

# A probabilistic study of neural complexity

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- ▶ G. Edelman, O. Sporns and G. Tononi [PNAS 1994] have proposed a definition of complexity for neural networks. This concept can be interpreted as a functional on probability laws on a finite space
- ▶ A complex (random) system should display a combination of high *differentiation* (local independence) and high *integration* (global correlation).
- ▶ The aim of our work is to explore this concept with the tools of mathematics and in particular to explain properties of random systems with high neural complexity.
- ▶ We study the order of magnitude of maximal neural complexity for fixed system size and the properties of maximizers as the system size grows to infinity.

Let  $X$  be a  $E$ -valued r.v. with  $E$  finite, e.g.  $E = \{0, 1\}$ . The entropy of  $X$  is

$$H(X) := - \sum_{x \in E} P_X(x) \log(P_X(x)), \quad P_X(x) := \mathbb{P}(X = x),$$

where we adopt the convention

$$0 \cdot \log(0) = 0 \cdot \log(+\infty) = 0.$$

The entropy is a measure of the randomness of  $X$ . We recall that

$$0 \leq H(X) \leq \log |E|,$$

with  $H(X) = 0$  iff  $X$  is constant and  $H(X) = \log |E|$  iff  $X$  is uniform.

# Mutual Information

Given a couple  $(X, Y)$  we have

$$H(X) \leq H(X, Y) \leq H(X) + H(Y)$$

and

1.  $H(X, Y) = H(X)$  iff  $Y$  is a function of  $X$
2.  $H(X, Y) = H(X) + H(Y)$  iff  $(X, Y)$  is independent.

Then we define the Mutual Information of  $(X, Y)$

$$MI(X, Y) := H(X) + H(Y) - H(X, Y) \geq 0.$$

MI is a measure of the dependence between  $(X, Y)$ , more precisely of the randomness shared by the couple.

Edelman-Sporns-Tononi consider a finite system of  $N = |I|$  r.v.  $X = (X_i)_{i \in I}$  with  $X_i \in \{0, 1\}$  and define the neural complexity as

$$\sum_{k=0}^N \frac{1}{\binom{N}{k}} \sum_{S \subset I, |S|=k} \text{MI}(X_S, X_{S^c}),$$

where

$$X_S := (X_i, i \in S), \quad X_{S^c} := (X_i, i \in S^c).$$

By convention,  $\text{MI}(X_\emptyset, X_I) = \text{MI}(X_I, X_\emptyset) = 0$ .

The neural complexity of  $X$  is zero whenever

1.  $X$  is an independent family (*chaos*)
2.  $X$  is a deterministic family (*order*).

We adopt rather the following definition

$$\mathcal{I}(X) := \frac{1}{N+1} \sum_{k=0}^N \frac{1}{\binom{N}{k}} \sum_{S \subset I, |S|=k} \text{MI}(X_S, X_{S^c}).$$

With this normalization, we have a first result

1.  $\mathcal{I}$  is exchangeable, i.e.  $\mathcal{I}(X)$  is invariant under permutations of  $(X_i)_{i \in I}$
2.  $\mathcal{I}$  is weakly additive, i.e.  $\mathcal{I}(X, Y) = \mathcal{I}(X) + \mathcal{I}(Y)$  whenever  $X$  and  $Y$  are independent

# Maximal neural complexity

It is easy to find systems with minimal (null) neural complexity. But what about systems with *maximal* neural complexity? This is harder. Let

$$\mathcal{I}_N := \sup\{\mathcal{I}(X) : X = (X_i)_{i \in I}, |I| = N\}.$$

By super-additivity we find  $\lim_{N \rightarrow \infty} \frac{\mathcal{I}_N}{N} = \sup_{N \geq 1} \frac{\mathcal{I}_N}{N}$ .

What is this limit?

We define

1.  $X = (X_1, \dots, X_N)$  is a maximizer if  $\mathcal{I}(X) = \mathcal{I}_N$
2.  $(X^N)_N$  is an approximate maximizer if

$$\lim_{N \rightarrow \infty} \frac{\mathcal{I}(X^N)}{N} = \lim_{N \rightarrow \infty} \frac{\mathcal{I}_N}{N}.$$

What do maximizers and approximate maximizers look like?

We can characterize maximizers only for  $N = 2, 3$ , since in this case it is possible to maximize each mutual information separately. For large  $N$  we know that

1. Exchangeable systems have small neural complexity. More precisely

$$\sup_{(X_1, \dots, X_N) \text{ exchangeable}} \mathcal{I}(X) = o(N^{2/3+\epsilon}), \quad N \rightarrow +\infty,$$

for any  $\epsilon > 0$ . In particular maximizers are neither unique nor exchangeable.

2. if  $X$  is a maximizer, then its support does not exceed a fixed proportion of the configuration space.

The first property is an example of a spontaneous symmetry breaking.

# Main result

1. For any sequence  $X^N = (X_1^N, \dots, X_N^N)$

$$\limsup_{N \rightarrow \infty} \frac{\mathcal{I}(X^N)}{N \log 2} \leq \frac{1}{4}.$$

2. For any sequence  $X^N = (X_1^N, \dots, X_N^N)$  such that

$$\lim_{N \rightarrow \infty} \frac{H(X^N)}{N \log 2} = x \in [0, 1],$$

we have

$$\limsup_{N \rightarrow \infty} \frac{\mathcal{I}(X^N)}{N \log 2} \leq x(1-x).$$

3. For all  $x \in [0, 1]$  there is at least a sequence  $X^N$  such that

$$\lim_{N \rightarrow \infty} \frac{H(X^N)}{N \log 2} = x, \quad \lim_{N \rightarrow \infty} \frac{\mathcal{I}(X^N)}{N \log 2} = x(1-x).$$

# Random Sparse Configurations

Let  $N \geq 2$  and  $1 \leq M \leq N$  be an integer. We denote

$$\Lambda_n := \{0, 1\}^n, \quad \forall n \geq 1.$$

We consider a family  $(W_i)_{i \in \Lambda_M}$  of i.i.d. variables, each uniformly distributed on  $\Lambda_N$ . We define a *random* probability measure on  $\Lambda_N$

$$\mu^{N,M}(x) := 2^{-M} \sum_{i \in \Lambda_M} \mathbb{1}_{(x=W_i)}, \quad x \in \Lambda_N.$$

## Theorem

Let  $x \in ]0, 1[$ . We have a.s. and in  $L^1$

$$\lim_{N \rightarrow +\infty} \frac{H(\mu^{N, \lfloor xN \rfloor})}{N \log 2} = x \tag{1}$$

$$\lim_{N \rightarrow +\infty} \frac{\mathcal{I}(\mu^{N, \lfloor xN \rfloor})}{N \log 2} = x(1-x). \tag{2}$$

Given  $X \mapsto \{0, 1\}^N$ , its **entropy profile** is the function  $h_X : [0, 1] \rightarrow [0, 1]$  such that  $h_X(0) = 0$ ,

$$h_X\left(\frac{k}{N}\right) = \frac{1}{\binom{N}{k}} \sum_{S \subset I, |S|=k} \frac{H(X_S)}{N \log 2}, \quad k \in I := \{1, \dots, N\}$$

and  $h_X$  is affine on each interval  $[\frac{k-1}{N}, \frac{k}{N}]$ ,  $k \in I$ . Then

$$\frac{\mathcal{I}(X)}{N \log 2} = \frac{2}{N+1} \sum_{k=0}^N h_X(k/N) - h_X(1).$$

It is easy to see that  $h_X$  is 1-Lipschitz and increasing with  $h_X(0) = 0$ . Then for  $x = h_X(1)$  we have  $h_X(y) \leq h_X^*(y) := x \wedge y$ .

Therefore for  $x := \frac{H(X)}{N \log 2}$

$$\frac{I(X)}{N \log 2} \leq \left\{ \frac{2}{N+1} \sum_{k=0}^N x \wedge (k/N) \right\} - x =: i_N(x).$$

It is easy to see that  $\lim_N i_N(x) = x(1-x)$ . It follows that

1. For any sequence  $X^N = (X_1^N, \dots, X_N^N)$

$$\limsup_{N \rightarrow \infty} \frac{I(X^N)}{N \log 2} \leq \frac{1}{4}.$$

2. For any sequence  $X^N = (X_1^N, \dots, X_N^N)$  such that

$$\lim_{N \rightarrow \infty} \frac{H(X^N)}{N \log 2} = x \in [0, 1],$$

we have

$$\limsup_{N \rightarrow \infty} \frac{I(X^N)}{N \log 2} \leq x(1-x).$$

Since

$$\frac{H(\mu^{N, \lfloor xN \rfloor})}{N \log 2} \leq \frac{\lfloor xN \rfloor}{N}$$

and

$$\frac{\mathcal{I}(\mu^{N, \lfloor xN \rfloor})}{N \log 2} \leq i_N \left( \frac{H(\mu^{N, \lfloor xN \rfloor})}{N \log 2} \right)$$

then in order to prove  $L^1$  convergence it is enough to prove that

$$\lim_{N \rightarrow +\infty} \mathbb{E} \left( \frac{H(\mu^{N, \lfloor xN \rfloor})}{N \log 2} \right) = x$$

and

$$\lim_{N \rightarrow +\infty} \mathbb{E} \left( \frac{\mathcal{I}(\mu^{N, \lfloor xN \rfloor})}{N \log 2} \right) = x(1 - x).$$

By the symmetries

$$\mathbb{E} \left( \frac{\mathcal{I}(\mu^{N, \lfloor xN \rfloor})}{N \log 2} \right) = \frac{2}{N+1} \sum_{k=0}^N h_k - h_N$$

where

$$h_k := 2^k \mathbb{E} \left( \varphi \left( B_k 2^{-M} \right) \right)$$

and  $B_k$  is a binomial variable with parameters  $(2^M, 2^{-k})$  and

$$\varphi(x) := -\frac{x \log x}{\log 2}, \quad x > 0, \quad \varphi(0) := 0.$$

There is an interesting transition between three regimes:

1. For  $k \ll M$  we have  $B_k 2^{-M} \approx 2^{-k}$  (Gaussian regime)
2. For  $k = M$  we have  $B_k \approx \text{Poisson}(1)$  (Poisson regime)
3. For  $k \gg M$  we have  $B_k \approx \text{Poisson}(2^{M-k})$  (Null regime)

This translates into a transition for  $h_k = 2^k \mathbb{E}(\varphi(B_k 2^{-M}))$

1. For  $k \leq M$  we have  $k - 2^{\frac{k-M}{2}} \leq h_k \leq k$
2. For  $k > M$  we have  $M - 2^{k-M} \leq h_k \leq M$

# Approximate maximizers

This sequence satisfies the following property: as  $N \rightarrow +\infty$ ,

1. if  $y \in ]0, x]$  then for *almost* all subsets  $S$  with  $|S| = \lfloor yN \rfloor$ ,  $X_S$  is *almost* uniform;
2. if  $y \in [x, 1[$  then for *almost* all subsets  $S$  with  $|S| = \lfloor yN \rfloor$ ,  $X$  is *almost* a function of  $X_S$ .

It turns out that the same property is shared by any sequence of approximate maximizers.

This property describes the interplay between *differentiation* and *integration* that biologists expect to find in complex systems.