Undecidable Dynamical Properties of Cellular Automata

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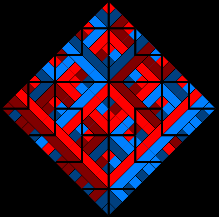
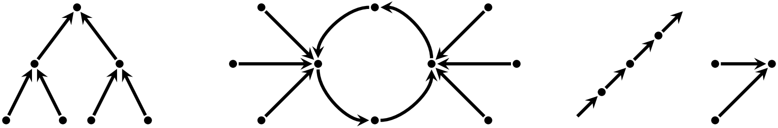


Discrete dynamical systems

Definition A **DDS** is a pair (X, F) where X is a **topological space** and $F : X \longrightarrow X$ is a **continuous** map.



The **orbit** of $x \in X$ is the sequence $(F^n(x))$ obtained by iterating F.

In this talk, $X = S^{\mathbb{Z}}$ where S is a finite alphabet and X is endowed with the **Cantor topology** (product of the discrete topology on S), and F is a continuous map that **commutes with the shift map** σ: $F \circ \sigma = \sigma \circ F$ where $\sigma(x)(z) = x(z + 1)$.

Two dynamical properties

We consider two simple dynamical properties (as opposed to more computational properties like reachability questions).

Definition A DDS (X, F) is **periodic** if for all $x \in X$ there exists $n \in \mathbb{N}$ such that $F^n(x) = x$.

Definition A DDS (X, F) is **nilpotent** if there exists $0 \in X$ such that for all $x \in X$ there exists $n \in \mathbb{N}$ such that $F^n(x) = 0$.

Question With a **proper recursive encoding** of the DDS, can we decide given a DDS if it is periodic? if it is nilpotent?

Contents of the talk

1. cellular automata

   more combinatorial definitions

2. domino and immortality problems

   some undecidability tools

3. undecidable dynamical properties

   applying tools to cellular automata

1. cellular automata

Cellular automata

Definition A **CA** is a triple (S, r, f) where S is a **finite set of states**, $r \in \mathbb{N}$ is the **radius** and $f : S^{2r+1} \longrightarrow S$ is the **local rule**.

A **configuration** $c \in S^{\mathbb{Z}}$ is a coloring of $\mathbb{Z}$ by S.

The **global map** $F : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$ applies f uniformly and locally:

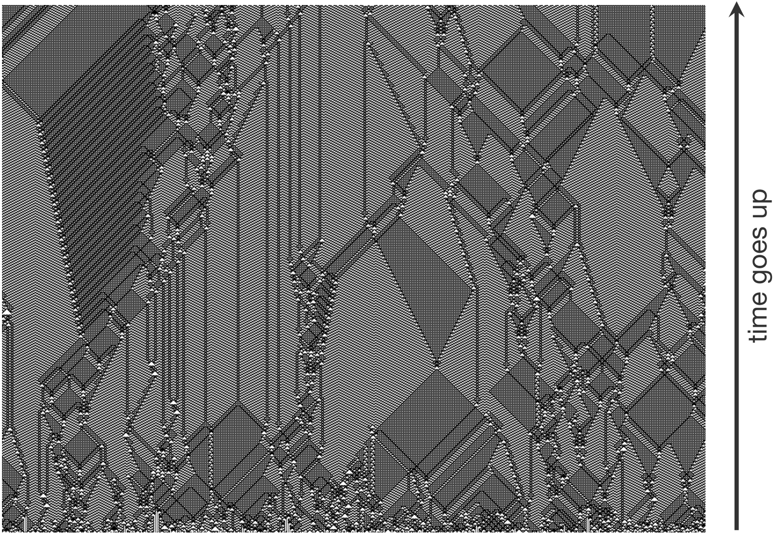
$$\forall c \in S^{\mathbb{Z}}, \forall z \in \mathbb{Z}, \qquad F(c)(z) = f(c(z-r), \ldots, c(z+r)).$$

A **space-time diagram** $\Delta \in S^{\mathbb{N}\times\mathbb{Z}}$ satisfies, for all $t \in \mathbb{Z}^+$,

$$\Delta(t+1) = F(\Delta(t)).$$

The associated DDS is $(S^{\mathbb{Z}}, F)$.

Space-time diagram



$$S = \{0, 1, 2\}, r = 1, f(x, y, z) = \lfloor 6430564760289/3^{9x+3y+z} \rfloor \pmod 3$$

König's lemma

König's lemma Every infinite tree with finite branching admits an infinite path.

For all $n \in \mathbb{N}$ and $u \in S^{2n+1}$, the **cylinder** $[u] \subseteq S^{\mathbb{Z}}$ is

$$[u] = \left\{ c \in S^{\mathbb{Z}} \middle| \forall i \in [-n, n]\ c(i) = u_{i+n} \right\} \quad .$$

For all $C \subseteq S^{\mathbb{Z}}$, the **König tree** $\mathcal{A}_C$ is the tree of cylinders of C.

The **topping** $\overline{\mathcal{A}_C} \subseteq S^{\mathbb{Z}}$ of a König tree is the set of configurations tagging an infinite path from the root (intersection of the cylinders on the path).

Definition The **König topology** over $S^{\mathbb{Z}}$ is the topology whose close sets are the toppings of König trees.

Curtis-Hedlund-Lyndon’s theorem

König and Cantor topologies coincide: their open sets are unions of cylinders. Compacity arguments have combinatorial counterparts.

The **clopen sets** are finite unions of cylinders.

Therefore in this topology **continuity** means **locality**.

Theorem [Hedlund 1969] The continuous maps commuting with the shift coincide with the global maps of cellular automata.

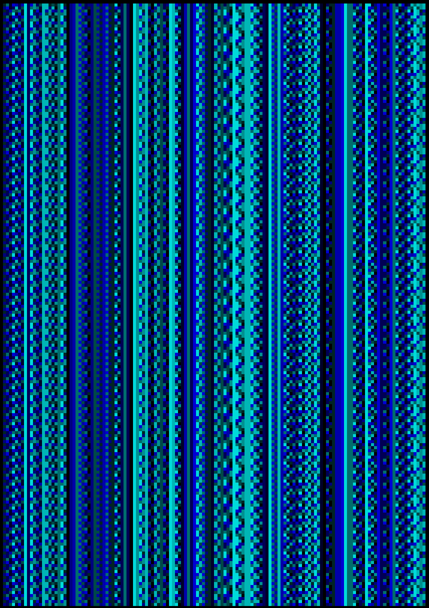
Cellular automata have a dual nature : topological maps with finite automata description.

Periodicity

A CA is periodic iff there exists a **uniform period** $n \in \mathbb{Z}^+$ such that F^n is the identity map.

Hint Take the period of a **universal configuration** containing all finite words on S.

The Periodicity Probem (**PP**) given a CA decide if it is periodic.

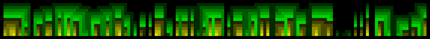


Nilpotency

A CA is nilpotent iff there exists a **uniform bound** $n \in \mathbb{Z}^+$ such that F^n is a constant map.

Hint Take the bound of a **universal configuration** containing all finite words on S.

The Nilpotency Probem (**NP**)
given a CA decide if it is nilpotent.



2. domino and immortality problems

Entscheidungsproblem: the ∀∃∀ case

Hilbert's Entscheidungsproblem (semantic version) To find a method which for every sentence of elementary quantification theory yields a decision as to whether or not the sentence is satisfiable.

In the 60s, the **classical decision problem** is studied with respect to classes of quantification types.

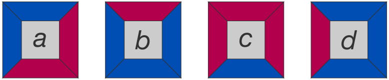
One big open class: the ∀∃∀ class. Wang and Büchi introduce in 1961 two decision problems in order to solve it.

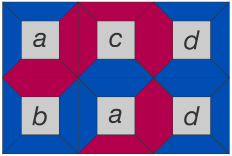
The problem is proved undecidable in 1962 by Kahr, Moore and Wang using a simpler reduction.

The Domino Problem (**DP**)

"Assume we are **given a finite set of square plates** of the same size with **edges colored**, each in a different manner. Suppose further there are infinitely many copies of each plate (plate type). We are **not permitted to rotate or reflect a plate**. The question is to find an effective procedure by which we can **decide**, for each given finite set of plates, **whether we can cover up the whole plane** (or, equivalently, an infinite quadrant thereof) **with copies of the plates subject to the restriction that adjoining edges must have the same color**."

(Wang, 1961)





Aperiodicity in **DP**

The set of tilings of a tile set T is a compact subset of $T^{\mathbb{Z}^2}$.

By compacity, if a tile set does not tile the plane, there exists a square of size $n \times n$ that cannot be tiled.

Tile sets without tilings are recursively enumerable.

A set of Wang tiles with a periodic tiling admits a biperiodic tiling.

Tile sets with a biperiodic tiling are recursively enumerable.

Undecidability is to be found in **aperiodic tile sets**, tile sets that only admit aperiodic tilings.

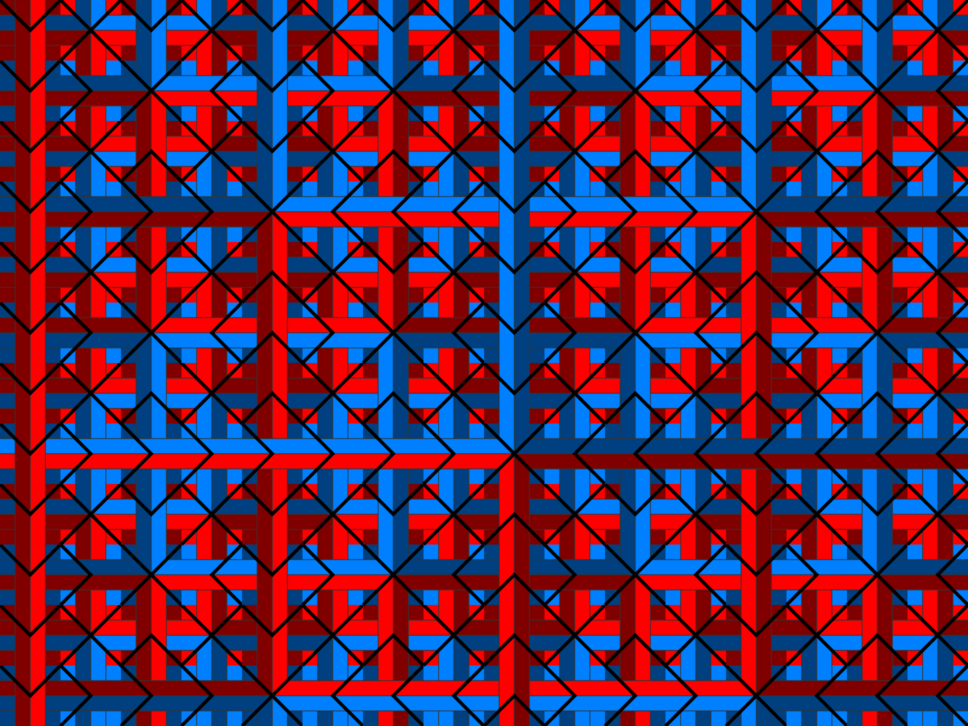
Undecidability of **DP**

Theorem [Berger 1964] DP is undecidable.

Composition technique [Robinson 1971, O 2008] Define an unambiguous substitution, encode it with local constraints to obtain an aperiodic tile set. Modify the tile set to insert everywhere prefixes of unbounded length of TM computation.

Fixpoint technique [Durand, Romashchenko, Shen 2008] Define a tile set with prototiles enforcing tiling constraints using a Turing machine. A fixpoint tile set is aperiodic. Modify the tile set to insert everywhere prefixes of unbounded length of TM computation.

Transducer and sturmian words [Kari 2007] Consider lines of tilings as a transducer coding a relation on biinfinite words. Encode tuples of real numbers in a sturmian way, the transducer enforcing affine relations. Reduce the immortality problem of Turing machines to the immortality problem of affine maps.



The Immortality Problem (**IP**)

"(T_2) To find an effective method, which for every Turing-machine M decides whether or not, for all tapes I (finite and infinite) and all states B, M will eventually halt if started in state B on tape I" *(Büchi, 1962)*

A **TM** is a triple (S, Σ, T) where S is a finite set of states, Σ a finite alphabet and $T \subseteq (S \times \{\leftarrow, \rightarrow\} \times S) \cup (S \times \Sigma \times S \times \Sigma)$ is a set of instructions.

(s, δ, t) : *"in state s move according to δ and enter state t."*
(s, a, t, b) : *"in state s, reading letter a, write letter b and enter state t."*

Partial DDS $(S \times \Sigma^{\mathbb{Z}}, G)$ where G is a partial continuous map.

A TM is **mortal** if all configurations are ultimately halting.

Aperiodicity in **IP**

As $S \times \Sigma^{\mathbb{Z}}$ is compact, G is continuous and the set of halting configurations is open, **mortality** implies **uniform mortality**.

Mortal TM are recursively enumerable.

TM with a periodic orbit are recursively enumerable.

Undecidability is to be found in **aperiodic TM**, TM whose infinite orbits are all aperiodic.

Undecidability of **IP**

Theorem [Hooper 1966] **IP** is undecidable.

Reduction reduce **HP** for 2-CM $\left(s, \underline{@}1^m \mathtt{x}2^n \mathtt{y}\right)$

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Problem unbounded searches produce immortal configurations.

Idea *by compacity, extract infinite failure sequence*

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Hooper's trick use bounded searches with **recursive calls** to initial segments of the simulation of increasing sizes:

$\underline{@}$111111111111x2222y *search* x $\longrightarrow$

s

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$@\underline{1}11111111111x2222y$ *bounded search 1*

s'_1

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$@1\underset{s'_2}{\underline{1}}11111111111x2222y$ *bounded search 2*

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Hooper's trick use bounded searches with **recursive calls** to initial segments of the simulation of increasing sizes:

$@11\underset{s'_3}{\underline{1}}1111111111x2222y$ *bounded search 3*

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$@\underline{@}_s xy1111111111x2222y$ *recursive call*

s_0

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$@@_s 11111x22222\underset{\overline{s}_c}{\underline{y}}x2222y$ *ultimately in case of collision...*

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$@\underline{@}_s xy1111111111x2222y$ *...revert and clean*

s_b

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$\texttt{@111}\underset{s'_1}{\underline{\texttt{1}}}\texttt{11111111x2222y}$ *pop and continue bounded search 1*

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$@1111\underset{s_2'}{\underline{1}}1111111x2222y$ *bounded search 2*

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$\mathtt{@111}\underline{\mathtt{@}}_{\mathtt{s}}\mathtt{xy1111111x2222y}$ *recursive call*

$\mathtt{s}_{\emptyset}$

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s_0

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s_0

The TM is immortal iff the 2-CM halts from $(s_0, (0, 0))$.

3. undecidable dynamical properties

Undecidability of the nilpotency problem

A tile set is **NW-deterministic** if, for each pair of colors, there exists at most one tile with these colors on N and W sides.

Theorem [Kari 1992] NW-deterministic **DP** is undecidable.

The **limit set** Λ_F of a CA F is the non-empty subshift $\Lambda_F = \bigcap_{n\in\mathbb{N}} F^n(S^{\mathbb{Z}})$ of configurations appearing in biinfinite space-time diagrams $\Delta \in S^{\mathbb{Z}\times\mathbb{Z}}$ such that $\forall t \in \mathbb{Z}, \Delta(t+1) = F(\Delta(t))$.

NW-deterministic **DP** reduces to **NP**.

Theorem [Kari 1992] NP is undecidable.

Undecidability of the periodicity problem

A TM is **reversible** if it is deterministic with a deterministic inverse.

Theorem [Kari O 2008] reversible **IP** is undecidable.

This implies to prove Hooper's result again with more constraints
(no easy reduction to the reversible case preserving mortality).

Reversible **IP** reduces to **PP**.

Theorem [Kari O 2008] PP is undecidable.

Undecidability of dynamical properties

Undecidability is not necessarily a negative result:

it is a **hint of complexity**.

There exists non trival nilpotent and periodic CA with a very large bound for quite simple CA (the bound grows faster than any recursive function).

Next step is to consider dynamical properties from topological dynamics, like Kůrka's classification.

Open Problem Is positive expansivity decidable?