Proving Languages not to be Regular

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- Regular languages has at least four different descriptions:
  1. Languages accepted by DFA
  2. Languages accepted by NFA
  3. Languages accepted by ϵ-NFA
  4. Languages defined by regular expressions
- **NOT every language is a regular language**
- Powerful technique for showing certain languages not to be regular: Pumping Lemma

Theorem: The pumping lemma for regular languages

Let L be a regular language. Then there exists a constant n (which depends on L) such that for every string w in L such that $\mid w \mid \geq n$, we can break w into three strings, $w = xyz$, such that:

1. $y \neq \epsilon$
2. $\mid xy \mid \leq n$
3. For all $k \geq 0$, the string xy^kz is also in L

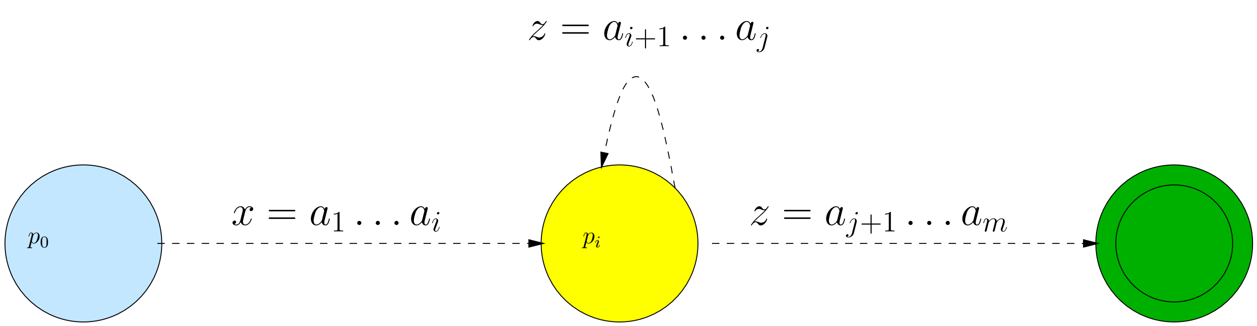
That is, we can always find a nonempty string y not too far from the beginning of w that can be ”pumped”; that is, repeating y any number of times, or deleting it (case $k = 0$, keeps the resulting string in the language L $\Box$

Proof

- Suppose L is regular and recognized by a DFA A. Suppose A has n states.
- Let w be a string of length n or more, *i.e.*, $w = a_1a_2 \ldots a_m$ where $m \geq n$ and each a_i is an input symbol.
- For $i = (0, 1, \ldots n)$ define state p_i to be $\hat{\delta}(q_0, a_1a_2 \ldots a_i)$, where δ is the transition function of A and q_0 is the start state of A.
- **That is, p_i is the state A is after reading the first i symbols of w.** Note that $p_0 = q_0$.
- It is not possible for the $n+1$ different p_i (for $i = (0, 1, \ldots n)$). to be distinct, since there are only n different states (Pigeonhole principle).
- Thus we can find two different integers i and j (with $0 \leq i < j \leq n$) such that $p_i = p_j$.

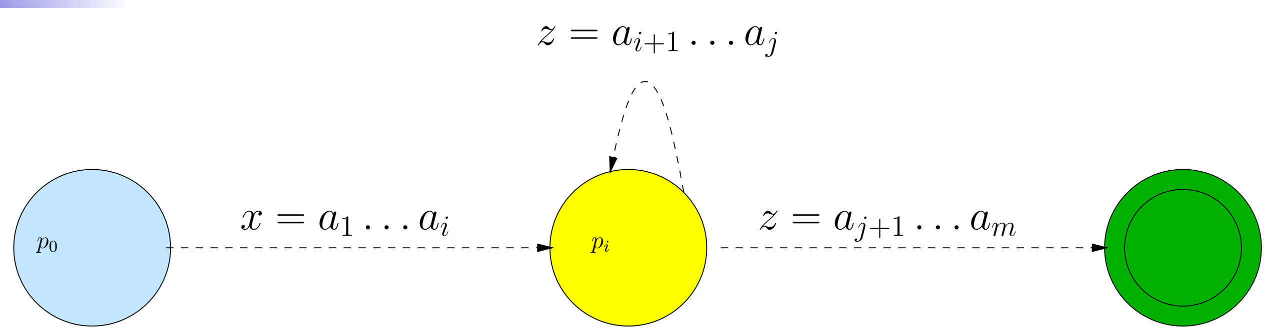
Proof (cont.)

- Now we can break $w = xyz$ as follows:
  1. $x = a_1 a_2 \ldots a_i$
  2. $y = a_{i+1} a_{i+2} \ldots a_j$
  3. $z = a_{j+1} a_{j+2} \ldots a_m$



- That is, x takes us to p_i once; y takes us from p_i back to p_i (since p_i is also p_j), and z is the balance of w.
- Note that x and z may be empty.
- However y cannot be empty (i is strictly less than j).

Proof (cont.)



- Now consider when A receives input $xy^k z$ for $k \geq 0$:
  - If $k = 0$ then the automaton goes from the start start state to p_i on input x. Since p_i is also p_j, it must be that A goes from p_i to the accepting state on input z. Thus A accepts xz.
  - If $k > 0$ then A goes from q_0 to p_i on input x, circles from p_i to p_i, k times on input y^k, and then goes to the accepting state on input z. Thus for any $k \geq 0$, $xy^k z$ is also accepted by A.

The Pumping Lemma as an Adversarial Game

Theorems whose statement involves several alternatives of *for all* and *there exists* quantifiers can be though of as a game between two players. The pumping lemma is an important example of this type of theorem.

We can see the application of the pumping lemma as a game, in which

1. **Player 1** picks the language L to be proved non regular.
2. **Player 2** picks n, but does not reveal to player 1 what n is; player 1 must devise a play for all possible n.
3. **Player 1** picks w, which may depend on n and which must be of length at least n.
4. **Player 2** divides w into x, y and z, obeying the constraints that are stipulated in the pumping lemma; $y \neq \epsilon$ and $\mid x.y \mid \leq n$. Again, player 2 does not have to tell player 1 what x, y and z are, although they must obey the constraints.
5. **Player 1** wins by picking k, which may be a function of n, x, y and z, such that $x.y^k.z$ is NOT in L.