Cartan sub-C*-algebras in C*-algebras

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1. **C*-algebra constructions.**
2. **Effective versus topologically principal.**
3. **Cartan subalgebras in C*-algebras.**
4. **Examples and counterexamples.**

Groupoids

Definition

A groupoid is a small category $(G, G^{(0)})$ such that every arrow is invertible.

$$r, s : G \to G^{(0)}$$

$$\begin{array}{ccc} G^{(2)} & \to & G \\ (\gamma, \gamma') & \mapsto & \gamma\gamma' \end{array} \qquad \begin{array}{ccc} G & \to & G \\ \gamma & \mapsto & \gamma^{-1} \end{array}$$

Example: action of a group Γ on a space X: $\begin{array}{ccc} X \times \Gamma & \to & X \\ (x, g) & \mapsto & xg \end{array}$

$$G = \{(x, g, y) \in X \times \Gamma \times X : y = xg\}$$

$$r(x, g, y) = x \qquad s(x, g, y) = y$$

$$(x, g, y)(y, h, z) = (x, gh, z) \qquad (x, g, y)^{-1} = (y, g^{-1}, x)$$

Haar systems

Definition

Let G be locally compact Hausdorff topological groupoid. A Haar system $\lambda = (\lambda^x)$ is a family of measures λ^x with support $G^x = r^{-1}(x)$ satisfying

- (continuity) $\forall f \in C_c(G), x \mapsto \int f d\lambda^x$ is continuous;
- (left invariance) $\forall \gamma \in G$, $\gamma\lambda^{s(\gamma)} = \lambda^{r(\gamma)}$.

In the previous example of a group action (Γ, X), a left Haar measure λ on Γ defines a Haar system (λ^x) on G such that
$\int f d\lambda^x = \int f(x, g, xg) d\lambda(g)$.

Definition

We say that the topological groupoid G is étale if the range map $r : G \to G^{(0)}$ is a local homeomorphism.

An étale locally compact Hausdorff groupoid has a natural Haar system, given by $\int f d\lambda^x = \sum_{r(\gamma)=x} f(\gamma)$.

The C*-algebra $C_r^*(G)$

Let (G,λ) be a locally compact Hausdorff groupoid endowed with a Haar system. The following operations turn the space $C_c(G)$ of compactly supported complex-valued continuous functions on G into an involutive algebra:

$$f*g(\gamma)=\int f(\gamma\gamma')g(\gamma'^{-1})d\lambda^{s(\gamma)}(\gamma');$$

$$f^*(\gamma)=\overline{f(\gamma^{-1})}.$$

For each $x\in G^{(0)}$, one defines the representation π_x of $C_c(G)$ on the Hilbert space $L^2(G_x,\lambda_x)$, where $G_x=s^{-1}(x)$ and $\lambda_x=(\lambda^x)^{-1}$, by $\pi_x(f)\xi=f*\xi$. One defines the reduced norm $\|f\|_r=\sup\|\pi_x(f)\|$. The reduced C*-algebra $C_r^*(G)$ is the completion of $C_c(G)$ for the reduced norm.

The C*-algebra $C_r^*(G, E)$

We shall need a slight generalization of the above construction.

Definition

Let G be a groupoid. A twist over G is a groupoid extension

$$\mathbb{T} \times X \rightarrowtail E \twoheadrightarrow G$$

where $X = G^{(0)} = E^{(0)}$ and $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

For example, a 2-Čech cocycle $\sigma = (\sigma_{ijk})$ relative to an open cover (U_i) of a topological space X defines a twist E_σ over the groupoid $G = \{(i, x, j) : x \in U_i \cap U_j\}$ of the open cover.

We replace the complex-valued functions by the sections of the associated complex line bundle. Essentially the same formulas as above provide the C*-algebra $C_r^*(G, E)$.

Reconstruction

When passing from the groupoid G to the C*-algebra $A = C_r^*(G)$, in general much information is lost. However, some extra piece of structure allows to recover G from A. In the case of a group, the coproduct does the job. When G is étale, the commutative C*-algebra $B = C_0(X)$, where $X = G^{(0)}$ is a subalgebra of $A = C_r^*(G, E)$. Thus our construction provides a pair (A, B) where A is a C*-algebra and B is a commutative sub-C*-algebra rather than just a C*-algebra. We shall study the case when the pair (A, B) completely determines the twisted groupoid (G, E).

Proposition (R 80)

Let G be an étale second countable locally compact Hausdorff groupoid. Then $B = C_0(X)$ is maximal abelian self-adjoint in $A = C_r^*(G, E)$ iff G is topologically principal.

Let X be a topological space. A partial homeomorphism of X is a homeomorphism $S : \mathrm{D}(S) \to \mathrm{R}(S)$, where $\mathrm{D}(S)$ and $\mathrm{R}(S)$ are open subsets of X. One defines the composition ST and the inverse S^{-1}. A **pseudogroup** on X is a family $\mathcal{G}$ of partial homeomorphisms of X closed under composition and inverse.

Given a partial homeomorphism S and $y \in \mathrm{D}(S)$, we denote by $[Sy, S, y]$ the germ of S at y.

One can associate to a pseudogroup $\mathcal{G}$ on X its **groupoid of germs** G. Its elements are the germs of $\mathcal{G}$. Its groupoid structure is:

$$r([x, S, y]) = x \quad s([x, S, y]) = y$$

$$[x, S, y][y, T, z] = [x, ST, z]$$

$$[x, S, y]^{-1} = [y, S^{-1}, x]$$

Its topology is the topology of germs. It turns G into a topological groupoid, which is locally compact if X is so, but not necessarily Hausdorff.

Conversely, let G be an étale groupoid with $G^{(0)} = X$. Its open bisections define a pseudogroup $\mathcal{G}$ on X, hence a groupoid of germs H. We have the groupoid extension:

$$\mathrm{Int}(G') \rightarrowtail G \twoheadrightarrow H$$

where $G' = \{\gamma \in G : r(\gamma) = s(\gamma)\}$ and $\mathrm{Int}(G')$ is its interior.

Definition

An étale groupoid G is said to be

- effective if it is isomorphic to its groupoid of germs;
- topologically principal if the set of units without isotropy is dense in $G^{(0)}$.

Proposition

Let G be an étale groupoid.

- *if G is Hausdorff and topologically principal, then it is effective;*
- *if G is second countable, if its unit space is a Baire space and if it is effective, then it is topologically principal.*

Examples

- Tranverse holonomy groupoids of foliated manifolds.
- The groupoid of a topologically free semi-group action

$$T : X \times N \to X :$$

$$G(X,T) = \{(x, m-n, y) : T(m)x = T(n)y\}.$$

- Minimal Cantor systems.
- Markov chains satisfying Cuntz-Krieger condition (I).
- One sided-shifts on infinite path spaces on graphs satisfying exit condition (L).

Cartan subalgebras

Definition (Kumjian 86)

Let B be a sub-C*-algebra of a C*-algebra A. One says that B is regular if its normalizer $N(B) = \{a \in A : aBa^* \subset B \quad a^*Ba \subset B\}$ generates A as a C*-algebra.

Definition (cf. Vershik, Feldman-Moore 77)

Let B be an abelian sub-C*-algebra of a C*-algebra A containing an approximate unit of A. One says that B is a Cartan subalgebra if

- B is maximal abelian self-adjoint (i.e. $B' = B$);
- B is regular;
- there exists a faithful conditional expectation of A onto B.

Main theorem

Theorem (R 08)

- *Let* (G, E) *be a twist with* G *étale, second countable locally compact Hausdorff and topologically principal. Then* $C_0(G^{(0)})$ *is a Cartan subalgebra of* $C_r^*(G, E)$.
- *Let* B *be a Cartan sub-algebra of a separable C*-algebra* A. *Then, there exists a twist* (G, E) *with* G *étale, second countable locally compact Hausdorff and topologically principal and an isomorphism of* $C_r^*(G, E)$ *onto* A *carrying* $C_0(G^{(0)})$ *onto* B.

This theorem is a C*-algebraic version of a well-known theorem of Feldman-Moore (77) about von Neumann algebras. The main difference is that the measured equivalence relation of the von Neumann case has to be replaced by a topologically principal groupoid.

It is an improvement of a theorem of Kumjian (86) who deals with the principal case (i.e. étale equivalence relation) and introduces the stronger notion of a diagonal.

Definition

One says that a sub-C*-algebra B of a C*-algebra A has the unique extension property if pure states of B extend uniquely to pure states of A. A Cartan subalgebra which has the unique extension property is called a diagonal.

Then one has

Proposition (Kumjian 86, R 08)

Let B be a Cartan sub-algebra of a separable C-algebra A. Let (G, E) be the associated twist. Then,*

$$G \text{ is principal} \quad \Leftrightarrow \quad B \text{ has the unique extension property.}$$

Corollary

Let B be a Cartan sub-algebra of a separable C-algebra A. Then, the conditional expectation of A onto B is unique.*

This is well-known when B has the unique extension property. There should be a direct proof of this result in the general case.

Existence and uniqueness of Cartan subalgebras

There are deep theorems about the existence and the uniqueness of Cartan subalgebras in the von Neumann algebras case. For example

- the hyperfinite factors have a Cartan subalgebra which is unique up to conjugacy (Krieger+Connes-Feldman-Weiss 81);
- the free group factors $L(\mathbf{F}_n)$ do not have Cartan subalgebras for $n \geq 2$ (Voiculescu 96);
- there are II_1 factors which have uncountably many non-conjugate Cartan subalgebras (Popa 90).
- Ozawa and Popa have recently (07/08) a class of II_1 factors which have a Cartan subalgebra unique up to inner conjugacay.

Much less is known about Cartan subalgebras in C*-algebras. I will give a few examples and counter-examples.

Two non-conjugate Cartan subalgebras

Here is an easy example which produces two non-conjugate diagonal subalgebras.

Let G, H be locally compact abelian groups and a continuous homomorphism $\varphi : G \to H$. Then G acts continuously on H and we can form the crossed product C*-algebra $G \ltimes C_0(H)$. By dualizing, we get $\hat{\varphi} : \hat{H} \to \hat{G}$ and the crossed product C*-algebra $\hat{H} \ltimes C_0(\hat{G})$. The Fourier transform gives an isomorphism of these C*-algebras.

If G is discrete and φ is one-to-one, $C_0(H)$ is a diagonal subalgebra. Similarly, if $\hat{H}$ is discrete and $\hat{\varphi}$ is one-to-one, $C_0(\hat{G})$ is another diagonal subalgebra. Both conditions happen simultaneously if G is discrete, H is compact, φ is one-to-one and has dense range. There are such examples where $C_0(H)$ and $C_0(\hat{G})$ are not isomorphic.

Example

$G = \mathbb{Z}^2, H = \mathbb{R}/\mathbb{Z}, \varphi(m, n) = \alpha m + \beta n + \mathbb{Z}$ where $(1, \alpha, \beta)$ are linearly independent over $\mathbb{Q}$.

AF C*-algebras

- AF C*-algebras have a privileged AF diagonal which is unique up to conjugacy (Krieger 80).
- An example of a Cartan subalgebra which is not a diagonal in an AF C*-algebras is given in my thesis.
- Blackadar (90) writes the CAR algebra as a crossed product $C(X) \rtimes \Gamma$, where Γ is a locally finite group acting freely on $X = \mathbf{T} \times$ Cantor space. This exhibits a diagonal of the CAR algebra which is not AF.

Cantor minimal systems

Theorem (Giordano-Putnam-Skau 95)

Two Cantor minimal systems (X, T) and (Y, S) are strongly orbit equivalent if and only if the C-algebras $C^*(X, T)$ and $C^*(Y, S)$ are isomorphic.*

On the other hand, the groupoids $G(X, S)$ and $G(X, T)$ are is isomorphic (which amounts to flip conjugacy) if and only if the Cartan pairs $(C^*(X, T), C(X))$ and $(C^*(Y, S), C(Y))$ are isomorphic.

Example (Boyle and Handelman 94)

The strong orbit equivalence class of the dyadic adding machine contains homeomorphisms of arbitrary entropy.

These will give the same C*-algebra but the corresponding Cartan subalgebras will not be conjugate.

Continuous trace C*-algebras

One has:

Proposition

Let B be a Cartan subalgebra of a continuous-trace C^*-algebra A. Then B has the unique extension property.

We recall that:

Theorem (Green 77, Muhly-R-Williams 94)

Let (G, λ) be a locally compact principal groupoid with a Haar system . TFAE

- $C_r^*(G, \lambda)$ *has continuous trace;*
- G *is a proper groupoid.*

Combining these results we see that if $A = C^*_r(G, E)$, where G is étale, second countable locally compact Hausdorff and topologically principal, has continuous trace, then G must be proper and principal with $G^{(0)}/G = \hat{A}$. Moreover,

Theorem (R 85)

Let (R, E) and (S, F) be two twists with R, S étale, second countable locally compact proper and principal. Suppose that $C^*(R, E)$ and $C^*(S, F)$ are isomorphic. Then

- *(R, E) and (S, F) are Morita equivalent.*
- *The Dixmier-Douady class of $C^*_r(G, E)$ is the image of $[E]$ in $H^3(G^{(0)}/G, \mathbb{Z})$.*

The proof exhibits (R, E) as a reduction of the dual groupoid $R(A), E(A)$ of $A = C^*(R, E)$. This is not quite the expected uniqueness.
Question: Are two Cartan subalgebras of a continuous trace C*-algebra necessarily conjugate?

Here are two results about the existence of Cartan subalgebras in continuous trace C*-algebras.

Proposition (Raeburn-Taylor 85, R 85)

Given a second countable locally compact Hausdorff space T and $\delta \in H^3(T, \mathbb{Z})$, there exists a continuous trace C-algebra A possessing a Cartan subalgebra and realizing δ as its Dixmier-Douady invariant.*

Example (Natsume in Kumjian 85)

There exists a continuous trace C*-algebra which does not possess a Cartan subalgebra.

Natsume's example is the C*-algebra of compact operators of a continuous field of Hilbert spaces $H \to T$, where T is connected and simply connected and H does not decompose as a direct sum of line bundles.

A quiz

Among the following subalgebras, which ones are Cartan subalgebras, which ones are diagonals?

$$A = \{f : [0,1] \to M_2(\mathbb{C}) \text{ continuous}\} \quad B = \{f \in A : \forall t, f(t) \in D_2(\mathbb{C})\}$$

$$A_1 = \{f \in A : f(0) = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\} \quad B_1 = B \cap A_1$$

$$A_2 = \{f \in A : f(0) = \begin{pmatrix} a & a \\ a & a \end{pmatrix}\} \quad B_2 = B \cap A_2$$

$$A_3 = \{f \in A : f(0) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\} \quad B_3 = B \cap A_3$$

Answer to Exercise 1

$$A_1 = \{f \in A : f(0) = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\}$$

B_1 is a Cartan subalgebra which does not have the unique extension property: the states $f \mapsto a \pm b$ both extend the pure state $f \mapsto a$ of B.

A_1 is the C*-algebra of the groupoid of germs of the map $T(x) = -x$ on $[-1, 1]$. Explicitly, $A_1 = C^*(G)$ where $G = \{(\pm x, \pm 1, x), x \in [-1, 1]\}$ is topologically principal but not principal.

Answer to Exercise 2

$$A_2 = \{f \in A : f(0) = \begin{pmatrix} a & a \\ a & a \end{pmatrix}\}$$

B_2 is a masa which is not a Cartan subalgebra.

The C*-algebra A_2 can be realized as $C^*(R, \lambda)$ where R is the graph of the equivalence relation $y = \pm x$ on $[-1, 1]$. This is a closed subset of the product $[0, 1] \times [0, 1]$. We endow it with the product topology. It is a proper groupoid which is not étale. It has the Haar system

$$\int f d\lambda^x = f(x, x) + f(x, -x).$$

Answer to Exercise 3

$$A_3 = \{f \in A : f(0) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\}$$

B_3 is a diagonal in A_3.

The C*-algebra A_3 can be realized as $C^*(R_\tau)$ where R_τ is again the graph R of the equivalence relation $y = \pm x$ on $[-1, 1]$. However, we endow it with a topology finer than the product topology to make it étale. Following Molberg 06, we consider the topology generated by the product topology and the diagonal $\{(x, x), x \in \mathbb{R}\}$. Then R_τ is étale but no longer proper.