Evolution equation on networks with stochastic inputs

Stefano Bonaccorsi (Univ. Trento)

January 19, 2009

Introduction

We shall discuss some mathematical models of a complete neuron subject to stochastic perturbations.

A reference model for the whole neuronal network has been recently introduced by Cardanobile and Mugnolo (2007).

We treat the neuron as a simple graph with different kind of (stochastic) evolutions on the edges and dynamic Kirchhoff-type condition on the central node (the soma).

This approach is made possible by recent developments of techniques of network evolution equations; as opposite to most of the papers in the literature, which concentrate on some parts of the neuron, could it be the dendritic network, the soma or the axon, we take into account the complete cell.

The model

In this talk, we schematize a neuron as a network by considering

- a FitzHugh-Nagumo (nonlinear) system on the axon, coupled with
- a linear (Rall) model for the dendritical tree, complemented with
- Kirchhoff-type rule in the soma.

Notes

Shortly after the publication of Hodgkin and Huxley's model for the diffusion of electric potential in the squid giant axon, a more analytically treatable model was proposed by FitzHugh and Nagumo; the model is able to catch the main mathematical properties of excitation and propagation using

- a voltage-like variable having cubic nonlinearity that allows regenerative self-excitation via a positive feedback, and
- a recovery variable having a linear dynamics that provides a slower negative feedback.

In our model the axon has length ℓ, i.e. the space variable x in the above equations ranges in an interval $(0, \ell)$, where the soma (the cell body) is identified with the point 0.

The model

In this talk, we schematize a neuron as a network by considering

- a FitzHugh-Nagumo (nonlinear) system on the axon, coupled with
- a linear (Rall) model for the dendritical tree, complemented with
- Kirchhoff-type rule in the soma.

Notes

It is commonly accepted that dendrites conduct electricity in a passive way. The well known Rall's model simplify the analysis of this part by considering a simpler, concentrated "equivalent cylinder" (of finite length ℓ_d) that schematizes a dendritical tree.

The model

In this talk, we schematize a neuron as a network by considering

- a FitzHugh-Nagumo (nonlinear) system on the axon, coupled with
- a linear (Rall) model for the dendritical tree, complemented with
- Kirchhoff-type rule in the soma.

Notes

The soma is assumed to be isopotential; it represents a boundary point both for the axon and for the dendritic tree and shall be complemented by a suitable dynamical condition.

The noise

There is a large evidence in the literature that realistic neurobiological models shall incorporate stochastic terms to model real inputs. It is classical to model the random perturbation with a Wiener process, as it comes from a central limit theorem applied to a sequence of independent random variables.

The noise

There is a large evidence in the literature that realistic neurobiological models shall incorporate stochastic terms to model real inputs. It is classical to model the random perturbation with a Wiener process, as it comes from a central limit theorem applied to a sequence of independent random variables.

However, there is a considerable interest in literature for different kind of noises: we shall mention long-range dependence processes and self-similar processes, as their features better model the real inputs. Further, they can be justified theoretically as they arise in the so called Non Central Limit Theorem, see for instance Taqqu (1975) or Dobrushin and Major (1979).

The noise

There is a large evidence in the literature that realistic neurobiological models shall incorporate stochastic terms to model real inputs. It is classical to model the random perturbation with a Wiener process, as it comes from a central limit theorem applied to a sequence of independent random variables.

However, there is a considerable interest in literature for different kind of noises: we shall mention long-range dependence processes and self-similar processes, as their features better model the real inputs. Further, they can be justified theoretically as they arise in the so called Non Central Limit Theorem, see for instance Taqqu (1975) or Dobrushin and Major (1979).

The fractional Brownian motion is of course the most studied process in the class of Hermite processes due to its significant importance in modeling. It is not only selfsimilar, but also exhibits long-range dependence, i.e., the behaviour of the process at time t does depend on the whole history up to time t, stationarity of the increments and continuity of trajectories.

The noise

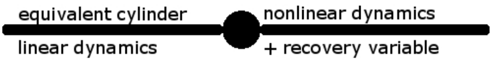
There is a large evidence in the literature that realistic neurobiological models shall incorporate stochastic terms to model real inputs. It is classical to model the random perturbation with a Wiener process, as it comes from a central limit theorem applied to a sequence of independent random variables.

However, there is a considerable interest in literature for different kind of noises: we shall mention long-range dependence processes and self-similar processes, as their features better model the real inputs. Further, they can be justified theoretically as they arise in the so called Non Central Limit Theorem, see for instance Taqqu (1975) or Dobrushin and Major (1979).

In different models, the electrical activity of background neurons is subject to a stochastic input of impulsive type, which takes into account the stream of excitatory and inhibitory action potentials coming from the neighbors of the network. The need to use models based on impulsive noise was already pointed out in several papers by Kallianpur and coauthors (1984–1995).

FitzHugh-Nagumo model

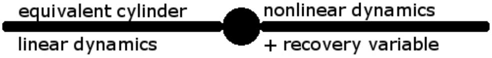
In the following, as long as we allow for variable coefficients in the diffusion operator, we can let the edges of the neuronal network to be described by the interval $[0, 1]$.



The general form of the equation we are concerned with can be written as a system in the space $\mathbb{X} = (L^2(0,1))^2 \times \mathbb{R} \times L^2(0,1)$ for the unknowns (u, u_d, d, v):

FitzHugh-Nagumo model

In the following, as long as we allow for variable coefficients in the diffusion operator, we can let the edges of the neuronal network to be described by the interval $[0, 1]$.



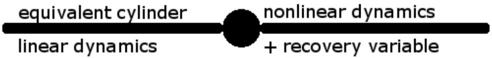
The general form of the equation we are concerned with can be written as a system in the space $\mathbb{X} = (L^2(0,1))^2 \times \mathbb{R} \times L^2(0,1)$ for the unknowns (u, u_d, d, v):

The Rall's model for the linear dynamics on the dendritic tree

$$\frac{\partial}{\partial t} u_d(t,x) = \frac{\partial}{\partial x}\left(c_d(x)\frac{\partial}{\partial x} u_d(t,x)\right) - p_d(x)u_d(t,x) + \frac{\partial}{\partial t}\zeta^d(t,x)$$

FitzHugh-Nagumo model

In the following, as long as we allow for variable coefficients in the diffusion operator, we can let the edges of the neuronal network to be described by the interval $[0, 1]$.



The general form of the equation we are concerned with can be written as a system in the space $\mathbb{X} = (L^2(0,1))^2 \times \mathbb{R} \times L^2(0,1)$ for the unknowns (u, u_d, d, v):

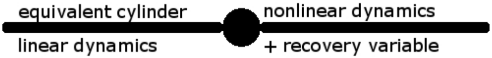
The FitzHugh-Nagumo model for the nonlinear dynamics along the axon

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial}{\partial x}\left(c(x)\frac{\partial}{\partial x}u(t,x)\right) - p(x)u(t,x) - v(t,x) + \theta(u(t,x)) + \frac{\partial}{\partial t}\zeta^u(t,x)$$

$$\frac{\partial}{\partial t}v(t,x) = u(t,x) - \epsilon v(t,x) + \frac{\partial}{\partial t}\zeta^v(t,x)$$

FitzHugh-Nagumo model

In the following, as long as we allow for variable coefficients in the diffusion operator, we can let the edges of the neuronal network to be described by the interval $[0, 1]$.



The general form of the equation we are concerned with can be written as a system in the space $\mathbb{X} = (L^2(0,1))^2 \times \mathbb{R} \times L^2(0,1)$ for the unknowns (u, u_d, d, v):

The continuity assumption on the soma

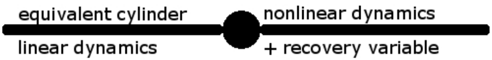
$$d(t) = u(t,0) = u_d(t,1), \qquad t \geq 0$$

and the corresponding dynamic boundary condition

$$\frac{\partial}{\partial t} d(t) = -\gamma d(t) - \left(c(0) \frac{\partial}{\partial x} u(t,0) - c_d(1) \frac{\partial}{\partial x} u_d(t,1) \right)$$

FitzHugh-Nagumo model

In the following, as long as we allow for variable coefficients in the diffusion operator, we can let the edges of the neuronal network to be described by the interval $[0, 1]$.



The general form of the equation we are concerned with can be written as a system in the space $\mathbb{X} = (L^2(0,1))^2 \times \mathbb{R} \times L^2(0,1)$ for the unknowns (u, u_d, d, v):

Neumann boundary conditions on the free ends

$$\frac{\partial}{\partial x} u(t,1) = 0, \qquad \frac{\partial}{\partial x} u_d(t,0) = 0, \qquad t \geq 0$$

FitzHugh-Nagumo model

The nonlinear term

The function $\theta : \mathbb{R} \to \mathbb{R}$, in the classical model of FitzHugh, is given by $\theta(u) = u(1-u)(u-\xi)$ for some $\xi \in (0,1)$; it satisfies a dissipativity condition of the following form: there exists $\lambda \geq 0$ such that

$$\begin{aligned} &\text{for } h(u) = -\lambda u + \theta(u) \text{ it holds} \\ &[h(u)-h(v)](u-v) \leq 0 \quad \forall\, u, v \in \mathbb{R}, \\ &|h(u)| \leq c(1+|u|^{2\rho+1}), \quad \rho \in \mathbb{N}, \end{aligned} \tag{1}$$

with $\lambda = \frac{1}{3}(\xi^2 - \xi + 1)$. Other examples of nonlinear conditions are known in the literature, see for instance Izhikevich (2004) and the references therein.

Abstract formulation

We aim to express the problem in an abstract form in the Hilbert space $\mathbb{X} = (L^2(0,1))^2 \times \mathbb{R} \times L^2(0,1)$. We also introduce the Banach space $\mathbb{Y} = (C([0,1]))^2 \times \mathbb{R} \times L^2(0,1)$ that is continuously (but not compactly) embedded in $\mathbb{X}$.

At first, we prove that the $\boxed{\text{linear part of the system}}$ defines a linear, unbounded operator $\mathbb{A}$ that generates on $\mathbb{X}$ an $\boxed{\text{analytic semigroup}}$.

On the domain

$$D(\mathbb{A}) := \left\{ \begin{aligned} &\mathfrak{v} := (u, v, d, u_d)^\top \in (H^2(0,1))^2 \times \mathbb{R} \times L^2(0,1) \\ &\qquad\text{s. th.} \quad u(0) = u_d(1) = d, \quad u'(1) = 0, \\ &\qquad u_d'(0) = 0, \quad c(0)u'(0) + c_d(1)u_d'(1) = 0 \end{aligned} \right\} \tag{2}$$

we define the operator $\mathbb{A}$ by setting

$$\mathbb{A}\mathfrak{v} := \begin{pmatrix} (cu')' - pu + \lambda u - v \\ (c_d u_d')' - p_d u_d \\ -\gamma d - (c(0)u'(0) - c_d(1)u_d'(1)) \\ u - \epsilon v \end{pmatrix} \tag{3}$$

Abstract formulation

Setting $B(t) = (\zeta^u(t), \zeta^v(t), 0, \zeta^d(t))^\top$, we obtain the abstract stochastic Cauchy problem

$$\begin{cases} \mathrm{d}\mathfrak{v}(t) = [\mathbb{A}\mathfrak{v}(t) + \mathbb{F}(\mathfrak{v}(t))\,\mathrm{d}t + \mathrm{d}B(t), & t \geq 0, \\ \mathfrak{v}(0) = \mathfrak{v}_0, \end{cases} \tag{4}$$

where the initial value is given by $\mathfrak{v}_0 := (u_0, v_0, u_0(0), u_{d;0})^\top \in \mathbb{X}$.

Theorem

The proposed model for a neuron cell, endowed with a stochastic input that satisfies certain natural conditions, has a unique solution on the time interval $[0, T]$, for arbitrary $T > 0$. In particular, it is a mean square continuous process which belongs to $L^2_{\mathcal{F}}(\Omega; L^2([0, T]; \mathbb{Y}))$ and depends continuously on the initial condition.

Abstract formulation

The above theorem does not have a unique reference. According to different kind of noises, it ir proved in B., Marinelli and Ziglio (2008), B. and Mugnolo (2008) or B. and Tudor (2009); this last paper, in particular, is the main reference for the model we are discussing here. Some examples of possible stochastic input which is treated in the above-mentioned papers are:

- a pure jump Lévy process $\{L_t,\ t \geq 0\}$, i.e., a stochastically continuous, adapted process starting almost surely from 0, with stationary and independent increments and càdlàg trajectories;
- a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$,
- or a bifractional Brownian motion with $H > \frac{1}{2}$ and $K \geq 1/2H$,
- or an Hermite process with selfsimilarity order $H > \frac{1}{2}$,
- or, more generally, a continuous process whise covariance function satisfies the following condition:

$$\left|\frac{\partial R}{\partial s \partial t}(s,t)\right| \leq c_1 |t-s|^{2H-2} + g(s,t)$$

for every $s, t \in [0,T]$ where $|g(s,t)| \leq c_2 (st)^\beta$ with $\beta \in (-1,0)$, $H \in (\frac{1}{2}, 1)$ and c_1, c_2 are strictly positive constant.

Well-posedness of the linear system

There exist some results in the literature concerning well-posedness and further qualitative properties of our system: the main references here are the papers by Cardanobile and Mugnolo (2007), Mugnolo and Romanelli (2007), Mugnolo (2007).

Our first remark is that, neglecting the recovery variable v, the (linear part of the) system for the unknown (u, u_d, d) is a diffusion equation on a network with *dynamical boundary conditions*:

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} c(x) u(t,x)\right) - p(x) u(t,x) + \lambda u(t,x) \\ \frac{\partial}{\partial t} u_d(t,x) &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} c_d(x) u_d(t,x)\right) - p_d(x) u_d(t,x) \\ \frac{\partial}{\partial t} d(t) &= -\gamma d(t) - \left(c(0)\frac{\partial}{\partial x} u(t,0) - c_d(1)\frac{\partial}{\partial x} u_d(t,1)\right) \end{aligned} \tag{5}$$

Well-posedness of the linear system

There exist some results in the literature concerning well-posedness and further qualitative properties of our system: the main references here are the papers by Cardanobile and Mugnolo (2007), Mugnolo and Romanelli (2007), Mugnolo (2007).

On the space $\mathcal{X} = (L^2(0,1))^2 \times \mathbb{R}$ we introduce the operator

$$\mathcal{A}\begin{pmatrix} u \\ u_d \\ d \end{pmatrix} = \begin{pmatrix} (cu')' - pu + \lambda u \\ (c_d u_d')' - p_d u_d \\ -\gamma_1 d - (c(0)u'(0) - c_d(1)u_d'(1)) \end{pmatrix}$$

with *coupled* domain

$$D(\mathcal{A}) = \{(u, u_d, d)^\top \in (H^2(0,1))^2 \times \mathbb{C} \,:\, u(0) = u_d(1) = d\}$$

Theorem

The operator $(\mathcal{A}, D(\mathcal{A}))$ is self-adjoint and dissipative and it has compact resolvent; by the spectral theorem, it generates a strongly continuous, analytic and compact semigroup $(\mathcal{S}(t))_{t\geq 0}$ on $\mathcal{X}$.

Well-posedness of the linear system

The next step is to discuss the operator $\mathbb{A}$ on the space $\mathbb{X} = \mathcal{X} \times L^2(0,1)$. We can think $\mathbb{A}$ as a matrix operator in the form

$$\mathbb{A} = \begin{pmatrix} \mathcal{A} & -P_1 \\ P_1^\top & -\epsilon \end{pmatrix}$$

where P_1 is the immersion on the first coordinate of $\mathcal{X}$: $P_1 v = (v, 0, 0)^\top$, while $P_1^\top (u, u_d, v)^\top = u$.

Well-posedness of the linear system

The next step is to discuss the operator $\mathbb{A}$ on the space $\mathbb{X} = \mathcal{X} \times L^2(0,1)$. We can think $\mathbb{A}$ as a matrix operator in the form

$$\mathbb{A} = \begin{pmatrix} \mathcal{A} & -P_1 \\ P_1^\top & -\epsilon \end{pmatrix}$$

where P_1 is the immersion on the first coordinate of $\mathcal{X}$: $P_1 v = (v, 0, 0)^\top$, while $P_1^\top (u, u_d, v)^\top = u$.

In order to prove the generation property of the operator $\mathbb{A}$, we introduce the Hilbert space

$$\mathbb{V} := \left\{ \begin{aligned} \mathfrak{v} := (u, u_d, d, v)^\top \in (H^1(0,1))^2 \times \mathbb{R} \times L^2(0,1) \quad \text{s. th.} \\ u(0) = u_d(1) = d \end{aligned} \right\}$$

and the sesquilinear form $\mathfrak{a} : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ associated, in a natural way, with $\mathbb{A}$. Using techniques from the theory of sesquilinear forms (see for instance Ouhabaz (2005)) we obtain the following result.

Well-posedness of the linear system

The next step is to discuss the operator $\mathbb{A}$ on the space $\mathbb{X} = \mathcal{X} \times L^2(0,1)$. We can think $\mathbb{A}$ as a matrix operator in the form

$$\mathbb{A} = \begin{pmatrix} \mathcal{A} & -P_1 \\ P_1^\top & -\epsilon \end{pmatrix}$$

where P_1 is the immersion on the first coordinate of $\mathcal{X}$: $P_1 v = (v, 0, 0)^\top$, while $P_1^\top (u, u_d, v)^\top = u$.

Theorem

The operator $\mathbb{A}$ generates a strongly continuous, analytic semigroup $(S(t))_{t \geq 0}$ on the Hilbert space $\mathbb{X}$ that is uniformly exponentially stable: there exist $M \geq 1$ and $\omega > 0$ such that $\|S(t)\|_{L(\mathbb{X})} \leq M e^{-\omega t}$ for all $t \geq 0$.

Well-posedness of the linear system

Notice that the operator $\mathbb{A}$ is not self-adjoint, as the corresponding form $\mathfrak{a}$ is not symmetric; also, since $\mathbb{V}$ is not compactly embedded in $\mathbb{X}$, it is easily seen that the semigroup generated by $\mathbb{A}$ is not compact hence it is not Hilbert-Schmidt.

The form domain

The form domain $\mathbb{V}$ is isometric to the fractional domain power $D((-A)^{1/2})$. This follows since the numerical range of the form $\mathfrak{a}$ is contained in a parabola, compare Cardanobile and Mugnolo (2007), Corollary 6.2, and then by an application of a known result of McIntosh (1982).

Well-posedness of the linear system

Notice that the operator $\mathbb{A}$ is not self-adjoint, as the corresponding form $\mathfrak{a}$ is not symmetric; also, since $\mathbb{V}$ is not compactly embedded in $\mathbb{X}$, it is easily seen that the semigroup generated by $\mathbb{A}$ is not compact hence it is not Hilbert-Schmidt.

The form domain

Coercivity

The form $\mathfrak{a}$ is real-valued and coercive, hence

$$\langle -\mathbb{A}u, u \rangle = \mathfrak{a}(u, u) \geq \omega \|u\|_{\mathbb{V}}^2$$

for some $\omega > 0$.

Well-posedness of the linear system

Notice that the operator $\mathbb{A}$ is not self-adjoint, as the corresponding form $\mathfrak{a}$ is not symmetric; also, since $\mathbb{V}$ is not compactly embedded in $\mathbb{X}$, it is easily seen that the semigroup generated by $\mathbb{A}$ is not compact hence it is not Hilbert-Schmidt.

The form domain

Coercivity

Although we shall not use directly the next result in this paper, we can characterize further the specturm of $\mathbb{A}$ in the complex plane.

Lemma

The spectrum of $\mathbb{A}$ in the complex plane is contained in the union of the (discrete, real and negative) spectrum of $\mathcal{A}$ and a bounded B.

Additive stochastic perturbation in the nodes

Let us sketch the model in B. and Mugnolo (2008). The starting point here is the analysis of non-deterministic aspects of sub-threshold stochastic behaviour, either in passive or active fibers. There seem to be good reasons to perform such an analysis: in particular, *many computations putatively performed in the dendritic tree (coincidence detection, multiplication, synaptic integration and so on) occur in the sub-threshold regime* (quoted from Manwani, Steinmetz and Koch (2000)).

We consider a further simplified model, where only the (equivalent cylinder for the) dentritic tree occours. We have only passive cable conduction on the edge and two nodes, of which one is passive (where we impose standard Kirchhoff's conditions) and the other is active, i.e., we impose conditions of the form

$$\frac{d}{dt} q(t) = \epsilon u_d'(t, \mathrm{v}_i) - bq(t) + c\dot{Z}(t)$$

Additive stochastic perturbation in the nodes

We express the problem in an abstract form, on the space $\mathbb{X} = L^2(0,1) \times \mathbb{R} \times \mathbb{R}$, with leading operator that can be given in matrix form as $\mathbb{A} = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{pmatrix}$ where $\mathcal{A} = \begin{pmatrix} \frac{d^2}{dx^2} - p & 0 \\ K & -b \end{pmatrix}$ is the operator on the space $\mathcal{X} = L^2(0,1) \times \mathbb{R}$ of diffusion on edges and active nodes only. It is possible to prove that, under some rather mild conditions on the coefficients, $\mathcal{S}(t)$ is a strongly continuous, analytic and compact semigroup, uniformly exponentially stable. Notice however that the semigroup $\mathbb{S}(t)$ is not even stable, since it acts as the identity on the last component.

Additive stochastic perturbation in the nodes

We express the problem in an abstract form, on the space $\mathbb{X} = L^2(0,1) \times \mathbb{R} \times \mathbb{R}$

We examine the case of a system featuring the presence of (stochastic) inputs (possibly also in the passive nodes). We introduce $Z(t)$ to be the 2-dimensional stochastic process which models the input in the (active and passive) nodes, and $\mathbb{C} = (0\ C_a\ C_p)^T$ be the covariance operator of $Z(t)$. Then the stochastic model can be written in the form

$$\begin{aligned} \mathrm{d}u(t) &= \mathbb{A}u(t)\,\mathrm{d}t + \mathbb{C}\,\mathrm{d}Z(t) \\ u(0) &= u_0 \end{aligned} \tag{6}$$

which is solved in mild form by

$$u(t) = \mathbb{S}(t)u_0 + \int_0^t \mathbb{S}(t-s)\mathbb{C}\,\mathrm{d}Z(s). \tag{7}$$

Additive stochastic perturbation in the nodes

Theorem

Assume that $u_0 \in L^2(0,1) \times \mathbb{R} \times \mathbb{R}$. Then the process $u = \{u(t),\ t \in [0,T]\}$ is a weak solution of (6).
Assume further that $u_0 \in D(\mathbb{A})$ and $H > 3/4$. Then the weak solution is a strong solution of (6).

(6) is a stochastic equation in infinite dimensional spaces with additive (finite dimensional) noise. Existence of strong solution is not usual even for Wiener noise.

1. Our result requires $H > 3/4$, together with the ambientation $W(t) \in D((-A)^\alpha)$ for any $\alpha < 1/4$, which leads to $H + \alpha > 1$. It is the analog of the assumption $W(t) \in D((-A)^\alpha)$ for some $\alpha > 1/2$ required in Barbu, Da Prato and Roeckner (2007) (since the Wiener case corresponds to $H = 1/2$).
2. Our techniques are more similar to those in Karchewska and Lizama (2008) altough they require the stronger condition $W(t) \in D(A)$.

Additive stochastic perturbation in the nodes

From the representation formula (7), using the explicit form of the semigroup $\mathbb{S}(t)$, we get: $u(t) = \begin{pmatrix} u_a(t) \\ C_p Z(t) \end{pmatrix}$ where

$$u_a(t) = \int_0^t \mathcal{S}(t-s)\,\mathcal{C}_a\,\mathrm{d}Z(s) + \int_0^t (I - \mathcal{S}(t-s))\mathcal{D}_0^{\mathcal{A},R} C_p\,\mathrm{d}Z(s).$$

Additive stochastic perturbation in the nodes

From the representation formula (7), using the explicit form of the semigroup $\mathbb{S}(t)$, we get: $u(t) = \begin{pmatrix} u_a(t) \\ C_p Z(t) \end{pmatrix}$ where

$$u_a(t) = \int_0^t \mathcal{S}(t-s)\,\mathcal{C}_a \,\mathrm{d}Z(s) + \int_0^t (I - \mathcal{S}(t-s))\mathcal{D}_0^{\mathcal{A},R} C_p \,\mathrm{d}Z(s).$$

Theorem

If the the matrix C_p is identically zero and some dissipativity acts in the system (for instance, $p > 0$), then there does exist a unique invariant probability measure for the system.

Conclusion

In different terms, there exists an equilibrium state (which can be thought of as resulting in the long term behavior) for the neuronal network. It is interesting, in this connection, to study large deviations to estimate the probability of onsets of chaotic impulses, compare Ringach and Malone (2007) or B. and Mastrogiacomo (2008).

Additive stochastic perturbation in the nodes

From the representation formula (7), using the explicit form of the semigroup $\mathbb{S}(t)$, we get: $u(t) = \begin{pmatrix} u_a(t) \\ C_p Z(t) \end{pmatrix}$ where

$$u_a(t) = \int_0^t \mathcal{S}(t-s)\,\mathcal{C}_a\,\mathrm{d}Z(s) + \int_0^t (I - \mathcal{S}(t-s))\mathcal{D}_0^{\mathcal{A},R} C_p\,\mathrm{d}Z(s).$$

Whenever the matrix C_p is not identically zero, we are again in presence of a gaussian process with finite trace class covariance operator at any time but not bounded as $t \to \infty$.

Corollary

If the behaviour of passive nodes is affected by some stochastic inputs (i.e., the matrix C_p is not identically zero), then there does not exist an invariant probability measure for the system.

Other properties of the system

Large deviation principle

The model in B. and Mastrogiacomo (2008) is a finite network. We identify every node with a soma while edges are equivalent cylinders (model the interactions between different neurons); we allow the drift term in the cable equation to include, in particular, dissipative functions of the FitzHugh-Nagumo type proposed in various models of neurophysiology.

The chaotic behaviour of the surroundings is modeled with a Wiener noise; the membrane's potential in the soma follows a nonlinear Ornstein-Uhlenbeck type process (dissipation + noise + Kirchhoff's type perturbation).

Our interest: small noise asymptotic of the system. Motivation: recent researches in *in-vivo* neuronal activities. Ringach and Malone (2007) states: "cortical cells behave like large deviation detectors". We estimate the probability that some of the neurons develop an action potential in presence of a background stochastic noise. We show that this probability decays exponentially as the intensity of the noise decreases.

Impulsive noise

Let us consider the case when the stochastic perturbation acting on the node, due to the external surrounding, is an additive, finite dimensional impulsive noise of the form $L(t) = \int x \, \tilde{N}(t, \mathrm{d}x)$ where we suppose that $L(t)$ has finite first and second moments.
For simplicity, we drop again the recovery variable; however, in this case it is of some interest to consider the nonlinear equation

$$\mathrm{d}X(t) = [\mathcal{A}X(t) + \mathcal{F}(X(t))] \, \mathrm{d}t + \Sigma \, \mathrm{d}L(t), \qquad X(0) = x_0 \tag{8}$$

where $\Sigma = (0, 1)^\star$. In accordance with the notation used before, we have $X(t) = (u(t), d(t))$.

Impulsive noise

Let us consider the case when the stochastic perturbation acting on the node, due to the external surrounding, is an additive, finite dimensional impulsive noise of the form $L(t) = \int x\,\tilde{N}(t, \mathrm{d}x)$ where we suppose that $L(t)$ has finite first and second moments.
For simplicity, we drop again the recovery variable; however, in this case it is of some interest to consider the nonlinear equation

$$\mathrm{d}X(t) = [\mathcal{A}X(t) + \mathcal{F}(X(t))]\,\mathrm{d}t + \Sigma\,\mathrm{d}L(t), \qquad X(0) = x_0 \tag{8}$$

where $\Sigma = (0, 1)^\star$.

Definition of solution

An $\mathcal{H}$-valued predictable process $X = \{X(t),\ t \in [0, T]\}$ is a *mild solution* of (8) if $\int_0^T |\mathcal{F}(X(t))|\,\mathrm{d}t < +\infty$ and

$$X(t) = \mathcal{S}(t)x_0 + \int_0^t \mathcal{S}(t-s)\mathcal{F}(X(s))\,\mathrm{d}s + \int_0^t \mathcal{S}(t-s)\Sigma\,\mathrm{d}L(s).$$

Existence of the solution

In previous slide we have seen the stochastic convolution $Z(t) = \int_0^t \mathcal{S}(t-s)\Sigma\,\mathrm{d}L(s)$. Notice that it is an infinite dimensional process (altough the noise is finite dimensional).

Lemma

$Z(t)$ is a predictable process, mean square continuous, taking values in the space $C(0,1)\times\mathbb{R}$. Further, $Z(t)$ has cádlág paths.

Existence of the solution

In previous slide we have seen the stochastic convolution $Z(t) = \int_0^t \mathcal{S}(t-s)\Sigma \, \mathrm{d}L(s)$. Notice that it is an infinite dimensional process (altough the noise is finite dimensional).

Lemma

$Z(t)$ is a predictable process, mean square continuous, taking values in the space $C(0,1) \times \mathbb{R}$. Further, $Z(t)$ has cádlág paths.

Now we are concerned with the existence of a solution for equation (8). We first consider the case $\mathcal{F}$ is a Lipschitz continuous mapping; the proof of the result follows from a fixed point argument.

Theorem

There exists a unique solution $X \in C([0,T]; L^2(\Omega; \mathcal{H}))$ which depends continuously on the initial data.

Existence of the solution

In previous slide we have seen the stochastic convolution $Z(t) = \int_0^t \mathcal{S}(t-s)\Sigma \, \mathrm{d}L(s)$. Notice that it is an infinite dimensional process (altough the noise is finite dimensional).

Lemma

$Z(t)$ is a predictable process, mean square continuous, taking values in the space $C(0,1) \times \mathbb{R}$. Further, $Z(t)$ has cádlág paths.

We next prove that the solution X has cádlág paths. We cannot adapt the factorization technique developed for Wiener integrals. We can appeal to Kotelenez (1984), since $\mathcal{A}$ is dissipative. One could also obtain this property proving the following a priori estimate, which might be interesting in its own right.

Theorem

Under previous assumptions, the unique mild solution of problem (8) verifies $\mathbb{E} \sup |X(t)|^2_{\mathcal{H}} < +\infty$.

We now consier the general case of a nonlinear quasi-dissipative drift term $\mathcal{F}$.

Theorem

Equation (8) has a unique mild solution X which satisfies the estimate $\mathbb{E}|X(t,x) - X(t,y)|^2 \leq e^{2\eta t}|x-y|^2$ *for all* $x, y \in \mathcal{H}$.

Remark

Existence of the solution

We now consier the general case of a nonlinear quasi-dissipative drift term $\mathcal{F}$.

Theorem

Equation (8) has a unique mild solution X which satisfies the estimate $\mathbb{E}|X(t,x) - X(t,y)|^2 \leq e^{2\eta t}|x-y|^2$ for all $x, y \in \mathcal{H}$.

Remark

The proof uses a monotonicity method and the existence result for Yosida approximations proved above.

Existence of the solution

We now consier the general case of a nonlinear quasi-dissipative drift term $\mathcal{F}$.

Theorem

Equation (8) has a unique mild solution X which satisfies the estimate $\mathbb{E}|X(t,x) - X(t,y)|^2 \le e^{2\eta t}|x - y|^2$ *for all* $x, y \in \mathcal{H}$.

Remark

There is in literature (see Da Prato and Zabczyk (1992) for the case of Wiener noise, and by Peszat and Zabczyk (2007) for the case of Lévy noise) an alternative approach, that consists essentially in the reduction of the stochastic PDE to a deterministic PDE with random coefficients, by “subtracting the stochastic convolution”.
Unfortunately, in the case of Lévy noise, it requires rather difficult conditions, that we have not been able to verify. On the other hand, our approach, while perhaps less general, yields the well-posedness result under seemingly natural assumptions.

Other properties of the system

In this section we discuss further properties of the system. We have only preliminary results so we shall only sketch them.

Optimal control

In the paper: B, Confortola and Mastrogiacomo (2008) we have studied an optimal control problem for the diffusion on an “equivalent cylinder” given by a nonlinear Rall’s model with dynamical boundary conditions and control acting on the boundary. This paper only deals with Wiener noise; extensions to general noises and general network dynamics are in progress.