representation

$$\psi(x) = 2\int_0^\infty (1 - \cos x\xi)\Pi(\xi)\,d\xi$$

and $\Pi(\xi)$ is a non-increasing

Thm: Assume that ψ^* is comparable to ψ

A: Lévy generator associated to ψ.

Let N and M related by

$$N(x) = \sup_{t>0} (xt - tM(1/t))$$

The following properties are equivalent :

1. There exist $c_1, c_2 \in (0, \infty)$ s.t.

$$\forall t > 0, \quad ||T_t||_{2,\infty} \leq \exp\left(c_1 M(c_2 t)\right) \quad (2)$$

2. There exist $c_3, c_4 \in (0, \infty)$: $|| f ||_1 \leq 1$,

$$c_3 || f ||_2^2 \, N\left(c_4 \log || f ||_2^2\right) \leq (Af, f). \quad (3)$$

IV. Very oscillating symbol

a) Construction of ψ

$$\psi = \psi_1 + \psi_2$$

For given $0 < \alpha' < \alpha$,

- $$\psi_1(x) = \sum_{k\geq 1} a_k \left[1 - \cos(x2^{-k})\right]$$

with $a_k = (\log k)^\alpha,\ k \geq 1.$

The Lévy measure Π is given by

$$d\Pi_1(x) = \sum_{k \geq 1} a_k \, \delta_{2-k}(x)$$

where δ_a the Dirac mass at point $a \in \mathbb{R}$.

- $$\psi_2(x) = \int_0^\infty [1 - \cos(x.\xi)] \, d\Pi(x)$$

with

$$d\Pi_2(x) = x^{-1} \Phi(x^{-1}) \, dx$$

$\Phi(\tau) = 0$ for $\tau \le e$ and

$\Phi(\tau) = (\log\log\tau)^{\alpha'}$ for $\tau > e$.

We have

- $\psi_2(x) \sim 2(\log x)(\log\log x)^{\alpha'}, \ x \to +\infty.$

- $\psi_2^*(x) \sim \psi_2(x), \ x \to +\infty.$

b) Properties of ψ

There exists a symmetric Lévy generator A with symbol

$$\psi \quad (= \psi_1 + \psi_2) \text{ s.t.}$$

(1) $\exists\, 0 < c_1 < c_2 < \infty$ s.t. for $x \gg 1$,

$$c_1 (\log x)(\log\log x)^{\alpha'} \le \psi(x) \le c_2 (\log x)(\log\log x)^{\alpha}$$

(2) There exists a sequence (x_n) s.t.

$$x_n \to +\infty$$

$$\lim_{n \to +\infty} \frac{\psi(x_n)}{(\log x_n)(\log\log x_n)^{\alpha'}} = c_3 > 0$$

(3) There exists a sequence (y_n) such that

$$y_n \to +\infty$$

$$\lim_{n \to +\infty} \frac{\psi(y_n)}{(\log y_n)(\log\log y_n)^{\alpha}} = c_4 > 0$$

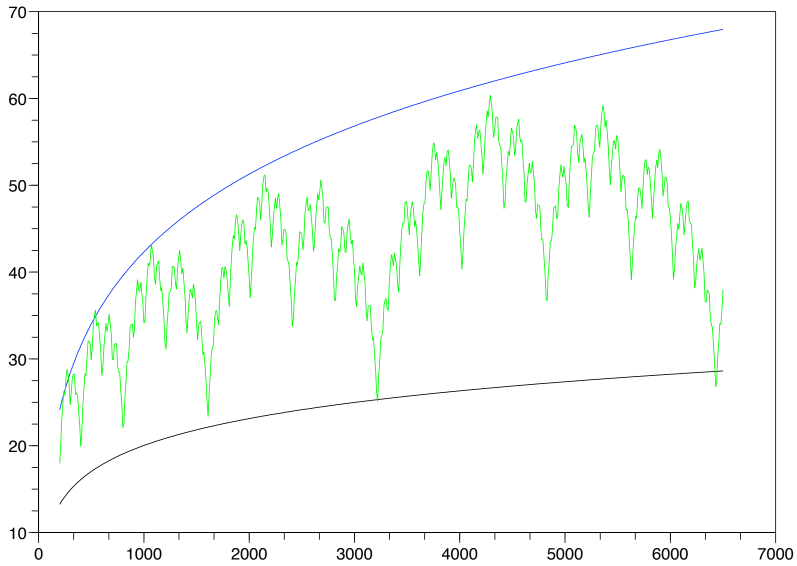
(4) The function ψ^* has the following behavior

$$\lim_{x \to +\infty} \frac{\psi^*(x)}{(\log x)(\log\log x)^{\alpha}} = c_5 > 0$$

(5) In particular, ψ and ψ^* are

not comparable

Picture 1 : ψ oscillating



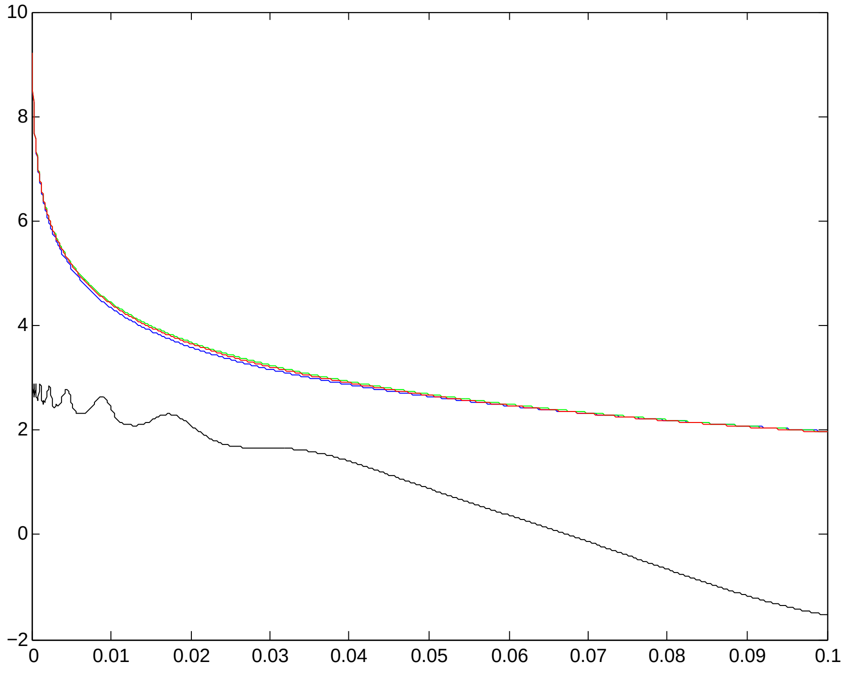
Picture 2

- Heat kernels

(decreasing curves)

- Ratio:

$$\log\log \mu_1(t) - \log\log \mu_{osc}(t) / \log\log \mu_2(t) - \log\log \mu_{osc}(t)$$



10
8
6
4
2
0
−2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1

Numerical simulations : M. Haddou/M.P.

Open problems :

• simulations of volume function and Nash function for ψ_{osc}

•$\Longrightarrow$ Conjecture about the behavior of μ_{osc}

and the behavior of Nash function:

close to the upper or lower level (Or nothing...)?

△