Fonctions primitives récusrsives sur les mots avec/sans concaténation

(On the power of recursive word-functions without concatenation)

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Well-known: Classical recursion (on natural numbers)

Functions from $\mathbb{N}^k$ to $\mathbb{N}$ constructed from

- constant 0 function,
- successor function
- projections (π_n^i)
- composition $\mathbf{Comp}(g, (h_i)_{1 \leq i \leq k})$
- recursion $f = \mathbf{Rec}(g, h)$ defined by:

$$\begin{aligned} f(0, \vec{y}) &= g(\vec{y}) \qquad \text{and} \\ f(n+1, \vec{y}) &= h(n, f(n, \vec{y}), \vec{y}) \end{aligned}$$

- (add minimisation to get all recursive functions)

Pros

- simple
- relate to arithmetic

Cons

- unfit for symbolic manipulation
- complexity blowup

Recursion on string/words

- $\Sigma = \{a_1, a_2, \cdots, a_r\}$
- ε empty word

Functions from $(\Sigma^*)^k$ to Σ^* constructed from

- constant $\widehat{\varepsilon}$,
- all left concatenation by one letter/symbol ${}_a\cdot(w) = a \cdot w = aw$
- projections (π_n^i)
- composition $\mathbf{Comp}(g, (h_i)_{1 \leq i \leq k})$
- (left) recursion $f = \mathbf{Rec}\big(g, (h_a)_{a \in \Sigma}\big)$ defined by:

$$\begin{aligned} f(\varepsilon, \vec{y}) &= g(\vec{y}) \qquad \text{and} \\ \forall a \in \Sigma, \quad f(a \cdot w, \vec{y}) &= h_a(w, f(w, \vec{y}), \vec{y}) \end{aligned}$$

- (what minimisation to get all recursive functions?)

Observations

1 letter alphabet corresponds to $\mathbb{N}$ (in unary)

- everything matches

r-adic encoding function from Σ^* to $\mathbb{N}$

- $\Sigma = \{a_1, a_2, \cdots, a_r\}$
- $\langle \varepsilon \rangle = 0$
- $a_k \cdot w$, $\langle a_k \cdot w \rangle = k + r \cdot \langle w \rangle$
- division, modulo, multiplication, addition...
  are primitive recursive (on $\mathbb{N}$)

Since the functions are the same (up to some encoding)...

- Why bother?

Why bother? indeed

Tropism

- culture and education stress on numbers, symbols are only to write sentences with
- proof by recursion and not induction
  (up to introducing measures like depth to do recursion)

Symbols are what is relevant

- in nowadays computations, computers. . .
- natural numbers are represented by *sequences of symbols*

Computability. . .

- is about symbol manipulation
- not natural numbers
- The term *Recursive* is getting replaced by *computable* (Soare, 2007)

State of the art... ancient and number oriented — 1

- *recursion on string*, *recursion on word*,
  recursive string-functions, *recursive word-functions*
- *recursion on representation*: representation of natural numbers by words in shortlex/military order, *non-trivial successor word-function*
- peak in the 1960's
- Most papers deal with hierarchies and is number-centric

Cook and Kapron (2017)

- m-adic notation of numbers (digits exlude 0) and relations on weak classes
- primitives $\{n \mapsto 10\,n + i\}_{0 \leq i \leq 9}$

State of the art... ancient and number oriented — 2

von Henke et al. (1975)

- survey on counterparts on words of classical results for primitive recursion on numbers

Variations

- infinite alphabet (Vučkovi, 1970), computation over finite sequences of numbers encoded by numbers
- restriction to unitary word-functions is considered in (Asser, 1987; Santean, 1990; Calude and Sântean, 1990)
- the nowhere defined function is added to primitive recursive word-functions in Khachatryan (2015)

Complexity measure

Needed

- formalism defines functions, not evaluation!
- what is a computation?
- what is the measure?

Dynamical computation

- store every result of evaluation
- do not recompute

Delayed evaluation

- compute value when need
- call by name

Complexity classes

Simulation of a Turing machine

- encoding: state \$ read symbol \$ word on left \$ word on right
- update in linear time

Class P is the same

- similar definition
- (one way) simulation of a Turing machine
- (other way) construction of the DAG in quasi-linear time

Same for higher classes

- NP (with certificate)
- EXP time...

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Strong limitation

Lemma

- the output is a suffix of an input

Corollary

- paring is not possible anymore!
- indeed $\{\varepsilon, \mathsf{a}, \mathsf{aa}\} \times \{\varepsilon, \mathsf{a}, \mathsf{aa}\}$
  has to be mapped one-to-one into $\{\varepsilon, \mathsf{a}, \mathsf{aa}\}$

Language decision

- $L = f^{-1}(\{\varepsilon\})$

Regular languages

- decided with the addition of constants

Boolean operators — closure properties

- $\top$ identified with ε

Ternary operator / test function

- $\text{if}_\varepsilon = \textbf{Rec}\left(\pi_2^1, (\pi_4^4, \pi_4^4)\right)$

Conjunction and disjunction

- $\wedge$ is $\text{and}_\varepsilon = \textbf{Comp}\left(\text{if}_\varepsilon, (\pi_2^1, \pi_2^2, \pi_2^1)\right)$
- $\vee$ is $\text{or}_\varepsilon = \textbf{Comp}\left(\text{if}_\varepsilon, (\pi_2^1, \widehat{\varepsilon}, \pi_2^2)\right)$

Negation — non-ε argument is needed

- $\neg$ is $\textbf{Comp}\left(\text{if}_\varepsilon, (\pi_2^1, \pi_2^2, \widehat{\varepsilon})\right)$ — arity is 2

Equality test to palindrome decision

$$\mathbf{Comp}\left(\mathbf{Rec}\left(\pi_2^1 \;\middle|\; \begin{matrix}\mathbf{Comp}\left(\mathbf{Rec}\left(\mathrm{id} \;\middle|\; \begin{matrix}\pi_3^1\\ \pi_3^3\end{matrix}\right) \;\middle|\; \begin{matrix}\pi_4^2\\ \pi_4^4\end{matrix}\right)\\ \mathbf{Comp}\left(\mathbf{Rec}\left(\mathrm{id} \;\middle|\; \begin{matrix}\pi_3^3\\ \pi_3^1\end{matrix}\right) \;\middle|\; \begin{matrix}\pi_4^2\\ \pi_4^4\end{matrix}\right)\end{matrix}\right) \;\middle|\; \begin{matrix}\pi_2^1\\ \pi_2^2\\ \pi_2^1\end{matrix}\right)$$

- test if on is the reverse of the other!
- $\rightsquigarrow$ palindrome test
- algebraic language, non-ambiguous but not deterministic

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Algebraic languages

$a_1^n a_2^n$

- non-ambiguous, deterministic
- read a_1 and stack functions to remove a_2

$a_1^n a_2^n a_1^m \cup a_1^n a_2^m a_1^m$

- ambiguous (non-deterministic)

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Non-algebraic languages

$a_1^n a_2^n a_1^n = a_1^n a_2^n a_1^m \cap a_1^n a_2^m a_1^m$

$a_1^n a_2^{P(n)}$ with P polynomial with positive coefficients

any boolean combination of the latter ones

- with prefixes and suffixes a_3^*

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Results

With concatenation

- computability: identical
- complexity: compatible (P and above)

Without concatenation, decides...

- all recursive languages
- some algebraic (deterministic/non ambiguous/ambiguous)
- some non algebraic
- languages with polynomial conditions on exponents/repetitions (unary encoding of natural numbers)

Perspectives — concatenation-less

- Test identity
- Polynomials in many variables, negative coefficients
- All algebraic languages (deterministic, non-ambiguous, ambiguous)

- Condition for not computability/decision

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