Nash Type Inequalities for Fractional Powers of Non-Negative Self-adjoint Operators

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Classical example:

Heat semigroup on $\mathbb{R}^n$ with dx the Lebesgue measure:

$$T_t f(x) = h_t \star f(x) = \int_{\mathbb{R}^n} h_t(x,y) f(y)\, dy$$

$$h_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

$$Af = \Delta f, \quad f \in \mathcal{C}_0^\infty(\mathbb{R}^n) \subset \mathcal{D}$$

$$\| T_t \|_{1\to\infty} := \sup\{\| T_t f \|_\infty, \quad \| f \|_1 = 1\}$$

$$(Ult) \quad \| T_t \|_{1\to\infty} = h_t(0,0) = \frac{1}{(4\pi t)^{n/2}}, \quad \forall t > 0$$

$$(Sob) \quad \| f \|^2_{2n/n-2} \leq c(\Delta f, f),$$

$$(Nash) \quad \| f \|_2^{2+4/n} \leq c(\Delta f, f) \| f \|_1^{4/n},$$

Let $(T_t)_{t>0}$ be a symmetric submarkovian semigroup acting on $L^2(X,\mu)$ with μ a σ-finite measure on X and let $-A$ be its generator with domain $\mathcal{D}$.

$$T_{t+s} = T_t T_s \quad t,s>0$$

$$\lim_{t\to 0^+} T_t f = f \quad (\text{in } L^2)$$

$$0 \le f \le 1 \quad \Rightarrow \quad 0 \le T_t f \le 1 \qquad \forall t>0$$

$$f \in \mathcal{D} \Leftrightarrow \qquad Af = \lim_{t\to 0^+} \frac{T_t f - f}{t} \in L^2$$

$T_t = e^{-tA}$ acts on L^p $(1 \le p \le +\infty)$ space as a contraction semigroup:

$$\| T_t f \|_p \le \| f \|_p, \quad t>0$$

Theorem[V,C-K-S] Let $\nu > 2$. The following conditions are equivalent :

$$(Sob) \qquad \| f \|^2_{2\nu/\nu-2} \leq c\,(Af, f),$$

$$\forall f \in \mathcal{D}$$

$$(Nash) \qquad \| f \|_2^{2+4/\nu} \leq c\,(Af, f) \, \| f \|_1^{4/\nu},$$

$$\forall f \in \mathcal{D} \cap L^1$$

$$(Ult) \qquad \| T_t \|_{1 \to \infty} \leq c\, t^{-\nu/2}, \quad \forall t > 0$$

Fact: Nash inequality for A implies Nash inequality for all $\alpha \in (0,1)$ (by subordination):

$$\| f \|_2^{2+4\alpha/\nu} \leq c\,(A^\alpha f, f)\, \| f \|_1^{4\alpha/\nu}$$

Nash-type (or generalized) inequality(Def)

Let $B : [0,+\infty[\to [0,+\infty[$ be a non-decreasing function.

A satisfies a Nash-type inequality if

$$\|f\|_2^2\, B(\|f\|_2^2) \leq \, (Af,f), \quad \forall f \in \mathcal{D} \quad \|f\|_1 \leq 1$$

Classical case in $\mathbb{R}^d$

$$(Nash) \quad c\|f\|_2^2 \left(\| f \|_2^2\right)^{2/n} \, \leq \, (Af,f)$$

with

$$B(x) = cx^{2/n}$$

Theorem [T.Coulhon. J.F.A 141]

$$(Ult) \qquad \| T_t \|_{1\to\infty} \le m(t), \quad \forall t > 0.$$

implies Nash type inequality

$$\| f \|_2^2 \ B(\| f \|_2^2) \le (Af, f), \quad \forall f \in \mathcal{D}, \ \| f \|_1 \le 1$$

$$B(x) = \sup_{s>0} [\, s \log x - sG(s)]$$

with

$$G(s) = \log m(1/2s)$$

Classical example: Polynomial decay:
Under the general assumptions above on the semigroup:

$$\| T_t \|_{1\to\infty} \leq \frac{c}{t^{\nu/2}} =: m(t)$$

then

$$B(x) = cx^{2/\nu}$$

Another example: One exponential decay

Proposition Let $\gamma > 0$. Assume that

$$\| T_t \|_{1\to\infty} \leq c\,\exp(1/t^{\gamma}) \quad \forall t > 0$$

Then for all $f \in \mathcal{D}$ with $\| f \|_1 \leq 1$,

$$\| f \|_2^2 \left[\log_+\left(\| f \|_2^2\right)\right]^{1+1/\gamma} \leq (Af, f)$$

Example : On $\mathbb{T}^{\infty}$,

$$A = \sum_{k=1}^{+\infty} k^{\frac{1}{\gamma}} \frac{\partial^2}{\partial_k^2}$$

Another example: Double exponential decay

Proposition Let $\beta > 0$. Assume that

$$\| T_t \|_{1\to\infty} \leq c\, e^{e^{1/t^\beta}} \quad \forall t > 0$$

Then for all $f \in \mathcal{D}$ with $\| f \|_1 \leq 1$,

$$\| f \|_2^2 \ \log_+ \left(\| f \|_2^2 \right) \ \left[\log_+ \left(\| f \|_2^2 \right) \right]^{1/\beta} \leq (Af, f)$$

Example : On $\mathbb{T}^\infty$,

$$A = \sum_{k=1}^{+\infty} (\ln k)^\gamma \, \frac{\partial^2}{\partial_k^2}$$

An example of Nash-type inequality **without Ultracontractivity**:

The Ornstein-Uhlenbeck operator $A = \Delta + x\nabla$ on $L^2(\mathbb{R}^n, \gamma_n)$ with the Gaussian measure:

$$\gamma_n(x) = \frac{1}{(2\pi)^{n/2}} \exp\left(\frac{-|x|^2}{2}\right) dx$$

Proposition

$$|| f ||_2^2 \log_+\left(|| f ||_2^2\right) \leq (Af, f), \qquad || f ||_1 \leq 1$$

then

$$\Theta(x) = x \log_+(x)$$

But (T_t) is not ultracontractive!

Proof. Deduce from Log-Sobolev inequality of Gross (but slightly weaker).

Spectral Theory

Let $A = A^*$ be a non negative self-adjoint

operator (unbounded) on $L^2(X,\mu)$

with μ a positive σ-finite

Spectral theory: $A = \int_0^{+\infty} \lambda \, dE_\lambda$ (E_λ spectral measure).

Functional calculus: Let $f : [0,+\infty[\to \mathbb{R}$ be a Borel function,

$$f(A) = \int_0^{+\infty} f(\lambda) \, dE_\lambda$$

Examples:

• $(t > 0)$ $f_t(\lambda) = e^{-t\lambda}$: $T_t = e^{-tA}$ semi-group

• $f(\lambda) = \lambda^\alpha, 0 < \alpha < +\infty$: A^α fractional power

• $f(\lambda) = \log(1+\lambda)$: $\log(I + A)$

General Question:

- Assume a Nash type inequality holds for A:

$$\| f \|_2^2 \, B(\| f \|_2^2) \leq (Af, f), \quad \forall f \in \mathcal{D}, \ \| f \|_1 \leq 1$$

- Is it true that $g(A)$ satisfies a Nash type inequality for some function $g \geq 0$?

If yes: describe the function B_g s.t:

$$\| f \|_2^2 \ B_g(\| f \|_2^2) \leq (g(A)f, f)$$

$$\forall f \in \mathcal{D}_g, \quad \| f \|_1 \leq 1$$

In particular with:

$g(x) = x^\alpha, \quad 0 < \alpha < 1$: Fractional powers

$g(x) = \log(1 + x)$: log-semigroup

• Why are we so interested by $g(A)$?

If $g : [0, +\infty[\to [$ is real and $g \geq 0$

then $g(A)$ is also non negative self adjoint

operator (defined by spectral theory).

Then, we can define the associated semigroup

$$T_t^g = e^{-tg(A)}$$

Locally Compact Abelian Groups Setting

Let G be a l.c.a group with dual group $\widehat{G}$.

(g is Bernstein function)

• g is a Bernstein function if $g :]0, +\infty[\to [0, +\infty[$,

g is C^∞ and $(-1)^p f^{(p)} \leq 0 \quad \forall p \in \mathbb{N} - \{0\}$.

• **Examples:**

$g(x) = 1 - e^{-sx}, \quad s > 0$: Poisson sgr. with jump s

$g(x) = x^\alpha, \quad 0 < \alpha < 1$: Fractional powers

$g(x) = \log(1 + x)$: Γ-semigroup

• Why are we so interested by $g(A)$?

Let μ_t be a convolution semigroup on G given by

$$\widehat{\mu_t}(y) = e^{-t\psi(y)}, \;\; y \in \widehat{G}$$

with ψ the associated **continuous negative**

definite function.

Then $(go\psi)$ is a **continuous negative definite function** $\Leftrightarrow \left(\mu_t^g\right)^{\widehat{}}(y) = e^{-t\,(go\psi)(y)}, \;\; y \in \widehat{G}$

is also a convolution semigroup !

Representation formula for Berstein function

$$g(x) = a + bx + \int_0^{+\infty} (1 - e^{-xs})\, d\mu(s)$$

with μ a positive measure on $[0, +\infty[$

$$\int_0^1 s\, d\mu(s) < \infty \quad \int_1^{+\infty} d\mu(s) < \infty$$

Examples:

- $g(x) = x^\alpha$ $(0 < \alpha < 1)$ then $g(A) = A^\alpha$ is the fractional operator.

- $g(x) = \log(1 + x)$ i.e $g(A) = \log(I + A)$.

The operator $g(A)$ generates a Markov semigroup.

Is it true that $g(A)$ satisfies a Nash type inequality?

The answer is yes with A^α.

Open problem: Nash inequality for $\log(I + A)$

(Partial results in $\mathbb{R}^n$ and also in the abstract setting)

The Assumptions:

• Let (X, μ) be a measure space with σ-finite measure μ.

• Let A be a non-negative self-adjoint operator with domain $\mathcal{D}(A) \subset L^2(X, \mu)$.

• Suppose that the semigroup $T_t = e^{-tA}$ acts as a contraction on $L^1(X, \mu)$

• Let $B : [0, +\infty[\to [0, +\infty[$ be a non-decreasing function which tends to infinity at infinity.

Theorem(A.Bendikov,P.M)

If the operator A satisfies

$$|| f ||_2^2 \, B \left(|| f ||_2^2 \right) \leq (Af, f)$$

$$\forall f \in \mathcal{D}(A), \;\; || f ||_1 \leq 1$$

Then, for any $\alpha > 0$,

$$|| f ||_2^2 \left[B \left(|| f ||_2^2 \right) \right]^{\alpha} \leq (A^{\alpha} f, f)$$

$$\forall f \in \mathcal{D}(A^{\alpha}), \;\; || f ||_1 \leq 1.$$

"Picture" :

Operators: Functions:

$$A \quad - - - - - - - > B(x)$$

$$A^{\alpha} \quad - - - - - - - > B^{\alpha}(x)$$