Minimum contrast parametric estimator. Application to stochastic neuron models

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Introduction

Neuron models

- Integrate-fire model
- Morris-Lecar model
- Hodgkin-Huxley model
- ...

General problem

- Unknown parameters
- $\Rightarrow$ parametric estimation

Our approach

- Motivation: stochastic Morris-Lecar model
- General statistical method for parametric estimation of diffusion coefficient

Morris-Lecar model

Morris-Lecar model

$$\frac{dV_t}{dt} = -I_0(V_t, W_t) + I(t)$$
$$\frac{dW_t}{dt} = \phi\frac{w_\infty(V_t) - W_t}{\tau(V_t)}$$

$$I_0 = C^{-1}\left[g_{Ca}m_\infty(V_t)(V_t - V_{Ca}) + g_K W_t(V_t - V_K) + g_L(V_t - V_L)\right]$$

$$m_\infty(v) = \frac{1}{2}\left(1 + \tanh\left(\frac{v - V_1}{V_2}\right)\right)$$

Unknown parameters

- $\theta_1 = (C, g_{Ca}, V_{Ca}, g_K, V_k, g_L, V_L, V_1, V_2)$
- $\theta_2 =$ parameters of the functions $w_\infty(v)$, $\tau(v)$

Morris-Lecar model

Other version

$$\frac{dV_t}{dt} = f_{\theta_1}(V_t, W_t, I_t)$$
$$\frac{dW_t}{dt} = \alpha_{\theta_2}(V_t)(1 - W_t) - \beta_{\theta_2}(V_t)W_t$$

with $\alpha_{\theta_2}(V_t)$, $\beta_{\theta_2}(V_t)$ = rates of opening/closing gates in ionic channels

$$\alpha_{\theta_2}(V_t) = \phi\frac{w_\infty(V_t)}{\tau(V_t)}, \quad \beta_{\theta_2}(V_t) = \phi\frac{1 - w_\infty(V_t)}{\tau(V_t)}$$

Morris-Lecar with noise

Pakdaman, Thieullen, Wainrib (2009)

- family of N hybrid stochastic processes indexed by N, number of channels
- When $N \to \infty$
  - Morris-Lecar as deterministic limit
- When N finite and large
  - Langevin approximation
  - Stochastic differential system

Stochastic Morris-Lecar (SML) model

Stochastic Morris-Lecar model

$$
\begin{aligned}
dV_t &= f_{\theta_1}(V_t, W_t, I_t)dt \\
dW_t &= \left(\alpha_{\theta_2}(V_t)(1 - W_t) - W_t\beta_{\theta_2}(V_t)\right) dt \\
&\quad + \frac{1}{\sqrt{N}}\sqrt{\alpha_{\theta_2}(V_t)(1 - W_t) + W_t\beta_{\theta_2}(V_t)}dB_t
\end{aligned}
$$

with (B_t) Brownian process

Remarks

- Choice of volatility function
- Hypoelliptic system
- Integrated diffusion process

Observations

More general model

$$\begin{aligned} dV_t &= f_{\theta_1}(V_t, W_t, I_t)dt \\ dW_t &= b_{\theta_2}(V_t, W_t)dt + \sigma_{\theta_2}(V_t, W_t)dB_t \end{aligned}$$

Observations

- Incomplete: (W_t) not observed
- Discrete: (V_{t_i}) observed at times $t_0 = 0 \leq t_1 \leq \ldots \leq t_n = T$

Objective

- Estimation of parameters θ_1 and/or θ_2

Litterature

Parametric estimation of SDE

- Hypoelliptic system
  - Pokern and Stuart, 2009: Itô-Taylor approximation (no theoretical results)
- Integrated diffusion process
  - Gloter, 2000, 2001, 2006: minimum contrast estimator
  - Ditlevsen and Sørensen, 2004: prediction-based estimating functions
  - Baltazar-Larios and Sørensen, 2009: observations with measurement noise, EM algorithm

Non parametric estimation of SDE

- Höpfner, 2006: kernel estimator
- Comte et al, 2009: projection estimator

Parametric estimation

Classical approach: Maximum likelihood estimator

- Likelihood function: how much the parameters are likely to fit the data
- Maximum in the true parameter
- Likelihood for SDE
  - Explicit if continuous observations (Girsanov theorem)
  - Generally not explicit for discrete observations

Alternative to the likelihood

- Contrast function, minimal in the true parameter
- Minimum contrast estimator consistent for completely observed SDE

Difficulty with SML model

- Bias of the classical contrast due to incomplete observations
- Modified contrast

Discrete observations of SDE

Classical (multi-dimensional) SDE

$$dX_t = b_\theta(X_t)dt + \sigma_\theta(X_t)dB_t$$

with $\sigma_\theta(X_t) \neq 0$

Discrete observations (X_{t_i}) at times $t_0 = 0 \leq t_1 \leq \ldots \leq t_n = T$

Assumptions

- $t_{i+1} - t_i = \Delta$, for all $i = 1, \ldots, n$. Thus

$$t_i = i\Delta \quad \text{with} \quad \Delta = \Delta_n = \frac{T}{n}$$

- $n \longrightarrow \infty$ which implies $\Delta_n \longrightarrow 0$

Motivation case

Assumption: $b_\theta(x) \equiv 0$

$$dX_t = \sigma_\theta(X_t)dB_t$$

$\Rightarrow (X_t)$ is a Gaussian process

$$(X_{i\Delta} - X_{(i-1)\Delta}) \sim \mathcal{N}(0, \Delta\sigma_\theta^2(X_{(i-1)\Delta}))$$

Equivalent

$$X_{i\Delta}|X_{(i-1)\Delta} \sim \mathcal{N}(X_{(i-1)\Delta}, \Delta\sigma_\theta^2(X_{(i-1)\Delta}))$$

Diffusion coefficient estimator

Assumption: $b_\theta(x) \equiv 0$

Log-Likelihood for discrete observations

$$\log L_n(\theta) = \log p(X_{t_1}, \ldots, X_{t_n}; \theta) = \sum_{i=1}^{n} \log p(X_{t_i} | X_{t_{i-1}}; \theta)$$

$$\propto -\frac{1}{2} \sum_{i=1}^{n} \left[\frac{(X_{i\Delta} - X_{(i-1)\Delta})^2}{\Delta \sigma_\theta^2 \left(X_{(i-1)\Delta}\right)} + \log \left(\sigma_\theta^2 \left(X_{(i-1)\Delta}\right)\right) \right]$$

Maximum likelihood estimator

$$\hat{\theta}_n = \arg\max_\theta \log L_n(\theta)$$

Diffusion coefficient estimator

When $b_\theta(x) \neq 0$, likelihood untractable

Contrast

$$U_n(\theta) \propto \sum_{i=1}^{n} \frac{(X_{i\Delta} - X_{(i-1)\Delta})^2}{\Delta \sigma_\theta^2(X_{(i-1)\Delta})} + \sum_{i=1}^{n} \log(\sigma_\theta^2(X_{(i-1)\Delta}))$$

Minimum contrast estimator

$$\hat{\theta}_n = \arg\min_\theta U_n(\theta)$$

Tools for the convergence of contrast estimator

To prove the convergence of the contrast, we have to study quantities of the form

$$\sum_{i=1}^{n}(X_{i\Delta_n} - X_{(i-1)\Delta_n})^2 f(X_{(i-1)\Delta_n})$$

Heuristic

If $b_\theta(x) \equiv 0$

$$(X_{i\Delta} - X_{(i-1)\Delta})^2 \approx \Delta\sigma_\theta^2(X_{(i-1)\Delta}),$$

which gives

$$\sum_{i=1}^{n}(X_{i\Delta} - X_{(i-1)\Delta})^2 \approx \sum_{i=1}^{n}\Delta\sigma_\theta^2(X_{(i-1)\Delta}) \xrightarrow[n\to\infty]{} \int_0^T \sigma_\theta^2(X_s)ds$$

Quadratic variation

General SDE

$$dX_t = b_\theta(X_t)dt + \sigma_\theta(X_t)dB_t$$

Quadratic variation of X

If $\Delta_n \to 0$ when $n \to \infty$

$$\sum_{i=1}^{n}(X_{i\Delta_n} - X_{(i-1)\Delta_n})^2 \xrightarrow[n\to\infty]{\mathbb{P}} \int_0^T \sigma_\theta^2(X_s)ds$$

If f is a continuous function

$$\sum_{i=1}^{n}(X_{i\Delta_n} - X_{(i-1)\Delta_n})^2 f(X_{(i-1)\Delta_n}) \xrightarrow[n\to\infty]{\mathbb{P}} \int_0^T \sigma_\theta^2(X_s)f(X_s)ds$$

Properties of minimum contrast estimator

(Genon-Catalot and Jacod, 1993)

If $\Delta = \Delta_n = T/n \to 0$, θ_0 the true parameter

- the minimum contrast estimator is convergent

$$\hat{\theta}_n \xrightarrow[n\to\infty]{\mathbb{P}} \theta_0$$

- the minimum contrast estimator is asymptotically Gaussian

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \Gamma(\theta_0))$$

Incomplete observations

Problem: only incomplete observations of Stochastic Morris-Lecar model

New model, one step closer to SML

$$\begin{aligned} dV_t &= X_t dt \\ dX_t &= b_\theta(X_t)dt + \sigma_\theta(X_t)dB_t \end{aligned}$$

Observations (V_{t_i}) at times $t_0 = 0, t_1, \ldots, t_n = T$

- Aim: estimation of θ
- Problem with the classical contrast: $(X_{i\Delta})$ not observed

Idea: approximation of $(X_{i\Delta})$

(V_t) Integrated diffusion process

$$dV_t = X_t dt$$

gives

$$V_{t+h} - V_t = \int_t^{t+h} X_s ds$$

Thus

$$V_{i\Delta} - V_{(i-1)\Delta} \approx \Delta X_{(i-1)\Delta}$$

and

$$X_{(i-1)\Delta} \approx \frac{V_{i\Delta} - V_{(i-1)\Delta}}{\Delta} =: \frac{J_i}{\Delta}$$

Contrast

Classical contrast

$$U_n(\theta) \propto \sum_{i=1}^{n} \frac{(X_{i\Delta} - X_{(i-1)\Delta})^2}{\Delta \sigma_\theta^2(X_{(i-1)\Delta})} + \sum_{i=1}^{n} \log\left(\sigma_\theta^2(X_{(i-1)\Delta})\right)$$

Naive contrast

$X_{(i-1)\Delta}$ replaced by $\frac{J_i}{\Delta}$

$$U_n(\theta) \propto \sum_{i=1}^{n} \frac{\left(\frac{J_{i+1}}{\Delta} - \frac{J_i}{\Delta}\right)^2}{\Delta \sigma_\theta^2\left(\frac{J_i}{\Delta}\right)} + \sum_{i=1}^{n} \log\left(\sigma_\theta^2\left(\frac{J_i}{\Delta}\right)\right)$$

Bias of the naive contrast

Recall for the classic contrast

$$\sum_{i=1}^{n}(X_{i\Delta_n} - X_{(i-1)\Delta_n})^2 f(X_{(i-1)\Delta_n}) \xrightarrow[n\to\infty]{\mathbb{P}} \int_0^T \sigma_\theta^2(X_s) f(X_s) ds$$

Bias of the naive contrast (Gloter, 2006)

$$\sum_{i=1}^{n}\left(\frac{J_{i+1}}{\Delta_n} - \frac{J_i}{\Delta_n}\right)^2 f\left(\frac{J_i}{\Delta_n}\right) \xrightarrow[n\to\infty]{\mathbb{P}} \frac{2}{3}\int_0^T \sigma_\theta^2(X_s) f(X_s) ds$$

Modified contrast

Contrast (Gloter, 2006)

$$U_n(\theta) = \frac{3}{2} \sum_{i=1}^{n} \frac{\left(\frac{J_{i+1}}{\Delta_n} - \frac{J_i}{\Delta_n} \right)^2}{\Delta_n \sigma_\theta^2 \left(\frac{J_i}{\Delta_n} \right)} + \sum_{i=1}^{n} \log \left(\sigma_\theta^2 \left(\frac{J_i}{\Delta_n} \right) \right)$$

Minimum contrast estimator

$$\hat{\theta}_n = \arg\min_{\theta} U_n(\theta)$$

Minimum contrast estimator

Assumptions

- Regularity conditions on b and σ,
- $\Delta = \Delta_n = T/n \to 0$,

Properties (Gloter, 2006)

Set θ_0 the true parameter. Then

- the minimum contrast estimator is convergent

$$\hat{\theta}_n \xrightarrow[n\to\infty]{\mathbb{P}} \theta_0$$

- the minimum contrast estimator is asymptotically Gaussian

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \Gamma(\theta_0))$$

Extension for stochastic ML model

SML model

$$\begin{aligned} dV_t &= f_{\theta_1}(V_t, W_t, I_t)dt \\ dW_t &= b_{\theta_2}(V_t, W_t)dt + \sigma_{\theta_2}(V_t, W_t)dB_t \end{aligned}$$

Transformation of SML

Set

$$X_t = f_{\theta_1}(V_t, W_t, I_t)$$

Thus

$$dV_t = X_t dt$$

Problem: equation for X_t ?

SML as integrated diffusion process

Assumption

There exists g_{θ_1} such that

$$W_t = g_{\theta_1}(V_t, X_t, I_t)$$

SML viewed as integrated diffusion process

$$\begin{aligned} dV_t &= X_t dt \\ dX_t &= \tilde{b}_\theta(V_t, X_t, I_t)dt + \tilde{\sigma}_\theta(V_t, X_t, I_t)dB_t \end{aligned}$$

with

- $\theta = (\theta_1, \theta_2)$
- $\tilde{b}_\theta$ and $\tilde{\sigma}_\theta$ derived from functions $b_{\theta_2}, \sigma_{\theta_2}, g_{\theta_1}$ (Itô formula)

New contrast for SML model

Gloter contrast

$$U_n(\theta) = \frac{3}{2}\sum_{i=1}^{n} \frac{\left(\frac{J_{i+1}}{\Delta_n} - \frac{J_i}{\Delta_n}\right)^2}{\Delta_n \sigma_\theta^2\left(\frac{J_i}{\Delta_n}\right)} + \sum_{i=1}^{n} \log\left(\sigma_\theta^2\left(\frac{J_i}{\Delta_n}\right)\right)$$

New contrast

$$U_n(\theta) = \frac{3}{2}\sum_{i=1}^{n} \frac{\left(\frac{J_{i+1}}{\Delta_n} - \frac{J_i}{\Delta_n}\right)^2}{\Delta_n \tilde{\sigma}_\theta^2\left(V_{(i-1)\Delta_n}, I_{(i-1)\Delta_n}, \frac{J_i}{\Delta_n}\right)} + \sum_{i=1}^{n} \log\left(\tilde{\sigma}_\theta^2\left(V_{(i-1)\Delta_n}, I_{(i-1)\Delta_n}, \frac{J_i}{\Delta_n}\right)\right)$$

Minimum contrast estimator

Properties

If $\Delta_n = T/n \to 0$,

- the minimum contrast estimator is convergent

$$\hat{\theta}_n \xrightarrow[n\to\infty]{\mathbb{P}} \theta_0$$

- the minimum contrast estimator is asymptotically Gaussian

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \Gamma(\theta))$$

where

$$\Gamma(\theta) = \frac{9}{16}\left(\int_0^T \left(\frac{\frac{\partial}{\partial\theta}\tilde{\sigma}_{\theta_0}(V_s, I_s, X_s)}{\tilde{\sigma}_{\theta_0}(V_s, I_s, X_s)}\right)^2 ds\right)^{-1}$$

Discussion

Summary

- Stochastic Morris-Lecar model viewed as an integrated diffusion process
- Parametric estimation of diffusion coefficient

Perspectives

- Validation with simulated data
- Drift coefficient estimation (not the same asymptotic)
- Hypoelliptic approach
- Extension to the Hodgkin-Huxley model
- Extension to estimation of parameters for multi-time scale systems