ON THE STEADY OSEEN PROBLEM IN THE WHOLE SPACE

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Abstract. We deal with Oseen’s equations in the whole space. A class of existence, uniqueness and regularity results for both the scalar and the vectorial equations are given. The use of weighted Sobolev spaces for describing the growth or the decay of functions at infinity is at the heart of our approach.

Key words. Oseen’s equations, weighted Sobolev spaces, unbounded domains, fluid mechanics.

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1. Introduction. The Oseen’s equations are a linearized version of the Navier-Stokes equations describing a viscous and incompressible fluid in which a small body is moving. The purpose of this paper is to study the Oseen problem in the whole space \( \mathbb{R}^n, n \geq 2 \):

\[
-\nu \Delta u + k \frac{\partial u}{\partial x_1} + \nabla \pi = f \text{ in } \mathbb{R}^n,
\]

\[
\text{div } u = h \text{ in } \mathbb{R}^n.
\]

(1.1)

Here the unknowns are the velocity vector \( u \) of the fluid and the pressure \( \pi \). The data are the viscosity \( \nu \) of the fluid, the external force \( f \), the function \( h \) and the positive real \( k \). System (1.1) was proposed by Oseen (see [22]) in order to remove some physical paradoxes of the Stokes system which corresponds to the case \( k = 0 \). One of the first works devoted to these equations is due to Finn [13]. Specifically, Finn treated the Oseen’s equations in a three dimensional exterior domains when \( (1 + |x|) f \) is square integrable and \( h = 0 \). He proved the existence of solution \( u \) such that \( (1 + |x|)^{-1} u \) is square integrable. Farwig in [12] proved, among other results, the existence of a solution \( (u, \pi) \) of (1.1) when \( f \in L^p(\mathbb{R}^n) \) and \( h \in W^{1,p}(\mathbb{R}^n) \). In that case the solution \( (u, \pi) \) satisfies \( u \in L^p_{\text{loc}}(\mathbb{R}^n)^n, \partial_1 u \in L^p(\mathbb{R}^n), \partial_i \pi \in L^p(\mathbb{R}^n), i = 1, 2, ..., n \). In [14] Galdi stated that if \( f \in W^{m,p}(\mathbb{R}^n)^n, h \in W^{m+1,p}(\mathbb{R}^n), m \geq 0 \), then the problem (1.1) has a solution in \( W^{m+2,p}_{\text{loc}}(\mathbb{R}^n) \times W^{m+1,p}_{\text{loc}}(\mathbb{R}^n) \). In [10] Farwig investigates the system (1.1), set in three dimensional exterior domains, in anisotropically weighted \( L^2 \) spaces. The use of anisotropic weights seems to be a natural approach because of the anisotropy introduced by the term \( \partial_1 u \). However, such an approach contains some serious technical complications. The reader can refer to [19], [23] [11] [10], [6],
[5], [4] for existence results in anisotropic weighted spaces. To our knowledge, most of the existing results in the literature concern the case $f \in L^p(\mathbb{R}^n)$ or are around that case. Several questions concerning the existence, the uniqueness and regularity of the solution remain not treated, especially when the data $f$ and $h$ are slowly decreasing or have a polynomial behavior at infinity. Among the results we present in this paper, we shall prove that if $f = (f_1, \ldots, f_n)$ and $h$ satisfies the conditions

$$(1 + |x|^2)^{(|\mu| - m)/2} \partial^\mu f_k \in L^p(\mathbb{R}^n) \text{ for } |\mu| \leq m,$$

$$(1 + |x|^2)^{(|\mu| - m - 1)/2} \partial^\mu h \in L^p(\mathbb{R}^n) \text{ for } |\mu| \leq m + 1,$$

for some integer $m \geq 0$, then Problem (1.1) admits a solution $(u, \pi)$, unique up to a class of polynomials, and satisfying

$$(1 + |x|^2)^{(|\mu| - m - 2)/2} \partial^\mu u_k \in L^p(\mathbb{R}^n) \text{ for } |\mu| \leq m + 2,$$

$$(1 + |x|^2)^{(|\mu| - m - 1)/2} \partial^\mu \pi \in L^p(\mathbb{R}^n) \text{ for } |\mu| \leq m + 1.$$

In all the paper, we deal with following problem obtained from (1.1) by means of a simple scaling argument

$$\begin{align*}
-\Delta u + 2 \frac{\partial u}{\partial x_1} + \nabla \pi &= f \text{ in } \mathbb{R}^n, \\
\text{div } u &= h \text{ in } \mathbb{R}^n.
\end{align*}$$

(1.2)

We are interested also in the scalar equation

$$-\Delta u + 2 \frac{\partial u}{\partial x_1} = f \text{ in } \mathbb{R}^n,$$

(1.3)

which is intimately linked to the system (1.2). The relation between this scalar equation and the general vectorial system (1.2) as well as the relation between their fundamental solutions are discussed in section 3 hereafter. Observe for the moment that Oseen’s system (1.2) can be formally decomposed into two problems: a Laplace equation for the pressure

$$\Delta \pi = \text{div } f + \Delta h - 2 \frac{\partial h}{\partial x_1},$$

(1.4)

and a scalar equation on each component of the velocity

$$-\Delta u_i + 2 \frac{\partial u_i}{\partial x_1} = \tilde{f}_i,$$

(1.5)

where $\tilde{f}_i = f_i - \frac{\partial \pi}{\partial x_i}$. Consequently, one must choose a functional framework which allows the solving of both the equations (1.4) and (1.5) for several behaviors at infinity.
The use of weighted Sobolev spaces turned out to be convenient for treating problems in unbounded domains, and consequently seems to be the natural framework for treating Problem (1.4) (see for instance [3]). The main difficulty here lies on the choice of the weights since the convective term $\frac{\partial u}{\partial x_1}$ in Equation (1.5) induces an anisotropic behavior of the velocity, while the pressure keeps an isotropic behavior as in the Stokes problem. Another difficulty is due to divergence condition $\text{div} \ u = h$ which complicates seriously the problem. In section 3 we expose how the system (1.2) can be treated by solving only the scalar equation (1.3) is a such a way that the divergence condition is automatically fulfilled.

For all these reasons, we shall treat in a first time and independently the scalar equation (1.3). We prove that there exists at least two kinds of solutions; tempered solutions, which are tempered distributions, and quasi-tempered solutions which are not necessarily tempered distributions. Only tempered solution are useful for solving the vectorial system (1.2).

In a forthcoming paper, we will use our present results in order to solve the Oseen equations in an exterior domain.

In the sequel, we set

$$T = -\Delta + 2 \frac{\partial}{\partial x_1}$$

and

$$T^* = -\Delta - 2 \frac{\partial}{\partial x_1}$$

The remaining of this paper is organized as follows

– Section 2 is devoted to a brief presentation of some basic definitions and properties of weighted Sobolev spaces, used as a functional framework for solving both the scalar and the vectorial Oseen equations.

– In Section 3, the relation between the scalar equation (1.5) and the vectorial system (1.2) is discussed. Some properties of their fundamental solutions are showed.

– Section 4 deals with the scalar equation (1.5). Existence of solutions and the well posedness of the problem are treated in several functional frameworks.

– Section 5 is devoted to the study of the vectorial system (1.2). After giving a characterization of the kernel of the system, we prove a complete class of existence, uniqueness and regularity results.


2.1. Notations. In the sequel, $n \geq 2$ is an integer and $p$ is a real in the interval $[1, +\infty[$. The dual number of $p$ denoted $p'$ is defined by the relation $1/p + 1/p' = 1$. We use bold characters for vector functions or distributions. For $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$
we set
\[ r = |x| = (x_1^2 + \ldots + x_n^2)^{1/2}. \]

Given a real \( \alpha \), we denote by \( [\alpha] \) its integer part. For any \( k \in \mathbb{Z} \), \( \mathbb{P}_k \) stands for the space of polynomial of degree lower then \( k \) and \( \mathbb{P}_k^\Delta \) the subspace of harmonic polynomials of \( \mathbb{P}_k \). If \( k \) is a negative integer, we set by convention \( \mathbb{P}_k = \{0\} \). We recall that \( \mathcal{D}(\mathbb{R}^n) \) is the well-known space of \( C^\infty(\mathbb{R}^n) \) functions with a compact support and \( \mathcal{D}'(\mathbb{R}^n) \) its dual space, namely the space of distributions. We denote by \( \mathcal{S}(\mathbb{R}^n) \) the Schwartz space of functions \( \varphi \in C^\infty(\mathbb{R}^n) \) with rapid decrease at infinity, by \( \mathcal{S}'(\mathbb{R}^n) \) its dual, i.e. the space of tempered distributions, and by \( \mathcal{S}'_1(\mathbb{R}^n) \) be the space of all the distributions \( u \in \mathcal{D}'(\mathbb{R}^n) \) such that \( e^{-x_1}u \in \mathcal{S}'(\mathbb{R}^n) \).

The Fourier transform of any complex valued Lebesgue integrable function \( u : \mathbb{R} \rightarrow \mathbb{C} \) is defined by
\[ \hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} u(x) dx, \]
where \( \xi \in \mathbb{R}^n \). If \( u \in \mathcal{S}'(\mathbb{R}^n) \) then its Fourier distribution \( \hat{u} \in \mathcal{S}'(\mathbb{R}^n) \) is defined by \( \langle \hat{u}, \phi \rangle = \langle u, \hat{\phi} \rangle \) for any function \( \phi \in \mathcal{S}(\mathbb{R}^n) \). The Fourier transform is an invertible mapping of \( \mathcal{S}(\mathbb{R}^n) \) into \( \mathcal{S}(\mathbb{R}^n) \) and from \( \mathcal{S}'(\mathbb{R}^n) \) into \( \mathcal{S}'(\mathbb{R}^n) \).

Given a Banach space \( B \) with its dual space \( B' \) and a closed subspace \( X \) of \( B \), we denote by \( B' \perp X \) the subspace of \( B' \) orthogonal to \( X \), namely
\[ B' \perp X = \{ f \in B', \forall v \in X, \langle f, v \rangle = 0 \} = (B/X)' \]

For any real \( \alpha > 0 \), the Bessel kernel \( g_\alpha \) is defined as the function whose Fourier transform is
\[ \hat{g}_\alpha(\xi) = (2\pi)^{-n/2}(1 + |\xi|^2)^{-\alpha/2}. \]
The kernel \( g_\alpha \) can be expressed in terms of Bessel functions by the formula
\[ g_\alpha(x) = \gamma_\alpha |x|^{(\alpha-n)/2} K_{(n-\alpha)/2}(|x|), \]
with \( \gamma_\alpha = (2\pi)^{-n/2}2^{-\alpha/2+1}\Gamma(\alpha/2) \). Here \( \Gamma \) denotes the classical gamma function and \( K_\lambda \) denotes the modified Bessel function of third kind. Since for any integer \( m \geq 0 \), we know that
\[ K_{m+1/2}(t) = \sqrt{\frac{\pi}{2t}} e^{-t} \sum_{k=0}^{m} \frac{(m+k)!}{k!(m-k)!} \frac{1}{(2t)^k}, \]
we deduce an explicit expression of \( g_\alpha \) when \( \alpha < n \) and \( n-\alpha \) is odd. In particular if \( \alpha = 2 \) and \( n = 3 \) we get
\[ g_2(x) = \frac{1}{4\pi |x|} e^{-|x|}. \]
More generally, we have

\[ \forall \mathbf{x} \in \mathbb{R}^n, \ |g_\alpha(\mathbf{x})| \leq \frac{c_\alpha}{|\mathbf{x}|^{n-\alpha}(1 + |\mathbf{x}|^{(\alpha+1)/2})^{e^{-|\mathbf{x}|}}. \quad (2.1) \]

Hence,

\[ g_\alpha \in W^{0,p}_s(\mathbb{R}^n) \text{ for } s \in \mathbb{R}, \ 1 \leq p \leq +\infty \text{ and } (n-\alpha)p < n. \quad (2.2) \]

In the sequel, the notation \( a \lesssim b \) (resp. \( a \sim b \)) means that there exists a constant \( c \) not depending on the functions such that \( a \leq cb \).

### 2.2. Weighted Sobolev spaces. Some basic results

In the sequel, \( \rho \) denotes the basic weight defined by

\[ \rho(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{1/2}. \quad (2.3) \]

For \( 1 \leq p \leq +\infty \), \( L^p(\mathbb{R}^n) \) will denote the space of (equivalence classes of) all measurable functions that are \( p \)-th power integrable on \( \mathbb{R}^n \). This space is equipped with the norm

\[ \|u\|_{L^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |u|^p \, dx \right)^{1/p}. \]

Given two integer \( m \geq 0 \) and \( k \in \mathbb{Z} \), we consider the weighted spaces

\[ W^{m,p}(\mathbb{R}^n) = \{ u \in \mathcal{D}'(\mathbb{R}^n); \forall \mu \in \mathbb{N}^n, |\mu| \leq m, \partial^\mu u \in L^p(\mathbb{R}^n) \} \]

\[ V^{m,p}_k(\mathbb{R}^n) = \{ u \in \mathcal{D}'(\mathbb{R}^n); \forall \mu \in \mathbb{N}^n, |\mu| \leq m, \rho^k \partial^\mu u \in L^p(\mathbb{R}^n) \} \]

\[ W^{m,p}_k(\mathbb{R}^n) = \{ u \in \mathcal{D}'(\mathbb{R}^n); \forall \mu \in \mathbb{N}^n, |\mu| \leq m, \rho^{-m+|\mu|} \partial^\mu u \in L^p(\mathbb{R}^n) \} \]

\[ \mathcal{H}^{m,p}_k(\mathbb{R}^n) = \{ u \in \mathcal{D}'(\mathbb{R}^n); \forall \mu \in \mathbb{N}^n, |\mu| \leq m, e^{-x_1} \rho^k \partial^\mu u \in L^p(\mathbb{R}^n) \}. \]

These spaces are equipped with the norms

\[ \|u\|_{W^{m,p}(\mathbb{R}^n)} = \left( \sum_{|\mu| \leq m} \|\partial^\mu u\|_{L^p(\mathbb{R}^n)}^p \right)^{1/p}. \]

\[ \|u\|_{V^{m,p}_k(\mathbb{R}^n)} = \left( \sum_{|\mu| \leq m} \|\rho^k \partial^\mu u\|_{L^p(\mathbb{R}^n)}^p \right)^{1/p}. \]

\[ \|u\|_{W^{m,p}_k(\mathbb{R}^n)} = \left( \sum_{|\mu| \leq m} \|\rho^{-m+|\mu|} \partial^\mu u\|_{L^p(\mathbb{R}^n)}^p \right)^{1/p}. \]

\[ \|u\|_{\mathcal{H}^{m,p}_k(\mathbb{R}^n)} = \left( \sum_{|\mu| \leq m} \|\rho^k e^{-x_1} \partial^\mu u\|_{L^p(\mathbb{R}^n)}^p \right)^{1/p}. \]

The spaces \( W^{m,p}(\mathbb{R}^n), V^{m,p}_k(\mathbb{R}^n) \) and \( \mathcal{H}^{m,p}_k(\mathbb{R}^n) \) are Banach spaces. The space \( \mathcal{D}(\mathbb{R}^n) \) is dense in \( W^{m,p}(\mathbb{R}^n) \), in \( V^{m,p}_k(\mathbb{R}^n) \) and in \( W^{m,p}_k(\mathbb{R}^n) \) (see, e.g., Hanouzet [18]). We denote by \( W^{-m,p}(\mathbb{R}^n), V^{-m,p}_k(\mathbb{R}^n) \) and \( \mathcal{W}^{-m,p}_k(\mathbb{R}^n) \) the dual spaces of \( W^{m,p}(\mathbb{R}^n), W^{m,p}_k(\mathbb{R}^n) \) and \( V^{m,p}_k(\mathbb{R}^n) \) respectively. They are spaces of tempered distributions. It is quite clear that the local properties of the spaces \( W^{m,p}(\mathbb{R}^n) \) and
\( V_k^{m,p}(\mathbb{R}^n) \) coincide with those of the Sobolev space \( W_k^{m,p}(\mathbb{R}^n) \). We have also the obvious identities

\[
V_0^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n), \quad V_0^{0,p}(\mathbb{R}^n) = W_0^{0,p}(\mathbb{R}^n).
\]

The spaces \( W_k^{m,p}(\mathbb{R}^n) \) will play a particular role in this paper. For a detailed study of the spaces \( W_k^{m,p}(\mathbb{R}^n) \), we can refer to \([3]\), \([18]\) and \([20]\). In this paper we need the semi-norm

\[
|u|_{W_k^{m,p}(\mathbb{R}^n)} = \left( \sum_{|\mu|=m} \| \rho^k \partial^\mu u \|_{L^p(\mathbb{R}^n)} \right)^{1/p}.
\]

and the Green’s formula

\[
\forall u \in W_1^{1,p}(\mathbb{R}^n), \forall v \in W_{-1}^{1,p'}(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} \frac{\partial u}{\partial x_i} v dx = - \int_{\mathbb{R}^n} u \frac{\partial v}{\partial x_i} dx,
\]

where \( 1 \leq i \leq n \), \( 1 < p < +\infty \) and \( k \in \mathbb{Z} \). We have also the inclusion

\[
P_\ell \subset W_k^{m,p}(\mathbb{R}^n), \quad \text{if} \ \ell < m - n/p - k.
\]

In what follows, the space \( W_k^{m,p}(\mathbb{R}^n) \) will be considered often in the case

\[
\frac{n}{p} + k \notin \{1, \ldots, m\}.
\]

Indeed, this condition is sufficient to get some Hardy’s type inequalities. Namely, if (2.6) is fulfilled, then (see \([3]\))

\[
\forall u \in W_k^{m,p}(\mathbb{R}^n), \quad \inf_{\lambda \in \mathbb{P}_{j'}} \| u + \lambda \|_{W_k^{m,p}(\mathbb{R}^n)} \lesssim |u|_{W_k^{m,p}(\mathbb{R}^n)},
\]

where \( j' = \min(m-1, j) \) and \( j = -[k+n/p-m] - 1 \) is the highest degree of polynomials contained in \( W_k^{m,p}(\mathbb{R}^n) \). If (2.6) does not hold, namely if \( \frac{n}{p} + k \in \{1, \ldots, m\} \), then similar inequalities can be obtained by adding a logarithmic factor to the weights in the definition of \( W_k^{m,p}(\mathbb{R}^n) \) (see \([3]\)). In that case, all the forthcoming results remains valid provided some minor corrections are given.

We have the following algebraic and topological inclusions (\( m \geq 0 \))

\[
V_k^{m,p}(\mathbb{R}^n) \subset W_k^{m,p}(\mathbb{R}^n) \subset W_{k-1}^{m,p}(\mathbb{R}^n) \subset \ldots \subset W_0^{0,p}(\mathbb{R}^n).
\]

For any \( \mu \in \mathbb{N}^n \), the mapping

\[
u \in W_k^{m,p}(\mathbb{R}^n) \longrightarrow \partial^\mu \nu \in W_k^{m-|\mu|,p}(\mathbb{R}^n)
\]

is continuous (see \([18]\)).
The spaces $W^{m,p}_k(\mathbb{R}^n)$ turned out to be adequate for treating several elliptic problems in unbounded regions of spaces and for several kinds of behavior at infinity (see [3], [2], [7, 8], [16, 17]). Let us recall some basic but fundamental results concerning the Laplace equation in whole the space (see [3]):

**Theorem 2.1.** Let $m \in \mathbb{Z}$ and $\ell \in \mathbb{N}^*$.

1. If $n/p \not\in \{1, \ldots, \ell + 1\}$, then the operator

$$\Delta : W^{m+1,p}_m(\mathbb{R}^n)/P_{[\ell+1-n/p]} \rightarrow W^{m-1,p}_m(\mathbb{R}^n),$$

is an isomorphism.

2. If $n/p' \not\in \{1, \ldots, \ell + 1\}$, then the operator

$$\Delta : W^{m+1,p}_m(\mathbb{R}^n) \rightarrow W^{m-1,p}_m(\mathbb{R}^n) \perp P_{[\ell+1-n/p']}$$

is an isomorphism.

3. If $n/p' \neq 1$ and $n/p \neq 1$, then the operator

$$\Delta : W^{m+1,p}_m(\mathbb{R}^n) \perp P_{[1-n/p]} \rightarrow W^{m-1,p}_m(\mathbb{R}^n) \perp P_{[1-n/p']}$$

is an isomorphism.

**3. Relation between the scalar and the vectorial Oseen’s equations.**

**Properties of the fundamental solutions.** Our purpose here is to discuss briefly the relation between the scalar equation (1.3) and the vectorial Oseen’s system (1.2). Indeed, the difference between the equation (1.3) and the system (1.2) seems to reinforced by the presence of the pressure $\pi$ and the divergence equation $\text{div } u = h$.

This difference appears also in terms of the fundamental solutions. However, there is a simple method for solving the system (1.2) by solving the equation (1.3) and the Laplace equation, in such a way that the divergence condition is automatically fulfilled. This method reveals that the fundamental solutions are intimately linked by a simple relation.

Formally, let $\theta$ and $s = (s_1, \ldots, s_n)$ be solution of the Laplace equations

$$\Delta \theta = h \text{ in } \mathbb{R}^n, \quad \Delta s = f + \nabla(h - 2\frac{\partial \theta}{\partial x_1}) \text{ in } \mathbb{R}^n.$$ 

Consider in addition a vector function $\Phi = (\Phi_1, \ldots, \Phi_n)$ whose components $\Phi_i$, $1 \leq i \leq n$, are solution of the scalar equations $T\Phi_i = s_i$. Then the pair $(u, \pi)$ defined by

$$u = \nabla \theta + \Delta \Phi - \nabla(\text{div } \Phi),$$

$$\pi = \text{div } s,$$

is solution of the system (1.2). In other words, the existence of solutions of (1.2) can be obtained by treating the Laplace equation and the scalar equation (1.3).
On the other hand, concerning the fundamental solution \((O_{i,j}, e_j)\) of (1.2) we know that

\[
O_{ij} = (\delta_{ij} \Delta - \frac{\partial^2}{\partial x_i \partial x_j}) \varphi
\]

\[
e_j = \frac{\partial}{\partial x_j}(-\Delta + 2 \frac{\partial}{\partial x_1}) \varphi.
\]

Here, \(i, j = 1, \ldots, n\) and \(\varphi\) satisfies \(T\varphi = E\) where \(E\) is the fundamental solution of the Laplace equation. It is well known (see for instance [14]) that the fundamental solution \(O\) of the scalar equation (1.3) in \(\mathbb{R}^n\) is given by

\[
O(x) = (2\pi)^{-n/2} |x|^{1-n/2} e^{x_1} K_{n/2-1}(|x|).
\]

In particular, if the dimension is odd, namely \(n = 2m + 1 \quad (m \geq 1)\), then

\[
O(x) = \frac{1}{2} e^{-|x|} \text{ if } n = 1, \quad O(x) = \frac{1}{4\pi} e^{x_1} K_{n/2-1}(|x|) \text{ if } n = 3.
\]

In order to clarify the relation between \((O_{i,j}, e_j)\) and \(O\) it is convenient to use the notions of Riesz transforms (see for instance [24]). Recall that the Riesz transforms of a function \(u \in L^p(\mathbb{R}^n), \ p > 1\), are defined by

\[
\hat{R}_j u(\xi) = -i \xi_j \hat{u}, \ j = 1, \ldots, n,
\]

where \(\hat{R}_j u\) and \(\hat{u}\) denote the Fourier transform of \(R_j u\) and \(u\) respectively. Among the properties of the Riesz transforms let us recall that \(R_j\) preserves the class \(L^p(\mathbb{R}^n)\) and satisfies

\[
R_i \circ R_j (\Delta u) = -\frac{\partial^2 u}{\partial x_i \partial x_j}.
\]

Consequently the relation between \((O_{i,j}, e_j)\) and \(O\) is summarized in terms of the Riesz transforms as follows

**Lemma 3.1.** For each \(i, j \leq n\),

\[
O_{ij} = (\delta_{ij} I + R_i \circ R_j) O.
\]

Since \(R_i\) maps \(L^p(\mathbb{R}^3)\) into itself, one can easily get some properties of \(O_{ij}\) from those of \(O\). Let us display some of them. We state the following proposition whose proof is given in appendix A.

**Proposition 3.2.** We have
More precisely, Equation (4.1) can be written into the form
\[ \nabla \phi_1 \cdot \nabla u = f \] in \( \mathbb{R}^n \).

A first approach for treating this equation is based on a simple but efficient idea. It consists to rewrite Equation (4.1) in term of the new unknown \( w(x) = e^{-x_1} u(x) \). More precisely, Equation (4.1) can be written into the form
\[ -\Delta u + \nabla \phi_1 \cdot \nabla u = f \] in \( \mathbb{R}^n \),
where $\phi_1 = 2x_1$, and therefore $\nabla \phi_1 = 2e_1$. Hence, let us consider the more general equation

$$-\Delta u + \nabla \phi. \nabla u = f \text{ in } \mathbb{R}^n,$$

with $\Delta \phi = 0$. Setting $w(x) = e^{-\phi(x)/2}u(x)$, then $w$ satisfies the usual elliptic equation:

$$-\Delta w + a(x)w = F_1 \text{ in } \mathbb{R}^n, \quad (4.2)$$

with $a(x) = \frac{1}{4}|\nabla \phi|^2(x)$ and $F_1(x) = e^{-\phi(x)/2}f(x)$. The scalar equation (4.1) corresponds to the case $\phi(x) = 2x_1$. In this case, we get the classical equation

$$-\Delta w + w = F_1 \text{ in } \mathbb{R}^n. \quad (4.3)$$

The main advantage of this new formulation is that the anisotropic character of equation (4.1) has disappeared. However, as we shall see, the solutions obtained by this method could be different from those obtained by dealing directly with the equation (4.1). This difference is mainly due to the fact that the space of tempered distribution $S'(\mathbb{R}^n)$ is not preserved under multiplication by $e^{-x_1}$.

These considerations lead one to distinguish two kinds of solutions of (4.1): tempered solutions, which are tempered distributions, and quasi-tempered solutions. A solution $u$ of (4.1) will be called quasi-tempered if $e^{-x_1}u \in S'(\mathbb{R}^n)$. Quasi-tempered solutions are obtained by solving (4.3). As we shall see, when a quasi-tempered solution exists it is unique. The uniqueness is lost in general with tempered solutions which are obtained by dealing with equation (4.1). It is worth nothing that only tempered solutions of the equation (4.1) turn out to be useful in the treatment of the vectorial Oseen’s system (1.2).

**Remark (Relation with the Laplace equation).** Let $u \in S'_1(\mathbb{R}^n)$ be a solution of (4.1) and set $w = e^{-x_1}u$. Consider the finite measure $\mu_2$ defined by $\mu_2 = (2\pi)^{n/2}\delta_0 - (2\pi)^{n/2}g_2(x)dx$, with $\delta_0$ the Dirac measure at the origin. Let $\hat{\mu}_2$ be the Fourier transform of $\mu_2$. We have $\hat{\mu}_2(\xi) = |\xi|^2(1 + |\xi|^2)^{-1} = 1 - (1 + |\xi|^2)^{-1}$. Next, let $v \in S'(\mathbb{R}^n)$ be solution of the Laplace equation $-\Delta v = e^{-x_1}f = F$ in $\mathbb{R}^n$. Then, on the one hand, we have $|\xi|^2\hat{v} = \hat{F}$. On the other, since $-\Delta w + w = F$, we get $\hat{w}(\xi) = (1 + |\xi|^2)^{-1}\hat{F}$. Thus, $\hat{w}(\xi) = \hat{\mu}_2(\xi)\hat{v}$, and $w(x) = (2\pi)^{-n/2}\mu_2 \ast v(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} v(x - y)\mu_2(y)$. Namely, $w = v - g_2 \ast v$ and $u = e^{x_1}(v - g_2 \ast v) = (2\pi)^{-n/2/e^{x_1}}\mu_2 \ast v$.

### 4.1. Quasi-tempered solutions of scalar Oseen equation.

Our aim here is to show existence of quasi-tempered solutions of the scalar Oseen equation (4.1). The main result of this paragraph is the following
**Theorem 4.1.** Let $m, k \in \mathbb{Z}$ be two integers and $p > 1$ a real. Then, the operator

$$T : \mathcal{H}^{m+2,p}_k(\mathbb{R}^n) \longrightarrow \mathcal{H}^{m,p}_k(\mathbb{R}^n)$$

is an isomorphism.

**Proof of Theorem 4.1**

Firstly, observe that the mapping

$$v \mapsto w = e^{-x_1}v$$

is an isomorphism between $\mathcal{H}^{m,p}_k(\mathbb{R}^n)$ and $V^{m,p}_k(\mathbb{R}^n)$. Moreover, in the sense of distributions one can easily prove that

$$Tv = e^{x_1}(I - \Delta)e^{-x_1}v.$$  

This remark allows one to deal only with the operator $I - \Delta$. We start with the following lemma.

**Lemma 4.2.** Let $k \geq 0$ be an integer and $p > 1$ a real. Then, the operator

$$I - \Delta : W^{s+2,p}(\mathbb{R}^n) \longrightarrow W^{s,p}(\mathbb{R}^n)$$

is an isomorphism.

**Proof.** We need the following identity (see [9])

$$\mathcal{L}^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n), \quad \forall m \in \mathbb{N},$$

where $\mathcal{L}^{m,p}(\mathbb{R}^n)$ is the Lizorkin space defined by

$$\mathcal{L}^{m,p}(\mathbb{R}^n) = \{ u \in \mathcal{S}'(\mathbb{R}^n), u = g_m * v, v \in L^p(\mathbb{R}^n) \}.$$

In terms of Fourier transform, the unique solution of the equation

$$(I - \Delta)w = h,$$

is given by $w = \mathcal{F}^{-1}((1 + |\xi|^2)^{-1}\mathcal{F}(h)) = \mathcal{F}^{-1}(\hat{g}_2\mathcal{F}(h))$. Hence,

$$h \in W^{m,p}(\mathbb{R}^n) \iff h \in \mathcal{L}^{m,p}(\mathbb{R}^n)$$

$$\iff (\hat{g}_m)^{-1}\hat{h} \in \mathcal{F}(L^p(\mathbb{R}^n))$$

$$\iff (\hat{g}_m)^{-1}(\hat{g}_2)^{-1}\hat{w} \in \mathcal{F}(L^p(\mathbb{R}^n))$$

$$\iff (\hat{g}_{m+2})^{-1}\hat{w} \in \mathcal{F}(L^p(\mathbb{R}^n))$$

$$\iff w \in W^{m+2,p}(\mathbb{R}^n).$$  

(4.4)
Lemma 4.3. Let \( k \in \mathbb{Z} \) and \( m \in \mathbb{Z} \) be two integers and \( p > 1 \) a real. Then, the operator
\[
I - \Delta : V^{m+2,p}_k(\mathbb{R}^n) \longrightarrow V^{m,p}_k(\mathbb{R}^n)
\]
is an isomorphism.

Proof. We know that the operator \( I - \Delta \) is one to one from \( S'(\mathbb{R}^n) \) into \( S'(\mathbb{R}^n) \).

Hence, for \( F \in V^{m,p}_k(\mathbb{R}^n) \), there exists a unique \( w \in S'(\mathbb{R}^n) \) such that
\[
-\Delta w + w = F \text{ in } \mathbb{R}^n.
\]

Suppose first that \( k \geq 0 \). Let us prove by induction on \( k \) the following
\[
F \in V^{0,p}_k(\mathbb{R}^n) \implies \left( w \in V^{2,p}_k(\mathbb{R}^n) \text{ and } ||w||_{V^{2,p}_k(\mathbb{R}^n)} \lesssim ||f||_{V^{0,p}_k(\mathbb{R}^n)} \right) \tag{4.5}
\]
The latter follows immediately from Lemma 4.2 when \( k = 0 \). Suppose that (4.5) holds for \( 0, \ldots, k \) and suppose that \( F \in V^{0,p}_{k+1}(\mathbb{R}^n) \). Necessarily \( w \in V^{2,p}_k(\mathbb{R}^n) \). Setting \( \tilde{w} = \rho^{k+1}w \), we prove easily that
\[
-\Delta \tilde{w} + \tilde{w} = \rho^{k+1}F - \nabla \rho^{k+1} \cdot \nabla w - (\Delta \rho^{k+1})w.
\]
The right hand side belongs to \( L^p(\mathbb{R}^n) \) since \( w \in V^{2,p}_k(\mathbb{R}^n) \) and \( |\partial^\alpha \rho^k| \lesssim \rho^{k-|\alpha|} \) for any multi-index \( \alpha \). We deduce that \( \tilde{w} \in V^{2,p}_0(\mathbb{R}^n) \), and, consequently, \( w \in V^{2,p}_k(\mathbb{R}^n) \) since
\[
\forall |\alpha| \leq 2; |\partial^\alpha w| \lesssim \sum_{|\nu| \leq 2, \nu \leq \alpha} |\partial^\nu \rho^{-k-1} \partial^\alpha - \nu \tilde{w}| \lesssim \sum_{|\nu| \leq 2, \nu \leq \alpha} |\rho^{-k-1-|\nu|} \partial^\alpha - \nu \tilde{w}|.
\]
This ends the proof of (4.5). Similarly, let us prove by induction on \( k \geq 0 \) the following
\[
F \in V^{0,p}_{-k}(\mathbb{R}^n) \implies \left( w \in V^{2,p}_{-k}(\mathbb{R}^n) \text{ and } ||w||_{V^{2,p}_{-k}(\mathbb{R}^n)} \lesssim ||f||_{V^{0,p}_{-k}(\mathbb{R}^n)} \right) \tag{4.6}
\]
The latter holds clearly for \( k = 0 \). Suppose that it holds for \( 0, \ldots, k \) and let \( F \in V^{0,p}_{-k-1}(\mathbb{R}^n) \). Setting \( \tilde{F} = \rho^{-k-1}F \in L^p(\mathbb{R}^n) \), \( \tilde{w} = (I - \Delta)^{-1} \tilde{F} \in V^{2,p}_{0}(\mathbb{R}^n) \) and \( h = w - \rho^{-k-1} \tilde{w} \), we get after few calculation
\[
-\Delta h + h = (\Delta \rho^{k+1}) \tilde{w} + \nabla \rho^{k+1} \cdot \nabla \tilde{w}.
\]
The right hand side of the last identity belongs to \( V^{0,p}_{-k}(\mathbb{R}^n) \). From induction hypothesis we deduce that \( h \in V^{2,p}_{-k}(\mathbb{R}^n) \rightarrow V^{2,p}_{-k-1}(\mathbb{R}^n) \). It follows that \( w \in V^{2,p}_{-k-1}(\mathbb{R}^n) \) since \( \rho^{k+1} \tilde{w} \in V^{2,p}_{-k-1}(\mathbb{R}^n) \). This ends the proof of (4.6).

At this stage, the assertion of lemma 4.3 is proved when \( m = 0 \) and \( k \in \mathbb{Z} \). Now, suppose that \( m \geq 1 \), and let \( w \in S'(\mathbb{R}^n) \) be the unique solution of \( -\Delta w + w = F \) with \( F \in V^{m,p}_k(\mathbb{R}^n) \). Then, for each multi-index \( \alpha \), \( |\alpha| \leq m \), \( \partial^\alpha w \) satisfies
\[-\Delta (\partial^\alpha w) + \partial^\alpha w = \partial^\alpha F \in V^{0,p}_k(\mathbb{R}^n)\]. Hence, \(\partial^\alpha w \in V^{2,p}_k(\mathbb{R}^n)\) for each \(\alpha, |\alpha| \leq m\). We conclude that \(w \in V^{m+2,p}_k(\mathbb{R}^n)\) which ends the proof of Lemma 4.3 for \(m \geq 0\).

The proof for \(m \leq -2\) is based on a classical duality argument. It remains to treat the case \(m = -1\). Let \(F \in V^{-1,p}_k(\mathbb{R}^n)\) for some \(k \in \mathbb{Z}\). Then, \(F \in V^{-2,p}_k(\mathbb{R}^n)\).

Hence, there exists \(w \in V^{k,p}_k(\mathbb{R}^n)\) such that \(-\Delta w + w = F\) (here we used the result of lemma for \(m = -2\)). Since in addition, \(\partial_i F \in V^{-2,p}_k(\mathbb{R}^n), i = 1, \ldots, n\), we deduce that \(\partial_i w \in V^{k,p}_k(\mathbb{R}^n)\), \(i = 1, \ldots, n\) since \(-\Delta (\partial_i w) + \partial_i w = \partial_i F\). Hence, \(w \in V^{1,p}_k(\mathbb{R}^n)\). \(\blacksquare\)

4.2. Tempered solutions of the scalar Oseen equation. Our aim here is to look for solutions of the equation (4.1) which are tempered distributions. In the sequel, for each integer \(m \in \mathbb{Z}\), we consider the space

\[\widetilde{W}^{m,p}_k(\mathbb{R}^n) = \{ u \in W^{m,p}_k(\mathbb{R}^n), \frac{\partial u}{\partial x_1} \in W^{m-2,p}_k(\mathbb{R}^n) \},\]

equipped with the norm

\[||u||_{\widetilde{W}^{m,p}_k(\mathbb{R}^n)} = \left\{ ||u||^{p}_{W^{m,p}_k(\mathbb{R}^n)} + ||\frac{\partial u}{\partial x_1}||^{p}_{W^{m-2,p}_k(\mathbb{R}^n)} \right\}^{1/p}.\]

When \(m \geq 0\) and \(k \in \mathbb{Z}\), we set \(\widetilde{W}^{-m,p'}_k(\mathbb{R}^n)\) the dual space of \(\widetilde{W}^{m,p}_k(\mathbb{R}^n)\). Clearly, we have the imbeddings

\[\widetilde{W}^{-m,p'}_k(\mathbb{R}^n) \hookrightarrow W^{-m,p'}_k(\mathbb{R}^n) \hookrightarrow \widetilde{W}^{-m,p'}_k(\mathbb{R}^n).\]

In what follows, \(Q_k\) denotes the sum \(\mathcal{H}_k^\ell + \mathbb{P}_{k-1}\), where \(\mathcal{H}_k^\ell\) is the space of homogeneous polynomials of degree \(k\) and depending only on \(x_2, \ldots, x_n\). We denote by \(Q_\ell^+\) (resp. \(Q_\ell^-\)) the subspace of all the polynomials \(p \in Q_\ell\) satisfying \(Tp = 0\) (resp. \(T^*p = 0\)). Notice that the mapping \(p(x_1, x_2, \ldots, x_n) \rightarrow p(-x_1, x_2, \ldots, x_n)\) is one to one from \(Q_\ell^+\) into \(Q_\ell^-\). We have the lemma

**Lemma 4.4.** Let \(m \geq 0\), \(1 < p < +\infty\) and \(k \in \mathbb{Z}\) such that \(k + n/p \notin \{0, \ldots, m\}\). A function \(u \in \widetilde{W}^{m,p}_k(\mathbb{R}^n)\) satisfies \(Tu = 0\) if and only if \(u \in Q_\ell^+\) with \(\ell = -[k + n/p - m] - 1\).

**Proof.** If \(u\) is a tempered such that \(Tu = 0\), then \((|\xi|^2 - i\xi_1)\hat{u} = 0\). Since \(|\xi|^2 - i\xi_1\) vanishes only at \(\xi = 0\), we deduce that \(u\) is polynomial. If in addition \(u \in \widetilde{W}^{m,p}_k(\mathbb{R}^n)\) than necessarily belongs to \(Q_\ell^-\). \(\Box\)

**Theorem 4.5.** If \(n/p \neq 1\) and \(n/p' \neq 1\). The operator

\[T : \widetilde{W}^{1,p}_0(\mathbb{R}^n)/\mathbb{P}_{[1-n/p]} \rightarrow W^{-1,p}_0(\mathbb{R}^n) \perp \mathbb{P}_{[1-n/p']}\]

is an isomorphism.

**Theorem 4.6.** Let \(m \geq 2\) be an integer and suppose that \(n/p \neq \{1, \ldots, m\}\), then the operator

\[T : \widetilde{W}^{m,p}_0(\mathbb{R}^n)/Q_{[m-n/p]}^+ \rightarrow W^{m-2,p}_0(\mathbb{R}^n)\]
is an isomorphism.

**Theorem 4.7.** Let $m \geq 2$ be an integer suppose that $n/p' \neq \{1, ..., m\}$ and $n/p \neq \{1, 2\}$ if $m$ is even and $n/p \neq \{1\}$. Let $\widetilde{W}_0^{-m+2,p}(\mathbb{R}^n) \perp \perp P_{[m-n/p']}$ be the space of all the functions $u \in \widetilde{W}_0^{-m+2,p}(\mathbb{R}^n)$ satisfying the conditions
\[ \forall p \in P_{[m-2-n/p']}, \langle u, p \rangle = 0, \]
\[ \forall p \in P_{[m-n/p']}, \langle \frac{\partial u}{\partial x_1}, p \rangle = 0. \]

Then, the operator
\[ T : \widetilde{W}_0^{-m+2,p}(\mathbb{R}^n) \perp \perp P_{[m-n/p']} \to W_0^{-m,p}(\mathbb{R}^n) \perp P_{[m-n/p']} \]
is an isomorphism.

By duality and transposition, Theorem 4.6 yields

**Theorem 4.8.** Suppose that $m \geq 2$, $1 < p < +\infty$ and $n/p' \neq \{1, ..., m\}$, then the operator
\[ T : W_0^{-m+2,p}(\mathbb{R}^n) \to \widetilde{W}_0^{-m,p}(\mathbb{R}^n) \perp Q_{[m-n/p']} \]
is an isomorphism.

The following proposition plays a prominent role in the proof of theorems 4.5-4.6,

**Proposition 4.9.** Let $1 < p < +\infty$, $m \geq 2$ such that $n/p \not\in \{1, ..., m\}$, and set $\ell = \min(m-1, [m-n/p])$. Then,
\[ \inf_{q \in Q_{\ell}} \|u + q\|_{\widetilde{W}_0^{-m,p}(\mathbb{R}^n)} \leq \|u\|_{W_0^{-m,p}(\mathbb{R}^n)} + \|\frac{\partial u}{\partial x_1}\|_{W_0^{m-2,p}(\mathbb{R}^n)} \]
for each $u \in \widetilde{W}_0^{-m,p}(\mathbb{R}^n)$.

**Proof of Proposition 4.9.** Observe first that the semi-norm $[u]_{m,p} = \|u\|_{W_0^{-m,p}(\mathbb{R}^n)} + \|\frac{\partial u}{\partial x_1}\|_{W_0^{m-2,p}(\mathbb{R}^n)}$ is a norm on $\widetilde{W}_0^{-m,p}(\mathbb{R}^n)$. Indeed, if $[u]_{m,p} = 0$, then $|u|_{W_0^{-m,p}(\mathbb{R}^n)} = 0$. Hence, $u$ is a polynomial. Since $u$ belongs to $W_0^{m,p}(\mathbb{R}^n)$, its degree is necessarily less or equal to $\ell$. Moreover, since $\frac{\partial u}{\partial x_1}$ belongs to $W_0^{m-2,p}(\mathbb{R}^n)$, the degree of $\frac{\partial u}{\partial x_1}$ is less or equal to $\ell - 2 = \min(m-3, [m-2-n/p])$. Hence, $u \in Q_{\ell}$.

Now, let us prove that this semi-norm is equivalent to the norm $\widetilde{W}_0^{-m,p}(\mathbb{R}^n)$. We need the following lemma (see [3])

**Lemma 4.10.** Let $k \in \mathbb{Z}$ and $s \geq 1$ be two integers such that $n/p + k \notin \{1, ..., s\}$. Let $q = \min(s-1, [s-k-n/p])$. Then, the semi-norm $|v|_{W_0^{s,p}(\mathbb{R}^n)}$ defines on $W_0^{s,p}(\mathbb{R}^n)/P_q$ a norm which is equivalent to the quotient norm.
Suppose first $\ell \geq 0$, then from Lemma 4.10 it follows that

\begin{align*}
\forall v \in W_0^{m-\ell,p}(\mathbb{R}^n), \quad & \inf_{c \in \mathbb{R}} \|v + c\|_{W_0^{m-\ell,p}(\mathbb{R}^n)} \lesssim \|v\|_{W_0^{m-\ell,p}(\mathbb{R}^n)}, \\
\forall v \in W_0^{m-\ell-1,p}(\mathbb{R}^n), \quad & \|v\|_{W_0^{m-\ell-1,p}(\mathbb{R}^n)} \lesssim \|v\|_{W_0^{m-\ell-1,p}(\mathbb{R}^n)}, \\
\forall v \in W_0^{m-\ell,p}(\mathbb{R}^n), \quad & \inf_{q \in P_{\ell-1}} \|v - q\|_{W_0^{m,p}(\mathbb{R}^n)} \lesssim \|v\|_{W_0^{m,p}(\mathbb{R}^n)},
\end{align*}

(4.7)\quad(4.8)\quad(4.9)

Now, let $P'_k$, $k$ being an integer, be the space of all the polynomials of degree less or to $\ell$ and depending only on $x_1, \ldots, x_n$. Namely, if $k \geq 0$,

$$P'_k = \mathbb{H}'_0 + \ldots + \mathbb{H}'_k.$$

If $k < 0$, $P'_k = \{0\}$. We shall use the following Lemma.

**Lemma 4.11.** Let $m$, $\ell$ and $p$ be as in Proposition 4.9. Then,

$$\inf_{q \in P'_k} \|u + q\|_{W_0^{m,p}(\mathbb{R}^n)} \lesssim \|u\|_{W_0^{m,p}(\mathbb{R}^n)} + \|\partial u/\partial x_1\|_{W_0^{m-1,p}(\mathbb{R}^n)}$$

for each $u \in W_0^{m,p}(\mathbb{R}^n)$.

**Proof.** (of Lemma 4.11) For each $k \geq 0$, we set

$$\Lambda_k = \{\mu = (0, \mu_2, \ldots, \mu_n), \quad |\mu| = \mu_2 + \ldots + \mu_n = k\}.$$ 

The proof of lemma 4.11 is based on an identification between the space $\mathbb{H}_k$, $k \geq 0$, and $\mathbb{R}^\text{card}(\Lambda_k)$ by way of the mapping

$$q \in \mathbb{H}_k \mapsto (\partial^\mu p)_{\mu \in \Lambda_k} \in \mathbb{R}^\text{card}(\Lambda_k).$$

Now from Lemma 4.10, we can write

$$\inf_{q \in P'_k} \|u + q\|_{W_0^{m,p}(\mathbb{R}^n)} = \inf_{p_1 \in \mathbb{H}'_1} \inf_{p_2 \in \mathbb{H}'_2} \ldots \inf_{p_\ell \in \mathbb{H}'_\ell} \inf_{p_0 \in \mathbb{H}_0} \|u - (p_\ell + p_{\ell-1} + \ldots + p_0)\|_{W_0^{m,p}(\mathbb{R}^n)}$$

$$\lesssim \inf_{p_1 \in \mathbb{H}'_1} \ldots \inf_{p_\ell \in \mathbb{H}'_\ell} \left\{ \sum_{\mu \in \Lambda_1} \|\partial^\mu u - \partial^\mu (p_\ell + \ldots + p_1)\|_{W_0^{m-1,p}(\mathbb{R}^n)} + \|\partial u/\partial x_1\|_{W_0^{m-1,p}(\mathbb{R}^n)} \right\}.$$

Since the polynomial $p_1 \in \mathbb{H}'_1$ can be identified to the constants $(\partial^\mu p_1, |\mu| = 1)$, it follows that

$$\inf_{p_1 \in \mathbb{H}'_1} \ldots \inf_{p_\ell \in \mathbb{H}'_\ell} \left\{ \sum_{\mu \in \Lambda_1} \|\partial^\mu u - \partial^\mu (p_\ell + \ldots + p_1)\|_{W_0^{m-1,p}(\mathbb{R}^n)} + \|\partial u/\partial x_1\|_{W_0^{m-1,p}(\mathbb{R}^n)} \right\}$$

$$\lesssim \inf_{p_1 \in \mathbb{H}'_1} \ldots \inf_{p_\ell \in \mathbb{H}'_\ell} \left\{ \sum_{\mu \in \Lambda_2} \|\partial^\mu u - \partial^\mu (p_\ell + \ldots + p_2)\|_{W_0^{m-1,p}(\mathbb{R}^n)} + \|\partial u/\partial x_1\|_{W_0^{m-1,p}(\mathbb{R}^n)} \right\}.$$

More generally the polynomial $p_k \in \mathbb{H}'_k$ can be identified to the constants $(\partial^\mu p_k, |\mu| = k)$. Then, by repeating the argument we deduce that

$$\inf_{q \in P'_k} \|u + q\|_{W_0^{m,p}(\mathbb{R}^n)} \lesssim \|u\|_{W_0^{m,p}(\mathbb{R}^n)} + \|\partial u/\partial x_1\|_{W_0^{m-1,p}(\mathbb{R}^n)}.$$
which ends the proof of lemma 4.11.

Now, since the space $Q_\ell$ can be identified to the product $P'_\ell \times x_1P_{\ell-2}$, we can write for each $u \in \tilde{W}^{m,p}_0(\mathbb{R}^n)$

$$\inf_{p \in Q_\ell} \|u - p\|_{\tilde{W}^{m,p}_0(\mathbb{R}^n)} = \inf_{(p_1,p_2) \in P'_\ell \times x_1P_{\ell-2}} \|u - (p_1 + p_2)\|_{\tilde{W}^{m,p}_0(\mathbb{R}^n)}$$

$$= \inf_{(p_1,p_2) \in P'_\ell \times x_1P_{\ell-2}} \left\{ \|u - (p_1 + p_2)\|_{W^{m,p}_0(\mathbb{R}^n)} + \|\frac{\partial u}{\partial x_1} - \frac{\partial p_2}{\partial x_1}\|_{W^{m-2,p}_0(\mathbb{R}^n)} \right\}$$

Now, using Lemma 4.11 and the fact that the mapping $p \rightarrow \frac{\partial p}{\partial x_1}$ is one to one from $x_1P_{\ell-2}$ into $P_{\ell-2}$, we get

$$\inf_{p \in Q_\ell} \|u - p\|_{\tilde{W}^{m,p}_0(\mathbb{R}^n)} \lesssim \left( |u|_{W^{m,p}_0(\mathbb{R}^n)} + \inf_{p \in P_{\ell-2}} \left\{ \|\frac{\partial u}{\partial x_1} - \frac{\partial p_2}{\partial x_1}\|_{W^{m-1,p}_0(\mathbb{R}^n)} + \|\frac{\partial u}{\partial x_1} - \frac{\partial p_2}{\partial x_1}\|_{W^{m-2,p}_0(\mathbb{R}^n)} \right\} \right)$$

Observe that

$$\|\frac{\partial u}{\partial x_1} - p\|_{W^{m-1,p}_0(\mathbb{R}^n)} = \|\frac{\partial u}{\partial x_1} - p\|_{W^{m-2,p}_0(\mathbb{R}^n)} + \|\frac{\partial u}{\partial x_1}\|_{W^{m-1,p}_0(\mathbb{R}^n)}$$

$$\lesssim \|\frac{\partial u}{\partial x_1} - p\|_{W^{m-2,p}_0(\mathbb{R}^n)} + \|\frac{\partial u}{\partial x_1}\|_{W^{m-1,p}_0(\mathbb{R}^n)} \quad (4.10)$$

and, from Lemma 4.10,

$$\inf_{p \in P_{\ell-2}} \|\frac{\partial u}{\partial x_1} - p\|_{W^{m-2,p}_0(\mathbb{R}^n)} \lesssim \left| \frac{\partial u}{\partial x_1} \right|_{W^{m-2,p}_0(\mathbb{R}^n)}$$

we deduce that

$$\inf_{q \in Q_\ell} \|u + q\|_{\tilde{W}^{m,p}_0(\mathbb{R}^n)} \lesssim |u|_{W^{m,p}_0(\mathbb{R}^n)} + \left| \frac{\partial u}{\partial x_1} \right|_{W^{m-2,p}_0(\mathbb{R}^n)}$$

which ends the proof. ■

**Proof of Theorem 4.5**

$T$ is clearly continuous from $\tilde{W}^{1,p}_0(\mathbb{R}^n)/\mathbb{P}_{[1-n/p]}$ into $W^{-1,p}_0(\mathbb{R}^n) \perp \mathbb{P}_{[1-n/p']}$. It is also injective, thanks to Lemma 4.4. Let us prove that it is onto. Let $f \in W^{-1,p}_0(\mathbb{R}^n) \perp \mathbb{P}_{[1-n/p']}$. Then, according to Proposition 4.1 of [3] and since $n/p' \neq 1$, there exists a vector function $w = (w_1, \ldots, w_n) \in L^p(\mathbb{R}^n)^n$ such that $\text{div} w = f$. We set

$$z = \mathcal{F}^{-1} \left( (|\xi|^2 + i\xi_1)^{-1} \mathcal{F} f \right) = \sum_{k=1}^n \mathcal{F}^{-1} \left( \frac{i\xi_k}{|\xi|^2 + 2i\xi_1} \mathcal{F} w_k \right).$$
We use the following multiplier theorem due to Lizorkin [21] (see also [14], Lemma 4.2 Ch. VII)

**Lemma 4.12.** Let \( j, k \in \{1, \ldots, n\} \). The operators

\[
    h \mapsto F^{-1} \left( \frac{\xi_k \xi_j}{|\xi|^2 + 2i \xi_1} F h \right), \quad h \mapsto F^{-1} \left( \frac{\xi_1}{|\xi|^2 + 2i \xi_1} F h \right),
\]

are continuous from \( L^p(\mathbb{R}^n) \) into \( L^p(\mathbb{R}^n) \), \( 1 < p < +\infty \).

Hence, for each \( j \leq n \), we have

\[
    \frac{\partial z}{\partial x_j} = -\sum_{k=1}^n F^{-1} \left( \frac{\xi_j \xi_k}{|\xi|^2 + 2i \xi_1} F w_k \right) \in L^p(\mathbb{R}^n),
\]

and

\[
    \|\nabla z\|_{L^p(\mathbb{R}^n)} \lesssim \|w\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{W^{-1,p}(\mathbb{R}^n)}.
\]

**Lemma 4.13.** (see [3]) Let \( h \in D'(\mathbb{R}^n) \) such that \( \nabla h \in L^p(\mathbb{R}^n) \), with \( 1 < p < +\infty \) and \( p \neq n \). Then, there exists a constant \( K \) such that \( h + K \in W^{1,p}_0(\mathbb{R}^n) \) and

\[
    \|h + K\|_{W^{1,p}_0(\mathbb{R}^n)} \lesssim \|\nabla h\|_{L^p(\mathbb{R}^n)}
\]

From this lemma, it follows that there exists a constant \( K \), such that \( z + K \in W^{1,p}_0(\mathbb{R}^n) \) and

\[
    \|z + K\|_{W^{1,p}_0(\mathbb{R}^n)} \lesssim \|\nabla z\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{W^{-1,p}(\mathbb{R}^n)}.
\]

We set \( u = z + K \). Then, \( u \in W^{1,p}_0(\mathbb{R}^n) \) and satisfies

\[
    -\Delta u + 2 \frac{\partial u}{\partial x_1} = -\Delta z + 2 \frac{\partial z}{\partial x_1} = f.
\]

In addition,

\[
    2 \frac{\partial u}{\partial x_1} = f + \Delta u \in W^{-1,p}_0(\mathbb{R}^n) \perp \mathbb{P}_{[1-n/p]},
\]

since the range of the Laplacian \( \Delta : W^{1,p}_0(\mathbb{R}^n) \to W^{-1,p}_0(\mathbb{R}^n) \) is nothing but \( W^{-1,p}_0(\mathbb{R}^n) \perp \mathbb{P}_{[1-n/p]} \).

**Proof of Theorem 4.6**

We start with the lemma

**Lemma 4.14.** The operator \( T : \mathbb{Q}_{\ell+2} \to \mathbb{P}_\ell \) is onto.

*Proof.* If \( \ell = 0 \) and \( p = c \in \mathbb{P}_\ell = \mathbb{R} \), then \( p = T(\frac{1}{2} c x_1) \). Suppose that \( \ell \geq 1 \) and let \( p \in \mathbb{P}_\ell \). Set

\[
    q = \frac{1}{2} \int_0^{x_1} p(t, x_2, \ldots, x_n) dt.
\]
Then, \( p - Tq \in \mathbb{P}_{\ell - 1} \). The proof is ended by applying the hypothesis of induction. ■

**Lemma 4.15.** Let \( m \geq 2 \) and and set \( \ell = [m - n/p] \). Let \( T^* \) be the adjoint operator of \( T : \mathring{W}_0^{m,p}(\mathbb{R}^n)/Q_\ell \rightarrow W_0^{m-2,p}(\mathbb{R}^n)/\mathbb{P}_{\ell - 2} \). Then, \( T^* \) is injective.

**Proof.** The adjoint operator \( T^* \) is defined from \( W_0^{m+2,p}(\mathbb{R}^n) \perp \mathbb{P}_{\ell - 2} \) into \( \mathring{W}_0^{m,p}(\mathbb{R}^n) \perp Q_\ell \) as follows:

\[
\forall u \in W_0^{m+2,p}(\mathbb{R}^n) \perp \mathbb{P}_{\ell - 2}, \forall v \in \mathring{W}_0^{m,p}(\mathbb{R}^n) \quad \langle T^* u, v \rangle = \langle u, Tv \rangle.
\]

If \( T^* u = 0 \) then \( -\Delta u - 2 \partial u/\partial x_1 = 0 \). Since \( u \) is a tempered distribution, and using the same argument of lemma 4.4, we deduce that \( u \) is a polynomial. Further, \( u = 0 \) since \( W_0^{m+2,p}(\mathbb{R}^n) \) contains only the trivial polynomial.

\[
\square
\]

**Lemma 4.16.** Let \( m \geq 2 \) be an integer and set \( \ell = [m - n/p] \). Suppose that \( n/p \neq \{1, \ldots, m\} \), then the operator \( T : \mathring{W}_0^{m,p}(\mathbb{R}^n)/Q_\ell \rightarrow W_0^{m-2,p}(\mathbb{R}^n)/\mathbb{P}_{\ell - 2} \) is an isomorphism.

**Proof.** The linear mapping \( T \) is clearly bounded. It is also injective; indeed, let \( u \in \mathring{W}_0^{m,p}(\mathbb{R}^n) \) such that \( Tu \in \mathbb{P}_{\ell - 2} \). According to lemma 4.14, there exists \( \theta \in Q_\ell \) such that \( T\theta = Tu \). Hence, \( T(\theta - u) = 0 \). Thus, by virtue of Lemma 4.4, \( \theta - u \) belongs to \( Q_\ell \) and \( u \in Q_\ell \).

Let us prove that the range of \( T \) is a closed subspace of \( W_0^{m-2,p}(\mathbb{R}^n)/\mathbb{P}_{\ell - 2} \). Let \( \alpha \) be an arbitrary multi-index such that \( |\alpha| = m - 2 \). Then, according to lemma 4.12, we have

\[
\| \partial^2(D^\alpha u)/\partial x_i \partial x_j \|_{L^p(\mathbb{R}^n)} = \| F^{-1}(\xi_i \xi_j F \partial^\alpha u)/L^p(\mathbb{R}^n) \|_{L^p(\mathbb{R}^n)}
\]

\[
= \| F^{-1}(\xi_i \xi_j F T \partial^\alpha u)/L^p(\mathbb{R}^n) \|_{L^p(\mathbb{R}^n)}
\]

\[
\leq \| \partial^\alpha Tu \|_{L^p(\mathbb{R}^n)}
\]

\[
\leq \| Tu \|_{W_0^{m-2,p}(\mathbb{R}^n)/\mathbb{P}_{\ell - 2}}
\]

\[
\leq \| Tu \|_{W_0^{m-2,p}(\mathbb{R}^n)/\mathbb{P}_{\ell - 2}}
\]

Hence,

\[
|u|_{W_0^{m,p}(\mathbb{R}^n)} \lesssim \| Tu \|_{W_0^{m-2,p}(\mathbb{R}^n)/\mathbb{P}_{\ell - 2}}
\]

Combining with Proposition 4.9 gives

\[
\| u \|_{W_0^{m,p}(\mathbb{R}^n)/Q_\ell} \lesssim \| Tu \|_{W_0^{m-2,p}(\mathbb{R}^n)/\mathbb{P}_{\ell - 2}}.
\]
We conclude that the range of $T$ is a closed subspace of $W_0^{-m-2,p}(\mathbb{R}^n) / \mathbb{F}_{t-2}$. By means of the Closed range theorem of Banach, and since the adjoint of $T$ is injective, we deduce that this range is nothing but the whole space $W_0^{-m-2,p}(\mathbb{R}^n) / \mathbb{F}_{t-2}$. \hfill \blacksquare

Theorem 4.6 stems directly from Lemma 4.14 and 4.16.

Proof of Theorem 4.7
Firstly, let $u \in W_0^{-m+2,p}(\mathbb{R}^n) \perp \mathbb{P}_{[m-n/p']}$. Then, for each $p \in \mathbb{P}_{[m-n/p']}$, we have

$$
\langle Tu, p \rangle = -(\Delta u, p) + 2\frac{\partial u}{\partial x_1}, p\rangle
\quad = -\langle u, \Delta p \rangle + 2\frac{\partial u}{\partial x_1}, p\rangle
\quad = 0.
$$

Hence, $Tu \in W_0^{-m,p}(\mathbb{R}^n) \perp \mathbb{P}_{[m-n/p']}$. On the other hand, Theorem 2.1 asserts that the operator

$$
\Delta : W_0^{m,p'}(\mathbb{R}^n) / \mathbb{P}_{[m-n/p']} \longrightarrow W_0^{-m-2,p'}(\mathbb{R}^n) / \mathbb{P}_{[m-n/p']},
$$

is an isomorphism if $m \geq 2$ and $n/p' \notin \{1, ..., m\}$. It follows that $\Delta^k$ is an isomorphism between $W_0^{2k,p'}(\mathbb{R}^n) / \mathbb{P}_{[2k-n/p']} and L^p(\mathbb{R}^n)$ for $k \geq 1$ and $n/p' \notin \{1, ..., 2k\}$. By duality and transposition $\Delta^k$ is also an isomorphism between $L^p(\mathbb{R}^n)$ and $W_0^{-2k,p}(\mathbb{R}^n) \perp \mathbb{P}_{[2k-n/p']}$. Let $\Delta^{-k}$ be its inverse. From theorem 4.6, we know that $T$ is an isomorphism between $\tilde{W}_0^{2,p}(\mathbb{R}^n) / \mathbb{Q}_{[2-n/p]} and L^p(\mathbb{R}^n)$ if $n/p \notin \{1, 2\}$. Moreover, it is quite clear that $\Delta^k$ is a continuous operator from $\tilde{W}_0^{2,p}(\mathbb{R}^n)$ into $\tilde{W}_0^{-2k+2,p}(\mathbb{R}^n) \perp \perp \mathbb{P}_{[2k-2-n/p']}$. The operator $\Delta^k \circ T^{-1} \circ \Delta^{-k}$ is well defined and continuous from $W_0^{-2k,p}(\mathbb{R}^n) \perp \mathbb{P}_{[2k-n/p']} into \tilde{W}_0^{-2k+2,p}(\mathbb{R}^n) \perp \perp \mathbb{P}_{[2k-2-n/p']} (here \ T^{-1} is from $L^p(\mathbb{R}^n)$ into $\tilde{W}_0^{2,p}(\mathbb{R}^n) / \mathbb{Q}_{[2-n/p]}$). Moreover, $T \circ (\Delta^k \circ T^{-1} \circ \Delta^{-k}) = I$, we deduce that $T$, considered as an operator from $\tilde{W}_0^{-2k+2,p}(\mathbb{R}^n) \perp \perp \mathbb{P}_{[2k-n/p']} into W_0^{-2k,p}(\mathbb{R}^n) \perp \mathbb{P}_{[2k-n/p']}$, is onto. It is also injective. It follows that is an isomorphism, thanks to Banach Theorem. \hfill \blacksquare

5. The Oseen system in $\mathbb{R}^n$. In this section, we consider, the nonhomogeneous Oseen problem: Given a vector field $f$ and a function $h$, we look for a solution $(u, \pi)$ satisfying

$$
-\Delta u + 2\frac{\partial u}{\partial x_1} + \nabla \pi = f \quad \text{in} \quad \mathbb{R}^n,
\quad \text{div} \ u = h \quad \text{in} \quad \mathbb{R}^n.
$$

We start with a characterization of the kernel of the operator $(u, \mu) \mapsto (Tu + \nabla \mu, \text{div} \ u)$. \hfill \blacksquare
Proposition 5.1. Let $m \geq 1$ be an integer, and set $\ell = -\lfloor n/p - m \rfloor - 1$. Then, $(u, \pi) \in \tilde{W}_0^{m,p}(\mathbb{R}^n) \times W_0^{m-1,p}(\mathbb{R}^n)$ is solution of (5.1) if and only if $(u, \pi) \in \mathcal{N}_\ell$, where

$$\mathcal{N}_\ell = \left\{ (\lambda, \mu) \in (\mathbb{Q}_\ell)^n \times P_{\ell-1}; -\Delta \lambda + \frac{\partial \lambda}{\partial x_1} + \nabla \mu = 0, \text{ div } \lambda = 0 \right\}.$$

Moreover, a pair $(\lambda, \mu)$ belongs to $\mathcal{N}_\ell$ if and only if there exists a vector function $\Phi = (\Phi_1, \ldots, \Phi_n) \in (\mathbb{P}_{\ell+2})^n$ such that $\text{div } \Phi \in \mathbb{Q}_{\ell+1}$, $(\Delta \circ T)\Phi_i = 0$, $i = 1, \ldots, n$, and

$$\begin{align*}
\lambda &= \Delta \Phi - \nabla(\text{div } \Phi) \\
\mu &= T(\text{div } \Phi).
\end{align*}$$

(5.2)

Proof of Proposition 5.1

Let $(u, \pi) \in \mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ be a solution of (5.1), with $f = 0$. Then taking the divergence of the first equation of (5.1), we obtain

$$\Delta \pi = 0.$$ 

Thus, $\pi$ is a harmonic polynomial. Now, we have

$$\Delta \left( -\Delta u + 2 \frac{\partial u}{\partial x_1} \right) = -\Delta(\nabla \pi) = 0.$$ 

It follows that

$$|\xi|^2 (|\xi|^2 - i\xi_1) \hat{u}(\xi) = 0.$$ 

Hence the support of $\hat{u}$ is included in $\{0\}$ and consequently $u$ is a polynomial. If in addition $u \in \tilde{W}_0^{m,p}(\mathbb{R}^n)$ and $\pi \in W_0^{m-1,p}(\mathbb{R}^n)$, then necessarily $u \in (\mathbb{Q}_\ell)^n$ and $\pi \in P_{\ell-1}$. This ends the proof of the first assertion of the proposition. Now, according to the Lemma 4.14 there exists a polynomial $r \in \mathbb{Q}_{\ell+1}$ such that $Tr = \pi$. The vector function $u + \nabla r$ belongs to $(\mathbb{P}_\ell)^n$. Hence, there exists a vectorial function $\varphi \in (\mathbb{P}_{\ell+2})^n$ such that $\Delta \varphi = u + \nabla r$ (since $\Delta P_{\ell+2} = P_\ell$). Furthermore, by applying the divergence operator to this identity one deduces that the function $s = \text{div } \varphi - r$ is a harmonic polynomial, and consequently belongs to $\mathbb{P}_{\ell+1}^\Delta$. Since $\text{div } (\mathbb{P}_{k+1}^\Delta) = \mathbb{P}_k^\Delta$ (see [15] or [1]), there exists a polynomial $\theta \in (\mathbb{P}_{\ell+2}^\Delta)^n$ such that $\text{div } \theta = -s$. Set $\Phi = \varphi + \theta \in (\mathbb{P}_{\ell+2})^n$. Then, $\text{div } \Phi = r$, $\Delta \Phi = \Delta \varphi$. It follows that $u = \Delta \Phi - \nabla(\text{div } \Phi)$ and $\pi = T(\text{div } \Phi)$. Since $Tu_i + \partial_i \pi = 0$, we deduce that $\Delta(T\Phi_i) = 0$, $i = 1, \ldots, n$ which ends the proof of (5.2). The converse is straightforward. Indeed, let $\Phi = (\Phi_1, \ldots, \Phi_n) \in (\mathbb{P}_{\ell+2})^n$ such that $\text{div } \Phi \in \mathbb{Q}_{\ell+1}$, $(\Delta \circ T)\Phi_i = 0$, $i = 1, \ldots, n$, and consider the pair $(\lambda, \mu)$ given by (5.2). Since $T\mathbb{Q}_{\ell+1} = \mathbb{P}_\ell$ then obviously we have $\mu \in \mathbb{P}_{\ell-2}^\Delta$. Moreover, $\Delta \Phi \in (\mathbb{Q}_\ell)^n$ since $\Delta \Phi \in \mathbb{P}_\ell^n$ and

$$2 \frac{\partial (\Delta \Phi)}{\partial x_1} = T\Delta \Phi + \Delta^2 \Phi = \Delta^2 \Phi \in \mathbb{P}_{\ell-2}^n.$$
Let us notice that $\mathcal{N}_f = \{(0,0)\}$ if $\ell < 0$, $\mathcal{N}_0 = \mathbb{R} \times \{0\}$ and $\mathcal{N}_1 = \mathbb{Q}_1^+ \times \mathbb{R}$. Our first existence result is for $f \in W_0^{-1,p}((\mathbb{R}^n))$ and $g \in L^p(\mathbb{R}^n)$. Note that a different proof of the next theorem, in the particular case $n = 3$, is given in [4].

**Theorem 5.2.** Assume $n/p \neq 1$ and $n/p' \neq 1$.
Let $f \in W_0^{-1,p}((\mathbb{R}^n)) \perp L^p(\mathbb{R}^n)$ and $h \in \tilde{W}_0^{0,p}(\mathbb{R}^n)$ satisfying

$$\forall q \in P[2-n/p'], \langle \frac{\partial h}{\partial x_1}, q \rangle = 0 \quad (5.3)$$

Then the Oseen system (5.1) has a unique solution

$$(u, \pi) \in (\tilde{W}_0^{1,p}(\mathbb{R}^n) \times L^p(\mathbb{R}^n))/\mathcal{N}_{[1-n/p]}.$$

Moreover, the following estimate holds

$$\inf_{\lambda \in \mathbb{R}[1-n/p]} \|u + \lambda\|_{\tilde{W}_0^{1,p}(\mathbb{R}^n)} + \|\pi\|_{L^p(\mathbb{R}^n)} \lesssim \left( \|f\|_{W_0^{-1,p}(\mathbb{R}^n)} + \|h\|_{\tilde{W}_0^{0,p}(\mathbb{R}^n)} \right). \quad (5.4)$$

**Proof of Theorem 5.2**

1) Consider first $(u, \pi) \in (\tilde{W}_0^{1,p}(\mathbb{R}^n) \times L^p(\mathbb{R}^n))$. Then $-\Delta u + 2 \frac{\partial u}{\partial x_1} + \nabla \pi \in W_0^{-1,p}(\mathbb{R}^n)$. Thus, due to the density of $D(\mathbb{R}^n)$ in $\tilde{W}_0^{1,p}(\mathbb{R}^n)$, for any $\lambda \in W_0^{1,p}(\mathbb{R}^n)$, we have

$$\langle -\Delta u + 2 \frac{\partial u}{\partial x_1} + \nabla \pi, \lambda \rangle = \langle u, -\Delta \lambda - 2 \frac{\partial \lambda}{\partial x_1} \rangle_{W_0^{-1,p}(\mathbb{R}^n) \times W_0^{1,p}(\mathbb{R}^n)} + \langle \pi, \div \lambda \rangle_{L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)}.$$
Hence, $f - \nabla \pi \in W^{-1,p}_0(\mathbb{R}^n)$. Furthermore, since the polynomials of $\mathbb{P}_{1-n/p'}$ are at most constants, for any $\lambda \in \mathbb{P}_{1-n/p'}$, we can write

$$\langle \nabla \pi, \lambda \rangle_{W^{-1,p}_0(\mathbb{R}^n) \times W^{1,p'}_0(\mathbb{R}^n)} = \langle \pi, \text{div} \lambda \rangle_{L^p(\mathbb{R}^n) \times L^{p'}(\mathbb{R}^n)} = 0.$$  

We deduce that $f - \nabla \pi \in W^{-1,p}_0(\mathbb{R}^n)$. Furthermore, since the polynomials of $\mathbb{P}_{1-n/p'}$ are at most constants, for any $\lambda \in \mathbb{P}_{1-n/p'}$, we can write

$$\langle \nabla \pi, \lambda \rangle_{W^{-1,p}_0(\mathbb{R}^n) \times W^{1,p'}_0(\mathbb{R}^n)} = \langle \pi, \text{div} \lambda \rangle_{L^p(\mathbb{R}^n) \times L^{p'}(\mathbb{R}^n)} = 0.$$  

We deduce that $f - \nabla \pi \in W^{-1,p}_0(\mathbb{R}^n)$. Furthermore, since the polynomials of $\mathbb{P}_{1-n/p'}$ are at most constants, for any $\lambda \in \mathbb{P}_{1-n/p'}$, we can write

$$\langle \nabla \pi, \lambda \rangle_{W^{-1,p}_0(\mathbb{R}^n) \times W^{1,p'}_0(\mathbb{R}^n)} = \langle \pi, \text{div} \lambda \rangle_{L^p(\mathbb{R}^n) \times L^{p'}(\mathbb{R}^n)} = 0.$$  

We deduce that $f - \nabla \pi \in W^{-1,p}_0(\mathbb{R}^n)$. Furthermore, since the polynomials of $\mathbb{P}_{1-n/p'}$ are at most constants, for any $\lambda \in \mathbb{P}_{1-n/p'}$, we can write

$$\langle \nabla \pi, \lambda \rangle_{W^{-1,p}_0(\mathbb{R}^n) \times W^{1,p'}_0(\mathbb{R}^n)} = \langle \pi, \text{div} \lambda \rangle_{L^p(\mathbb{R}^n) \times L^{p'}(\mathbb{R}^n)} = 0.$$  

We deduce that $f - \nabla \pi \in W^{-1,p}_0(\mathbb{R}^n)$. Furthermore, since the polynomials of $\mathbb{P}_{1-n/p'}$ are at most constants, for any $\lambda \in \mathbb{P}_{1-n/p'}$, we can write

$$\langle \nabla \pi, \lambda \rangle_{W^{-1,p}_0(\mathbb{R}^n) \times W^{1,p'}_0(\mathbb{R}^n)} = \langle \pi, \text{div} \lambda \rangle_{L^p(\mathbb{R}^n) \times L^{p'}(\mathbb{R}^n)} = 0.$$  

We deduce that $f - \nabla \pi \in W^{-1,p}_0(\mathbb{R}^n)$. Furthermore, since the polynomials of $\mathbb{P}_{1-n/p'}$ are at most constants, for any $\lambda \in \mathbb{P}_{1-n/p'}$, we can write

$$\langle \nabla \pi, \lambda \rangle_{W^{-1,p}_0(\mathbb{R}^n) \times W^{1,p'}_0(\mathbb{R}^n)} = \langle \pi, \text{div} \lambda \rangle_{L^p(\mathbb{R}^n) \times L^{p'}(\mathbb{R}^n)} = 0.$$  

We deduce that $f - \nabla \pi \in W^{-1,p}_0(\mathbb{R}^n)$. Furthermore, since the polynomials of $\mathbb{P}_{1-n/p'}$ are at most constants, for any $\lambda \in \mathbb{P}_{1-n/p'}$, we can write

$$\langle \nabla \pi, \lambda \rangle_{W^{-1,p}_0(\mathbb{R}^n) \times W^{1,p'}_0(\mathbb{R}^n)} = \langle \pi, \text{div} \lambda \rangle_{L^p(\mathbb{R}^n) \times L^{p'}(\mathbb{R}^n)} = 0.$$  

We deduce that $f - \nabla \pi \in W^{-1,p}_0(\mathbb{R}^n)$. Furthermore, since the polynomials of $\mathbb{P}_{1-n/p'}$ are at most constants, for any $\lambda \in \mathbb{P}_{1-n/p'}$, we can write

$$\langle \nabla \pi, \lambda \rangle_{W^{-1,p}_0(\mathbb{R}^n) \times W^{1,p'}_0(\mathbb{R}^n)} = \langle \pi, \text{div} \lambda \rangle_{L^p(\mathbb{R}^n) \times L^{p'}(\mathbb{R}^n)} = 0.$$  

We deduce that $f - \nabla \pi \in W^{-1,p}_0(\mathbb{R}^n)$.
Next, set $\lambda = (\lambda_1, \ldots, \lambda_n)$ the polynomial such that $\lambda_1 = 0$ and for any integer $i \geq 2$,
\[
\lambda_i = \frac{1}{n-1} \int_0^{x_i} p(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n) dt + h_i.
\]
One can verifies that $\lambda \in Q^+_t$ and satisfies
\[
\text{div } \lambda = p,
\]
which ends the proof of the proposition. ■

We are now ready to prove our next result.

**Theorem 5.5.** Let $m \geq 2$ be an integer and suppose that $n/p \neq \{1, \ldots, m\}$ and $n/p' \neq 1$ if $m = 2$. Let $f \in W^{m-2,p}_0(\mathbb{R}^n)$ and $h \in \tilde{W}^{m-1,p}_0(\mathbb{R}^n)$. Then the Oseen system (5.1) has a unique solution $(u, \pi) \in (\tilde{W}^{m,p}_0(\mathbb{R}^n) \times \tilde{W}^{m-1,p}_0(\mathbb{R}^n))/\mathcal{N}_{[m-n/p]}$. Moreover, the following estimate holds
\[
\inf_{(\lambda, \mu) \in \mathcal{N}_{[m-n/p]}} \left( \|u + \lambda\|_{\tilde{W}^{m,p}_0(\mathbb{R}^n)} + \|\pi + \mu\|_{\tilde{W}^{m-1,p}_0(\mathbb{R}^n)} \right) 
\lesssim \left( \|f\|_{W^{m-2,p}_0(\mathbb{R}^n)} + \|h\|_{\tilde{W}^{m-1,p}_0(\mathbb{R}^n)} \right).
\]

**Proof of Theorem 5.5**

1) If $(u, \pi) \in \tilde{W}^{m,p}_0(\mathbb{R}^n) \times \tilde{W}^{m-1,p}_0(\mathbb{R}^n)$ satisfies (5.1), then from Proposition 5.1, $(u, \pi) \in \mathcal{N}_{[m-n/p]}$.

2) The beginning of the proof of existence is similar to that of the preceding theorem. Given $f \in W^{m-2,p}_0(\mathbb{R}^n)$ and $h \in \tilde{W}^{m-1,p}_0(\mathbb{R}^n)$, we have $\text{div } f - Th \in W^{m-3,p}_0(\mathbb{R}^n)$. Considering first the case $m \geq 3$, and using Theorem 2.1 1), we deduce the existence a function $\pi \in W^{m-1,p}_0(\mathbb{R}^n)$ such that
\[
\Delta \pi = \text{div } f - Th.
\]
If $m = 2$, then, we easily see that $\text{div } f - Th \in W^{-1,1,p}_0(\mathbb{R}^n) \perp \mathbb{P}_{[1-n/p]}$. Again from Theorem 2.1, there exists a unique function $\pi \in W^{1,p}_0(\mathbb{R}^n)$ satisfying the previous equality. Thus summarizing, we conclude that for $m \geq 2$, there exists a function $\pi \in W^{m-1,p}_0(\mathbb{R}^n)$ satisfying the previous Laplace equation. Next, we see that $f - \nabla \pi \in W^{m-2,p}_0(\mathbb{R}^n)$. Thanks to Theorem 4.6, there exists a vector field $u \in \tilde{W}^{m,p}_0(\mathbb{R}^n)$ satisfying
\[
-\Delta u + 2 \frac{\partial u}{\partial x_1} = f - \nabla \pi.
\]
It follows that $\text{div } u - h \in W^{m-1,p}_0(\mathbb{R}^n)$ verifies
\[
T(\text{div } u - h) = 0.
\]
Therefore, \( \text{div } \mathbf{u} - h = q \in \mathbb{Q}_m^{r-n/p} \). Proposition 5.4 implies that there exists a polynomial \( \lambda \in \mathbb{Q}_{m-n/p}^+ \subset W_0^{m,p}(\mathbb{R}^n) \) such that

\[
\text{div } \lambda = q.
\]

Thus, \( (\mathbf{u} - \lambda, \pi) \in \tilde{W}_0^{m,p}(\mathbb{R}^n) \times W_0^{m-1,p}(\mathbb{R}^n) \) is a solution of the system (5.1). \( \blacksquare \)

**Theorem 5.6.** Let \( m \geq 2 \) be an integer. Suppose that \( n/p' \neq \{1, \ldots, m\} \) and \( n/p \neq \{1, 2[m/2]+2-m\} \). Let \( f \in W_0^{-m,p}(\mathbb{R}^n) \perp \mathbb{P}_{m-n/p'} \) and \( h \in W_0^{-m+1,p}(\mathbb{R}^n) \perp \mathbb{P}_{m+n/p} \). Then the Oseen system (5.1) has a unique solution \( (\mathbf{u}, \pi) \in ((\tilde{W}_0^{-m+2,p}(\mathbb{R}^n) \perp \mathbb{P}_{m-n/p'}) \times W_0^{-m+1,p}(\mathbb{R}^n)) \). Moreover, the following estimate holds

\[
\| \mathbf{u} \|_{\tilde{W}_0^{-m+2,p}(\mathbb{R}^n)} + \| \pi \|_{W_0^{-m+1,p}(\mathbb{R}^n)} \leq \left( \| f \|_{W_0^{-m,p}(\mathbb{R}^n)} + \| h \|_{\tilde{W}_0^{-m+1,p}(\mathbb{R}^n)} \right). \quad (5.9)
\]

**Proof of Theorem 5.6**

The proof is again similar to that of the previous ones. Given \( f \in W_0^{-m,p}(\mathbb{R}^n) \perp \mathbb{P}_{m-n/p'} \) and \( h \in \tilde{W}_0^{-m+1,p}(\mathbb{R}^n) \perp \mathbb{P}_{m+n/p} \), then div \( f - Th \in W_0^{-m-1,p}(\mathbb{R}^n) \perp \mathbb{P}_{m+1-n/p} \).

Now using the isomorphism of the Laplace operator defined by

\[
\Delta : W_0^{-m+1,p}(\mathbb{R}^n) \rightarrow W_0^{-m-1,p}(\mathbb{R}^n) \perp \mathbb{P}_{m+1-n/p'},
\]

there exists a unique function \( \pi \in W_0^{-m+1,p}(\mathbb{R}^n) \) such that

\[
\Delta \pi = \text{div } f - Th.
\]

Now since \( \Delta \pi \in W_0^{-m-1,p}(\mathbb{R}^n) \perp \mathbb{P}_{m+1-n/p'} \), we deduce that \( \nabla \pi \in W_0^{-m,p}(\mathbb{R}^n) \perp \mathbb{P}_{m-n/p'} \) which implies that \( f - \nabla \pi \in W_0^{-m,p}(\mathbb{R}^n) \perp \mathbb{P}_{m-n/p'} \). Hence using Theorem 4.7, there exists a unique vector field \( \mathbf{u} \in \tilde{W}_0^{-m+2,p}(\mathbb{R}^n) \perp \mathbb{P}_{m-n/p'} \) such that

\[
-\Delta \mathbf{u} + \frac{\partial \mathbf{u}}{\partial x_1} = f - \nabla \pi.
\]

Finally, since the space \( W_0^{-m+1,p}(\mathbb{R}^n) \) does not contain polynomials, we easily deduce that \( \text{div } \mathbf{u} = h \) which ends the proof. \( \blacksquare \)

**Appendix A. Proof of Proposition 3.2.**

Suppose that \( n \geq 3 \)

(a) We have

\[
\int_{\mathbb{R}^n} |O(\mathbf{x})|^p d\mathbf{x} = \int_{|\mathbf{x}| \geq 2x_1} |O(\mathbf{x})|^p d\mathbf{x} + \int_{|\mathbf{x}| \leq 2x_1} |O(\mathbf{x})|^p d\mathbf{x}
\]

If \( |\mathbf{x}| \geq 2x_1 \), then (2.1) gives

\[
|O(\mathbf{x})| \leq c \frac{(1 + |\mathbf{x}|)^{(n-3)/2}}{|\mathbf{x}|^{n-2}} e^{x_1 - |\mathbf{x}|} \leq c \frac{(1 + |\mathbf{x}|)^{(n-3)/2}}{|\mathbf{x}|^{n-2}} e^{-|\mathbf{x}|/2},
\]
Thus,
\[
\int_{|x| \geq 2x_1} |O(x)|^p dx \leq \int_{\mathbb{R}^n} \frac{(1 + |x|)^{(n-3)/2}}{|x|^{p(n-2)}} e^{-\rho |x|/2} dx < +\infty,
\]
if \(p(n-2) < n\). Now, let \(\alpha\) be a real, \(0 < \alpha < 1\). Using (2.1) in each region
\[\{x; (1 + \alpha^{k+1})x_1 \leq |x| \leq (1 + \alpha^k)x_1\}\] gives
\[
\int_{|x| \leq 2x_1} |O(x)|^p dx \leq \sum_{k=0}^{+\infty} \int_{(1 + \alpha^{k+1})x_1 \leq |x| \leq (1 + \alpha^k)x_1} |O(x)|^p dx.
\]
\[
\leq c_2 \sum_{k=0}^{+\infty} \int_0^{+\infty} \left( \int_{(1 + \alpha^{k+1})x_1 \leq |x| \leq (1 + \alpha^k)x_1} p_1^{n-2} d\rho_1 \right) \frac{(1 + x_1)^{(n-3)/2} e^{-\rho_1 x_1}}{x_1^{(n-2)p-n+1}} dx_1,
\]
\[
\leq c_3(\alpha) \sum_{k=0}^{+\infty} \alpha^k \int_0^{+\infty} \frac{(1 + x_1)^{(n-3)/2} e^{-\rho_1 x_1}}{x_1^{(n-2)p-n+1}} dx_1,
\]
\[
\leq c_4(\alpha) \sum_{k=0}^{+\infty} \alpha^k \int_0^{+\infty} \frac{(1 + x_1)^{(n-3)/2} e^{-\rho_1 x_1}}{x_1^{(n-2)p-n+1}} dx_1,
\]
where \(\rho_1 = (x_2^2 + ... + x_n^2)^{1/2}\). Hence,
\[
\int_{|x| \leq 2x_1} |O(x)|^p dx < +\infty
\]
if \(2 < p < n/(n-2)\). This condition is possible only if \(n = 3\), and this ends the proof of part (a).

(b) If \(|x| \geq 2x_1\), then (2.1) gives again
\[
|O(x) - g_2(x)| \leq c \frac{(1 + |x|)^{(n-3)/2}}{|x|^{n-2}} e^{x_1 - |x|} |1 - e^{-x_1}| \leq c \frac{(1 + |x|)^{(n-3)/2}}{|x|^{n-3}} e^{-|x|/2},
\]
Similarly, in each region \(\{x; (1 + \alpha^{k+1})x_1 \leq |x| \leq (1 + \alpha^k)x_1\}\), we have
\[
|O(x) - g_2(x)| \leq c_1(\alpha) \frac{(1 + x_1)^{(n-3)/2}}{x_1^{n-2}} e^{x_1 - |x|} |1 - e^{-x_1}| \leq c_2(\alpha) \frac{(1 + x_1)^{(n-3)/2}}{x_1^{n-3}} e^{-\alpha x_1},
\]
where we used the inequality \((1 + x_1)|1 - e^{-x_1}| \leq cx_1\). The constants \(c_1\) and \(c_2\) do not depend on \(k\). We prove similarly that
\[
\int_{\mathbb{R}^n} |O(x) - g_2|^p dx < +\infty,
\]
if \(2 < p < n/(n-3)\), which is only possible if \(3 \leq n \leq 5\). If \(n = 3\), we have clearly \(O(x) - g_2 \in L^\infty(\mathbb{R}^3)\).

Appendix B. Proof of Corollary 3.4.
Part (a) follows from Proposition 3.2 and Young’s inequality
\[
\|O \ast f\|_{L^r(\mathbb{R}^3)} \lesssim \|O\|_{L^p(\mathbb{R}^3)} \|f\|_{L^p(\mathbb{R}^3)},
\]
with
\[
\frac{1}{\theta} = 1 + \frac{1}{r} - \frac{1}{p} \quad \text{(hence } 2 < \theta < 3)\,.
\]
When \(r = p^*_2, 1 < p < 3/2\), one can use the inequality \(|O \ast f| \leq I_2(|f|)\), where \(I_2(f)\) is the Riesz potential of \(f\) defined by (cf. [24])
\[
I_2(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-2}}. \tag{5.10}
\]
Part (b) comes also from Proposition 3.2 and Young’s inequality.
Part (c) follows from the following lemma combined with Marcinkiewicz interpolation theorem (see for instance [24], Appendix B).

**Lemma 5.7.** Suppose that \(1 < p < 3/2\) and \(n = 3\). If \(f \in L^p(\mathbb{R}^3)\) then
\[
m\{x; |(O(x) - g_2) \ast f| > \lambda\} \leq \left( A_p \frac{\|f\|_{L^p(\mathbb{R}^3)}}{\lambda} \right)^q,
\]
where \(m\) denotes the Lebesgue measure and
\[
q = \frac{p(p + 3)}{(3 - 2p)}.
\]

**Proof.** Following Stein (see [24], Chap. V. 1.2 Theorem 1), we set
\[
K_1(x) = O(x) - g_2(x) \quad \text{if } |x| \leq \mu, \quad K_1(x) = 0 \quad \text{if } |x| > \mu
\]
\[
K_2(x) = O(x) - g_2(x) \quad \text{if } |x| > \mu, \quad K_2(x) = 0 \quad \text{if } |x| \leq \mu.
\]
We suppose without loss of generality that \(\|f\|_{L^p(\mathbb{R}^n)} = 1\). Then,
\[
|K_1(x)| \leq c \frac{|e^{x_1 - |x|} - e^{-|x|}|}{|x|} \leq c \frac{|x_1|}{|x|} \leq c,
\]
and we deduce that \(K_1 \in L^1(\mathbb{R}^3)\). Moreover,
\[
\|K_1\|_{L^1(\mathbb{R}^3)} \leq c \int_{|x| \leq \mu} dx = \mu^3.
\]
On the other hand,
\[
\|K_2 \ast f\|_{L^\infty(\mathbb{R}^3)} \leq \|K_2\|_{L^{p'}(\mathbb{R}^3)} \|f\|_{L^p(\mathbb{R}^3)}.
\]
We have also \(|K_2(x)| \leq \frac{c}{|x|}\). Thus,
\[
\int_{\mathbb{R}^3} |K_2(x)|^{p'} dx \leq c \int_{|x| \geq \mu} \frac{1}{|x|^{p'}} dx = \mu^{3-p'}.
\]
We choose $\mu = \lambda^{p'/(3-p')}$. Hence,

$$\|K_2 * f\|_{L^\infty(\mathbb{R}^3)} \leq \|K_2\|_{L^{p'}(\mathbb{R}^3)} \|f\|_{L^p(\mathbb{R}^3)} \leq \lambda,$$

and

$$m(\{x; \ |K_2 * f| > \lambda\}) = 0.$$

Moreover,

$$\|K_1\|_{L^1(\mathbb{R}^3)} \leq \lambda^{3p'/(3-p')}.$$

We get

$$\|K_1 * f\|_{L^p(\mathbb{R}^2)}^p \leq \lambda^{3q}/(3-q) - p = \lambda^{-(p+3)p/(3-2p)}.$$

Thus,

$$m(\{x; \ |(O(x) - g_2) * f| > \lambda\}) \leq m(\{x; \ |K_1 * f| > \lambda\}) + m(\{x; \ |K_2 * f| > \lambda\}).$$

\[
\leq \lambda^{-p} \|K_1 * f\|_{L^p(\mathbb{R}^3)}^p \leq \lambda^{-p} \|K_1\|_{L^1(\mathbb{R}^3)}^p \|f\|_{L^p(\mathbb{R}^3)}^p \leq \lambda^{3p!(3-p') - p} = \lambda^{-(p+3)p/(3-2p)}.
\]

\[\blacksquare\]

REFERENCES


