On Aperiodic Reversible Turing Machines

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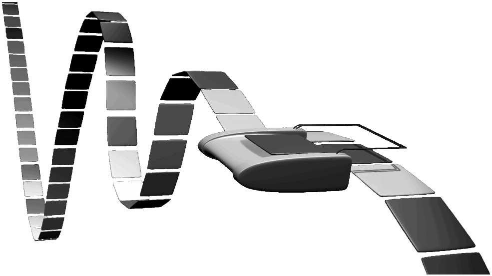
Turing machines

The classical **Turing machine**: finitely many **states**, a (bi-)infinite **tape**, a mobile i/o **head** pointing on a cell (optionally: blank symbol, starting and halting states).



Halting Problem[Σ^0_1-comp.] Given a TM and a finite starting configuration, decide if a halting state is eventually reached.

Reachability and similar questions

Reachability Problem[Σ^0_1-comp.] Given a TM and two states s and t, decide if state t is reachable from state s.

Totality Problem[Π^0_2-comp.] Given a TM, decide if it eventually halts starting from any **finite configuration**.

Mortality Problem[Σ^0_1-comp.] Given a TM, decide if it eventually halts starting from any configuration.

Periodicity Problem[Σ^0_1-comp.] Given a TM, decide if every configuration eventually loops by reaching itself again.

The Transitivity Problem

Transitivity Problem[Π^0_1-hard] Given a TM, decide if every **partial configuration** is reachable from every **partial configuration** (completed into a proper configuration).

A **partial configuration** specifies only a finite segment of the tape plus the state and head position.

$$???????ab.babaa???????$$
$$\mathbf{q}$$

Question How do we prove the undecidability of the Transitivity Problem?

The Transitivity Problem

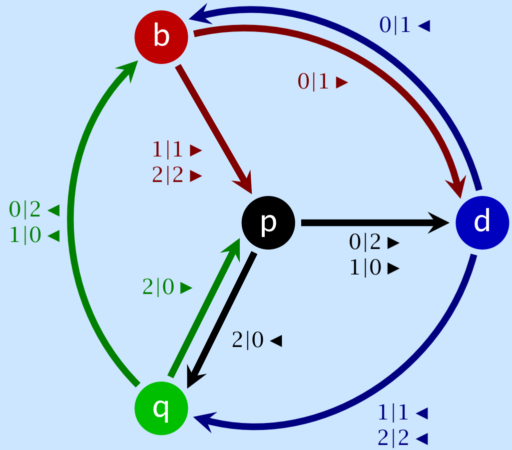
Transitivity Problem[Π^0_1-hard] Given a TM, decide if every **partial configuration** is reachable from every **partial configuration** (completed into a proper configuration).

A **partial configuration** specifies only a finite segment of the tape plus the state and head position.

$$???????ab.ba\underset{\mathbf{q}}{b}aa???????$$

Question How do we prove the undecidability of the Transitivity Problem?

Question ...and first, how do you build a transitive TM?



1. Dynamics of Turing machines

Turing machines

A **Turing machine** is a triple (Q, Σ, δ) where Q is the finite set of states, Σ is the finite set of tape symbols and $\delta : Q \times \Sigma \to Q \times \Sigma \times \{-1, 0, 1\}$ is the partial transition function.

A transition $\delta(s, a) = (t, b, d)$ means:
"in state s, when reading the symbol a on the tape,
replace it by b move the head in direction d and enter state t."

Configurations are triples $(s, c, p) \in Q \times \Sigma^{\mathbb{Z}} \times \mathbb{Z}$.

A **transition** transforms (s, c, p) into $(t, c', p + d)$ where $\delta(s, c(p)) = (t, b, d)$ and $c' = c$ everywhere but $c'(p) = b$.

Notation $(s, c, p) \vdash (t, c', p + d)$ and closures $\vdash^+$ and $\vdash^*$

Definitions

A configuration (s, c, p) is:

- **halting** if $\delta(s, c(p))$ is undefined, $(s, c(p))$ is a **halting pair**
- **periodic** if $(s, c, p) \vdash^+ (s, c, p)$

A TM (Q, Σ, δ) is:

- **complete** if δ is complete
- **aperiodic** if it has no periodic configuration
- **surjective** if every configuration has a preimage
- **injective** if every configuration has at most one preimage

For complete machines surjective is equivalent to injective.

Reversibility

Injective TM are in fact reversible TM.

Definition A **reversible** TM $M = (Q, \Sigma, \delta)$ is characterized by a partial injective map ρ and a map μ such that $\delta(s,a) = (t, b, \mu(t))$ where $\rho(s,a) = (t,b)$.

The **reverse** of M is M^{-1} where $\delta^{-1}(t,b) = (s, a, -\mu(s))$.

$$(s,c,p) \vdash_M (t,c',p+\mu(t)) \implies (t,c',p) \vdash_{M^{-1}} (s,c,p-\mu(s))$$

A **starting pair** is a halting pair of the reverse.

A **starting configuration** is a halting config of the reverse.

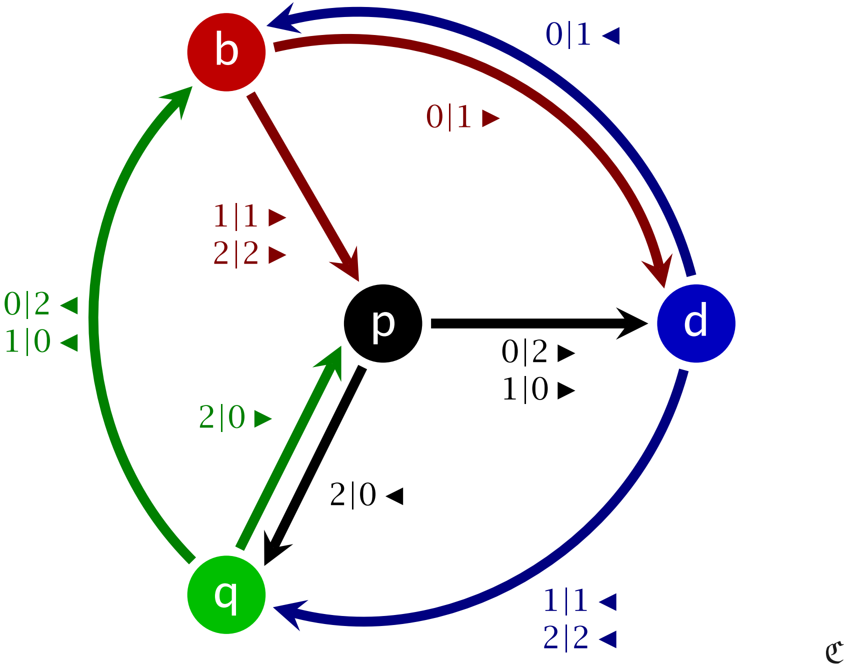
Naive dynamics

A **topological dynamical system** is a pair (X, T) where the topological space X is the **phase space** and the continuous function $T : X \to X$ is the **global transition function**.

The **orbit** of $x \in X$ is $\mathcal{O}(x) = (T^n(x))_{n\in\mathbb{N}}$.

Using the **product topology** one obtains a **topological dynamical system** (X, T) for a TM where the phase space is $X = Q \times \Sigma^{\mathbb{Z}} \times \mathbb{Z}$ and the transition function T is continuous.

Unfortunately, X is not **compact**, we follow Kůrka's alternative compact dynamical models TMH and TMT.



b
p
d
q
0|1 ◀
0|1 ▶
1|1 ▶
2|2 ▶
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1|0 ◀
0|2 ▶
1|0 ▶
2|0 ▶
2|0 ◀
1|1 ◀
2|2 ◀

Moving head vs moving tape dynamics

TMH

$$X_h \subset (Q \cup \Sigma)^{\mathbb{Z}}$$
$$T_h : X_h \to X_h$$

···000000**b**000000000···
···0000001**d**00000000···
···000000**b**110000000···
···0000001**p**10000000···
···00000010**d**0000000···
···0000001**b**01000000···
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···000000**b**101000000···
···0000001**p**01000000···
⋮

TMT

$$X_t = {}^{\omega}\Sigma \times Q \times \Sigma^{\omega}$$
$$T_t : X_t \to X_t$$

···0000000**b**00000000···
···0000001**d**00000000···
···0000000**b**11000000···
···0000001**p**10000000···
···0000010**d**00000000···
···0000001**b**01000000···
···0000011**d**10000000···
···0000001**q**11000000···
···0000000**b**10100000···
···0000001**p**01000000···
⋮

Trace-shift dynamics

ST

$$S_T \subseteq (Q \times \Sigma)^\omega$$

$$\sigma : S_T \to S_T$$

TMT

$$X_t = {}^\omega\Sigma \times Q \times \Sigma^\omega$$

$$T_t : X_t \to X_t$$

The column shift of TMT

0 0 1 1 0 0 1 1 1 0 . . .
b d b p d b d q b p

```
···0000000b00000000···
···0000001d00000000···
···0000000b11000000···
···0000001p10000000···
···0000010d00000000···
···0000001b01000000···
···0000011d10000000···
···0000001q11000000···
···0000000b10100000···
···0000001p01000000···
⋮
```

Topological transitivity

Definition A dynamical system (X,T) is **transitive** is it admits a **transitive point** x such that $\overline{\mathcal{O}(x)} = X$.

Proposition (X,T) is **transitive** iff for every pair of open sets $U,V \subseteq X$, there exists t such that $T^t(U) \cap V \neq \emptyset$.

TMH $\forall u,v,u',v' \; \exists w,z,w',z',n \;\; T_h^n(wu.vz) = w'u'.v'z'$

TMT $\forall u,v,\alpha,u',v',\beta \; \exists w,z,w',z',n \; T_t^n(wu,\alpha,vz) = (w'u',\beta,v'z')$

ST $\forall u,v{\in}S_T \;\; \exists w{\in}S_T \quad uwv \in S_T$

TMH transitive $\Longrightarrow$ TMT transitive $\Longrightarrow$ ST transitive.

TMH transitive implies **complete**, **reversible** and **aperiodic**

Transitivities

Definition A point $x \in X$ is **periodic** if it admits a **period** $p > 0$ such that $T^p(x) = x$.

Proposition A TM with a **periodic** point is **not ST transitive**.

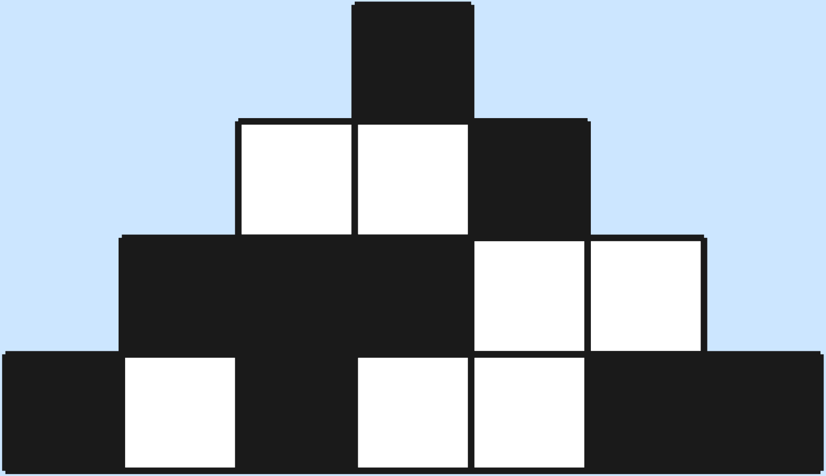
The single-state **shift** TM is **TMT transitive** but **not TMH**.

$$\delta(q, x) = (q, x, \blacktriangleright)$$

The single-state **eraser** TM is **ST transitive** but **not TMT**.

$$\delta(q, x) = (q, 0, \blacktriangleright)$$

Question How do we construct a complete reversible aperiodic TM?



2. Cellular Automata

Cellular automata

Definition A **CA** is a triple (S, r, f) where S is a **finite set of states**, $r \in \mathbb{N}$ is the **radius** and $f : S^{2r+1} \to S$ is the **local rule** of the cellular automaton.

A **configuration** $c \in S^{\mathbb{Z}}$ is a coloring of $\mathbb{Z}$ by S.

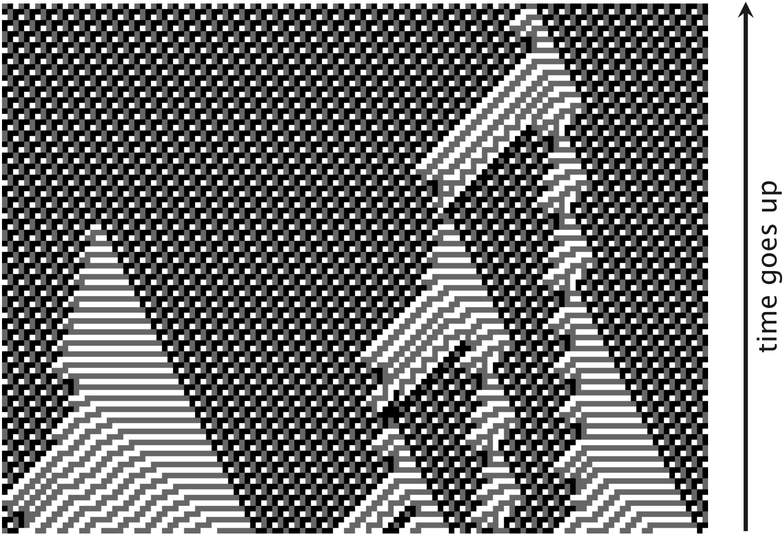
The **global map** $F : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ applies f uniformly and locally:

$$\forall c \in S^{\mathbb{Z}}, \forall z \in \mathbb{Z}, \qquad F(c)(z) = f(c(z-r), \ldots, c(z+r)).$$

A **space-time diagram** $\Delta \in S^{\mathbb{N}\times\mathbb{Z}}$ satisfies, for all $t \in \mathbb{Z}^+$,

$$\Delta(t+1) = F(\Delta(t)).$$

Space-time diagram



$S = \{0, 1, 2\}$, $r = 1$, $f(x, y, z) = \lfloor 6450288690466/3^{9x+3y+z} \rfloor \pmod 3$

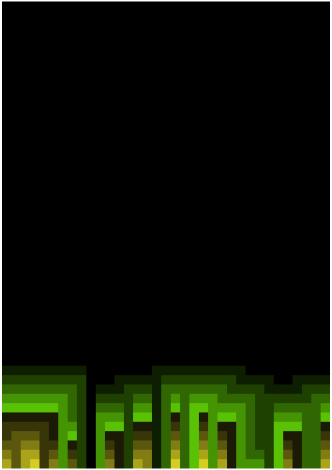
The nilpotency problem (Nil)

Definition A DDS is **nilpotent** if
$\exists z \in X, \forall x \in X, \exists n \in \mathbb{N}, F^n(x) = z.$

Given a recursive encoding of the DDS, can we **decide** nilpotency?

A DDS is **uniformly nilpotent** if
$\exists z \in X, \exists n \in \mathbb{N}, \forall x \in X, F^n(x) = z.$

Given a recursive encoding of the DDS, can we **bound recursively** n?



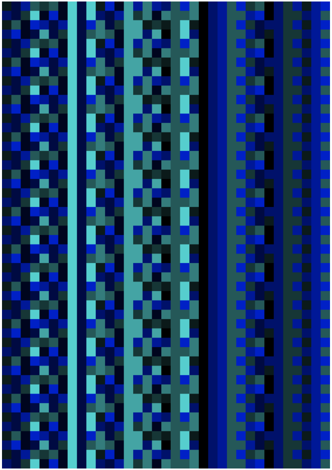
The periodicity problem (Per)

Definition A DDS is **periodic** if
$\forall x \in X, \exists n \in \mathbb{N}, F^n(x) = x$.

Given a recursive encoding of the DDS, can we **decide** periodicity?

A DDS is **uniformly periodic** if
$\exists n \in \mathbb{N}, \forall x \in X, F^n(x) = x$.

Given a recursive encoding of the DDS, can we **bound recursively** n?



Undecidability results

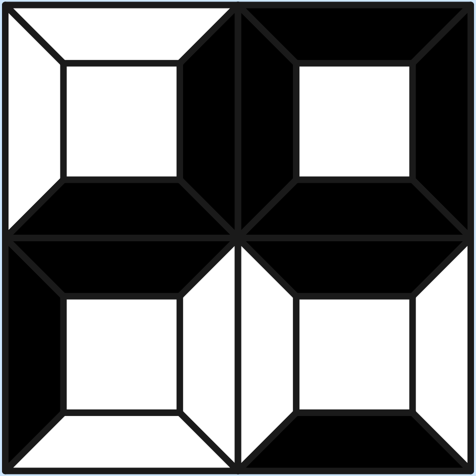
Theorem Both **Nil** and **Per** are **undecidable** for CA.

The proofs inject **computation** into **dynamics**.

Undecidability is not necessarily a negative result:
it is a **hint of complexity**.

Remark Due to **universe configurations** both nilpotency and periodicity are uniform.

The bounds grow **faster than any recursive function**: there exists simple nilpotent or periodic CA with huge bounds.

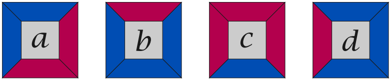


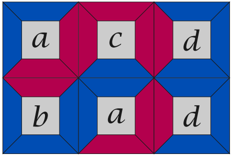
3. Nilpotency and tilings

The Domino Problem (DP)

*"Assume we are **given a finite set of square plates** of the same size with **edges colored**, each in a different manner. Suppose further there are infinitely many copies of each plate (plate type). We are **not permitted to rotate or reflect a plate**. The question is to find an effective procedure by which we can **decide**, for each given finite set of plates, **whether we can cover up the whole plane** (or, equivalently, an infinite quadrant thereof) **with copies of the plates subject to the restriction that adjoining edges must have the same color**."*

(Wang, 1961)





Undecidability of DP

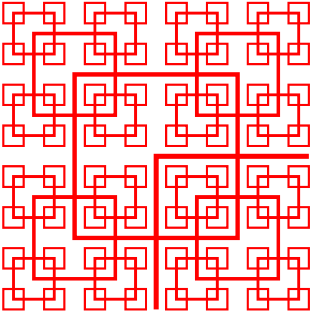
Theorem[Berger64] **DP** is **undecidable**.

Remark To prove it one needs **aperiodic** tile sets.

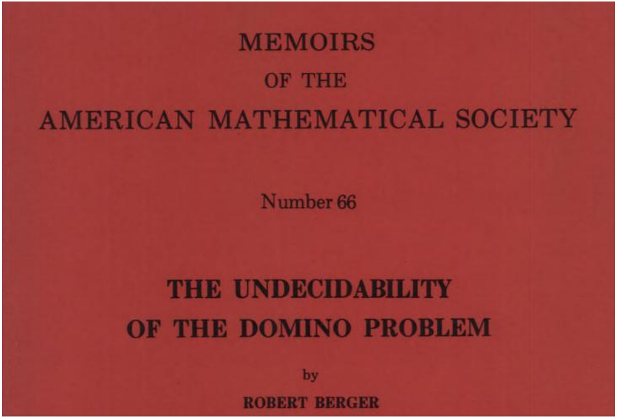
Idea of the proof

Enforce an (aperiodic) **self-similar structure** using local rules.

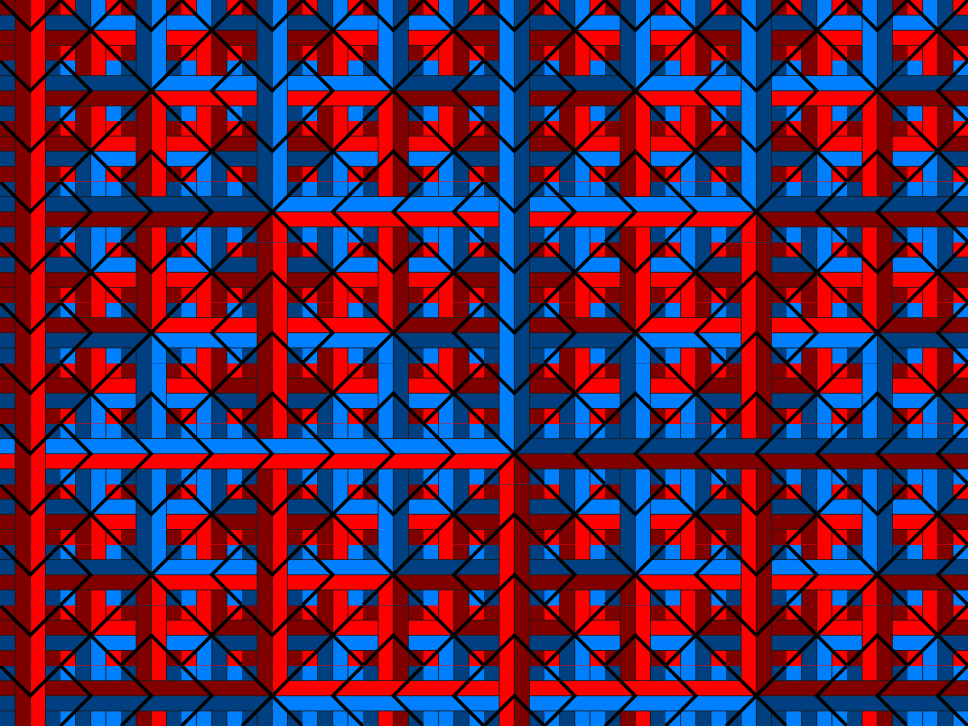
Insert a **Turing machine** computation **everywhere** using the structure.



Remark Plenty of different proofs!



"(. . .) In 1966 R. Berger discovered the first aperiodic tile set. It contains 20,426 Wang tiles, (. . .)
Berger himself managed to reduce the number of tiles to 104 and he described these in his thesis, though they were omitted from the published version (Berger [1966]). (. . .)" *[GrSh, p.584]*

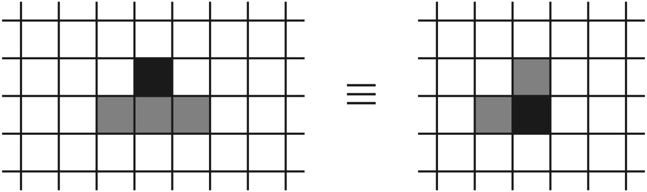


Reduction

A state $\perp \in S$ is **spreading** if $f(N) = \perp$ when $\perp \in N$.

A CA with a spreading state $\perp$ is not nilpotent iff it admits a biinfinite space-time diagram without $\perp$.

A tiling problem Find a coloring $\Delta \in (S \setminus \{\perp\})^{\mathbb{Z}^2}$ satisfying the tiling constraints given by f.



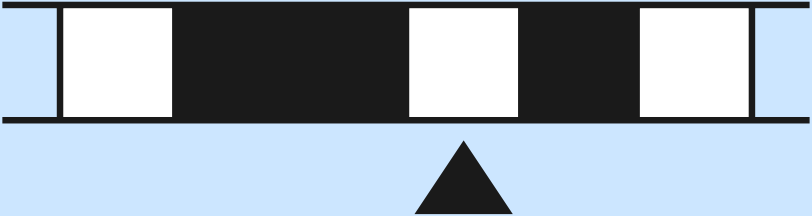
Theorem[Kari92] NW-DP $\leqslant_m$ Nil

Revisiting DP

Theorem[Kari92] **NW-DP** is **undecidable**.

Remark Reprove of undecidability of **DP** with the additionnal determinism constraint!

Corollary **Nil** is **undecidable**.



4. Periodicity and mortality

The Immortality Problem (IP)

“(T_2) To find an effective method, which for every Turing-machine M decides whether or not, for all tapes I (finite and infinite) and all states B, M will eventually halt if started in state B on tape I” *(Büchi, 1962)*

Definition A Turing machine is **mortal** if all configurations are ultimately halting.

Undecidability of IP

Theorem[Hooper66] **IP** is **undecidable**.

Remark To prove it one needs **aperiodic** TM.

Idea of the proof

Simulate 2-counters machines *à la* Minsky $(s, \underline{@}1^m x 2^n y)$

Replace **unbounded searches** by **recursive calls** to initial segments of the simulation.

Reduction: revisiting IP

Theorem[KO2008] **R-IP** $\leqslant_m$ **TM-Per** $\leqslant_m$ **Per**

Theorem[KO2008] **R-IP** is **undecidable**.

Remark Reprove of undecidability of **IP** with the additionnal reversibility constraint!

Immortality: a first attempt

"(T_2) To find an effective method, which for every Turing-machine M decides whether or not, for all tapes I (finite and infinite) and all states B, M will eventually halt if started in state B on tape I" *(Büchi, 1962)*

Immortality: a first attempt

"(T_2) *To find an effective method, which for every Turing-machine* M *decides whether or not, for all tapes* I *(finite and infinite) and all states* B, M *will eventually halt if started in state* B *on tape* I" *(Büchi, 1962)*

[Hooper66] IP is undecidable for TM.

Idea *TM with recursive calls! (we will discuss this)* ◊

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[Hooper66] IP is undecidable for TM.
Idea *TM with recursive calls! (we will discuss this)* $\Diamond$

[Lecerf63] Every TM is **simulated** by a RTM.
Idea *Keep history on a stack encoded on the tape.* $\Diamond$

Immortality: a first attempt

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Idea *TM with recursive calls! (we will discuss this)* ◊

[Lecerf63] Every TM is **simulated** by a RTM.
Idea *Keep history on a stack encoded on the tape.* ◊

Problem The simulation **does not** preserve immortality due to **unbounded searches**. We need to rewrite Hooper's proof for reversible machines.

Immortality: simulating RCM

Theorem 7 **IP** is undecidable for RTM.

Reduction reduce **HP** for 2-RCM $(s, \underline{@}1^m x 2^n y)$

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Problem unbounded searches produce immortality.
Idea *by compacity, extract infinite failure sequence*

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Hooper's trick use bounded searches with **recursive calls** to initial segments of the simulation of increasing sizes:

$\underset{s}{\underline{@}}$1111111111111x2222y *search x →*

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$@\underset{\overline{s}'_1}{\underline{1}}111111111111x2222y$ *bounded search 1*

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Hooper's trick use bounded searches with **recursive calls** to initial segments of the simulation of increasing sizes:

$@1\underset{\overline{s}_2'}{\underline{1}}11111111111x2222y$ *bounded search 2*

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Hooper's trick use bounded searches with **recursive calls** to initial segments of the simulation of increasing sizes:

$@11\underset{\overline{s}'_3}{\underline{1}}11111111111x2222y$ *bounded search 3*

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@$\underline{@}_s$**xy**111111111x2222y *recursive call*
$\overline{s}_0$

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Hooper's trick use bounded searches with **recursive calls** to initial segments of the simulation of increasing sizes:

@**$@_s$11111x22222$\underset{s_c}{y}$**x2222y *ultimately in case of collision...*

Immortality: simulating RCM

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Hooper's trick use bounded searches with **recursive calls** to initial segments of the simulation of increasing sizes:

$@\underline{@}_{s}\mathbf{xy}1111111111x2222y$ *...revert to clean*
$\underline{\mathbf{s}}_{\mathbf{b}}$

Immortality: simulating RCM

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Hooper's trick use bounded searches with **recursive calls** to initial segments of the simulation of increasing sizes:

$@111\underset{\overline{s}'_1}{\underline{1}}111111111x2222y$ *pop and continue bounded search 1*

Immortality: simulating RCM

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Hooper's trick use bounded searches with **recursive calls** to initial segments of the simulation of increasing sizes:

$@1111\underset{\bar{s}_2'}{\underline{1}}11111111x2222y$ *bounded search 2*

Immortality: simulating RCM

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Reduction reduce **HP** for 2-RCM $(s, \underline{@}1^m x2^n y)$

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Hooper's trick use bounded searches with **recursive calls** to initial segments of the simulation of increasing sizes:

$@11111\underset{\overline{s}_3'}{\underline{1}}1111111x2222y$ *bounded search 3*

Immortality: simulating RCM

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$@111\underset{s_0}{\underline{@}}_s\mathbf{xy}1111111x2222y$ *recursive call*

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@111$\underline{@}_s$**xy**1111111x2222y *recursive call*
s_0

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Problem unbounded searches produce immortality.
Idea *by compacity, extract infinite failure sequence*

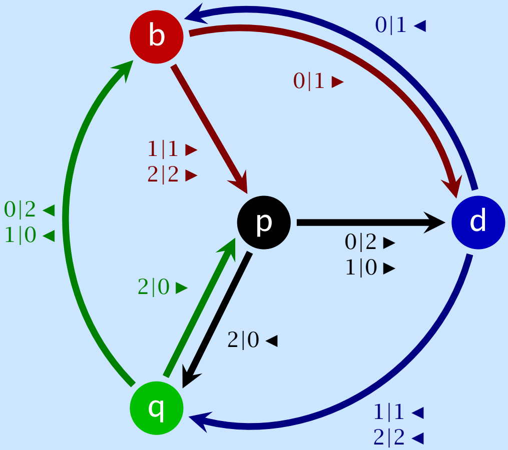
Hooper's trick use bounded searches with **recursive calls** to initial segments of the simulation of increasing sizes:

$@111\underset{s_0}{\underline{@}}_s xy1111111x2222y$ *recursive call*

The RTM is immortal iff the 2-RCM is mortal on $(s_0, (0, 0))$.

Program it!

```
def [s|search_1|t_0, t_1, t_2⟩ :
  s. @_α ⊢ @_α, l
  l. →, u
  u. x ⊢ x, t_0
   | 1x ⊢ 1x, t_1
   | 11x ⊢ 11x, t_2
   | 111 ⊢ 111, c
  call [c|check_1|p⟩ from 1
  p. 111 ⊢ 111, l

def [s|search_2|t_0, t_1, t_2⟩ :
  s. x ⊢ x, l
  l. →, u
  u. y ⊢ y, t_0
   | 2y ⊢ 2y, t_1
   | 22y ⊢ 22y, t_2
   | 222 ⊢ 222, c
  call [c|check_2|p⟩ from 2
  p. 222 ⊢ 222, l

def [s|test1|z, p⟩ :
  s. @_αx ⊢ @_αx, z
   | @_α1 ⊢ @_α1, p

def [s|endtest2|z, p⟩ :
  s. xy ⊢ xy, z
   | x2 ⊢ x2, p

def [s|test2|z, p⟩ :
  [s|search_1|t_0, t_1, t_2⟩
  [t_0|endtest2|z_0, p_0⟩
  [t_1|endtest2|z_1, p_1⟩
  [t_2|endtest2|z_2, p_2⟩
  ⟨z_0, z_1, z_2|search_1|z]
  ⟨p_0, p_1, p_2|search_1|p]

def [s|mark_1|t, co⟩ :
  s. y1 ⊢ 2y, t
   | yx ⊢ yx, co

def [s|endinc_1|t, co⟩ :
  [s|search_2|r_0, r_1, r_2⟩
  [r_0|mark_1|t_0, co_0⟩
  [r_1|mark_1|t_1, co_1⟩
  [r_2|mark_1|t_2, co_2⟩
  ⟨t_2, t_0, t_1|search_2|t]
  ⟨co_0, co_1, co_2|search_2|co]

def [s|inc2_1|t, co⟩ :
  [s|search_1|r_0, r_1, r_2⟩
  [r_0|endinc_1|t_0, co_0⟩
  [r_1|endinc_1|t_1, co_1⟩
  [r_2|endinc_1|t_2, co_2⟩
  ⟨t_0, t_1, t_2|search_1|t]
  ⟨co_0, co_1, co_2|search_1|co]

def [s|dec2_1|t⟩ :
  ⟨s, co|inc2_1|t]

def [s|mark_2|t, co⟩ :
  s. y2 ⊢ 2y, t
   | yx ⊢ yx, co

def [s|endinc_2|t, co⟩ :
  [s|search_2|r_0, r_1, r_2⟩
  [r_0|mark_2|t_0, co_0⟩
  [r_1|mark_2|t_1, co_1⟩
  [r_2|mark_2|t_2, co_2⟩
  ⟨t_2, t_0, t_1|search_2|t]
  ⟨co_0, co_1, co_2|search_2|co]

def [s|inc2_2|t, co⟩ :
  [s|search_1|r_0, r_1, r_2⟩
  [r_0|endinc_2|t_0, co_0⟩
  [r_1|endinc_2|t_1, co_1⟩
  [r_2|endinc_2|t_2, co_2⟩
  ⟨t_0, t_1, t_2|search_1|t]
  ⟨co_0, co_1, co_2|search_1|co]

def [s|dec2_2|t⟩ :
  ⟨s, co|inc2_2|t]

def [s|pushinc_1|t, co⟩ :
  s. x2 ⊢ 1x, c
   | xy1 ⊢ 1xy, pt
   | xyx ⊢ 1yx, pco
  [c|endinc_1|pt0, pco0⟩
  pt0. →, t0
  t0. 2 ⊢ 2, pt
  pt. ←, t
  pco0. x ⊢ 2, pco
  pco. ←, zco
  zco. 1 ⊢ x, co

def [s|inc1_1|t, co⟩ :
  [s|search_1|r_0, r_1, r_2⟩
  [r_0|pushinc_1|t_0, co_0⟩
  [r_1|pushinc_1|t_1, co_1⟩
  [r_2|pushinc_1|t_2, co_2⟩
  ⟨t_2, t_0, t_1|search_1|t]
  ⟨co_0, co_1, co_2|search_1|co]

def [s|dec1_1|t⟩ :
  ⟨s, co|inc1_1|t]

def [s|pushinc_2|t, co⟩ :
  s. x2 ⊢ 1x, c
   | xy2 ⊢ 1xy, pt
   | xyy ⊢ 1yy, pco
  [c|endinc_2|pt0, pco0⟩
  pt0. →, t0
  t0. 2 ⊢ 2, pt
  pt. ←, t
  pco0. x ⊢ 2, pco
  pco. ←, zco
  zco. 1 ⊢ x, co

def [s|inc1_2|t, co⟩ :
  [s|search_1|r_0, r_1, r_2⟩
  [r_0|pushinc_2|t_0, co_0⟩
  [r_1|pushinc_2|t_1, co_1⟩
  [r_2|pushinc_2|t_2, co_2⟩
  ⟨t_2, t_0, t_1|search_1|t]
  ⟨co_0, co_1, co_2|search_1|co]

def [s|dec1_2|t⟩ :
  ⟨s, co|inc1_2|t]

def [s|init_1|r⟩ :
  s. →, u
  u. 11 ⊢ xy, e
  e. ←, r

def [s|RCM_1|co_1, co_2⟩ :
  [s|init_1|s_0⟩
  [s_0|test1|s_1z, n⟩
  [s_1|inc1_1|s_2, co_1⟩
  [s_2|inc2_1|s_3, co_2⟩
  [s_3|test1|n', s_1p⟩
  ⟨s_1z, s_1p|test1|s_1]

def [s|init_2|r⟩ :
  s. →, u
  u. 22 ⊢ xy, e
  e. ←, r

def [s|RCM_2|co_1, co_2⟩ :
  [s|init_2|s_0⟩
  [s_0|test1|s_1z, n⟩
  [s_1|inc1_2|s_2, co_1⟩
  [s_2|inc2_2|s_3, co_2⟩
  [s_3|test1|n', s_1p⟩
  ⟨s_1z, s_1p|test1|s_1]

fun [s|check_1|t⟩ :
  [s|RCM_1|co_1, co_2, ...⟩
  ⟨co_1, co_2, ...|RCM_1|t]

fun [s|check_2|t⟩ :
  [s|RCM_2|co_1, co_2, ...⟩
  ⟨co_1, co_2, ...|RCM_2|t]
```



5. a SMART machine

The SMART machine $\mathfrak{C}$

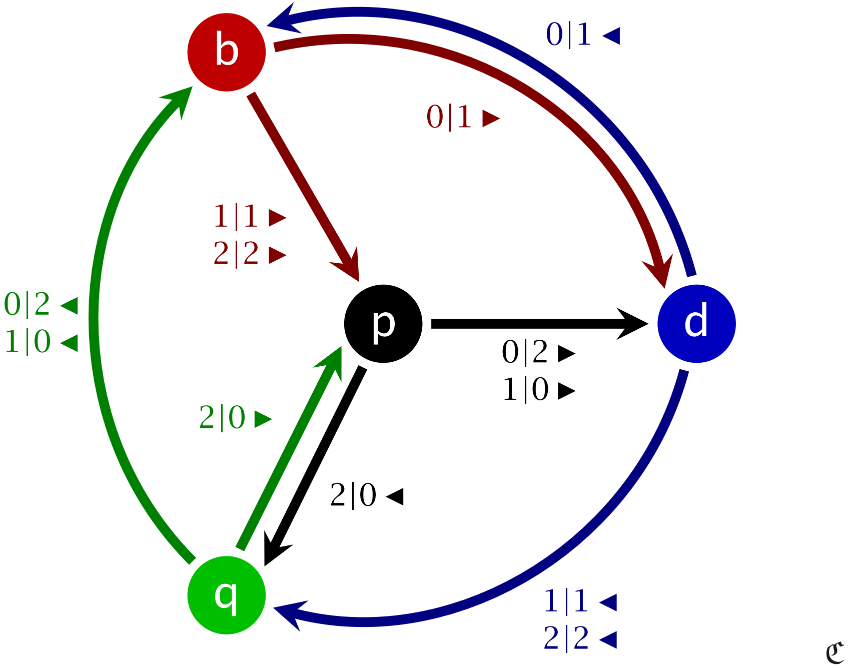
Conj[Kůrka97] Every **complete** TM has a **periodic** point.

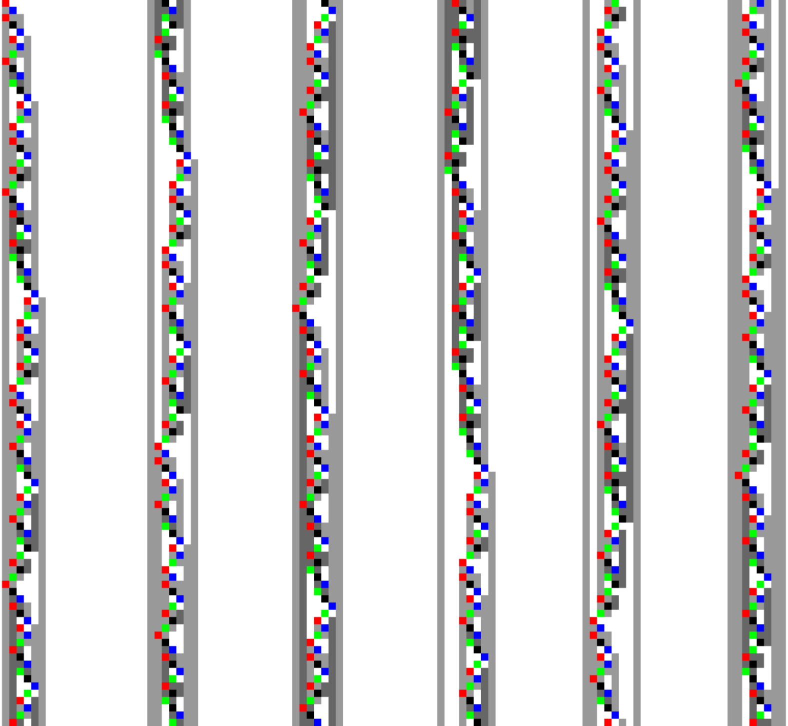
Thm[BCN02] No, here is an **aperiodic** complete TM.

Rk It relies on the **bounded search** technique [Hooper66].

In 2008, I asked **J. Cassaigne** if he had a reversible version of the BCN construction...

...he answered with a small machine $\mathfrak{C}$ which is a reversible and (drastic) simplification of the BCN machine.


b
0|1 ◀
0|1 ▶
1|1 ▶
2|2 ▶
0|2 ◀
1|0 ◀
p
d
0|2 ▶
1|0 ▶
2|0 ▶
2|0 ◀
q
1|1 ◀
2|2 ◀



The SMART machine $\mathfrak{C}$

A 4-state 3-symbols TM with nice properties:

complete no halting configuration

reversible reversed by a TM...

time-symetric ...essentially itself (up to details)

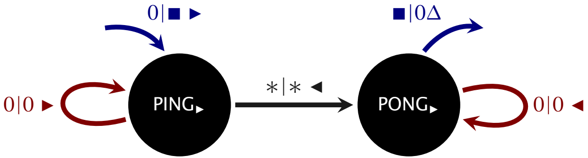
aperiodic no time periodic orbit

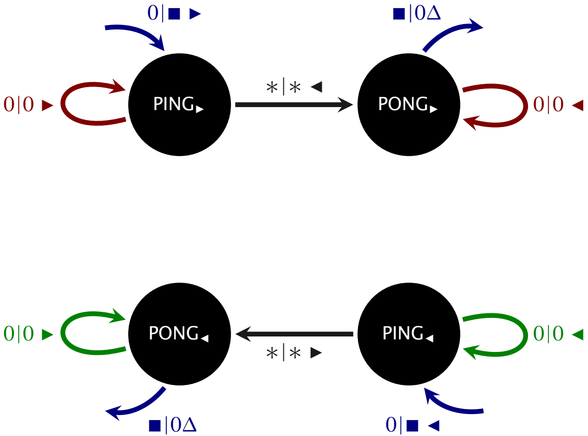
substitutive substitution-generated trace-shift language

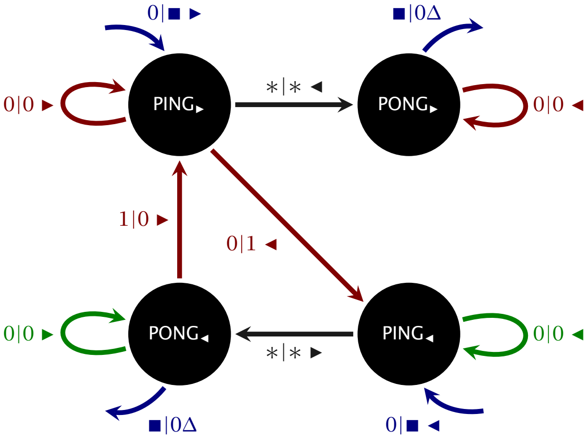
TMH-transitive dense orbits with moving head

TMT-minimal every orbit is dense with moving tape

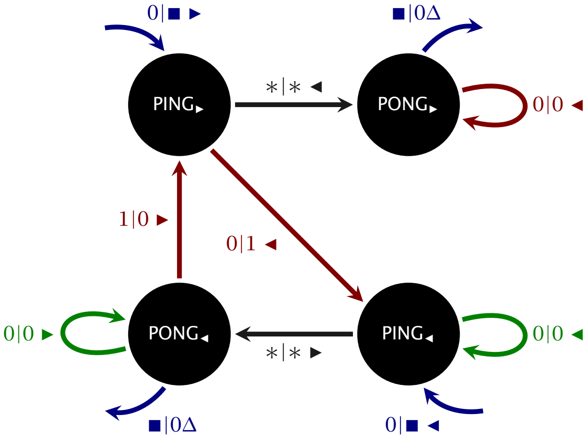
How does it work?


0|■ ▶
■|0Δ
0|0 ▶
PING▸
∗|∗ ◀
PONG▸
0|0 ◀

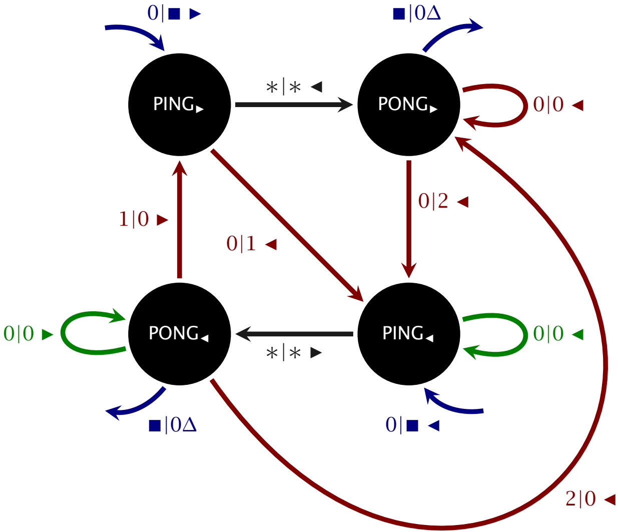

0|■ ▶
■|0Δ
0|0 ▶
PING▶
∗|∗ ◀
PONG▶
0|0 ◀
0|0 ▶
PONG◀
PING◀
0|0 ◀
∗|∗ ▶
■|0Δ
0|■ ◀

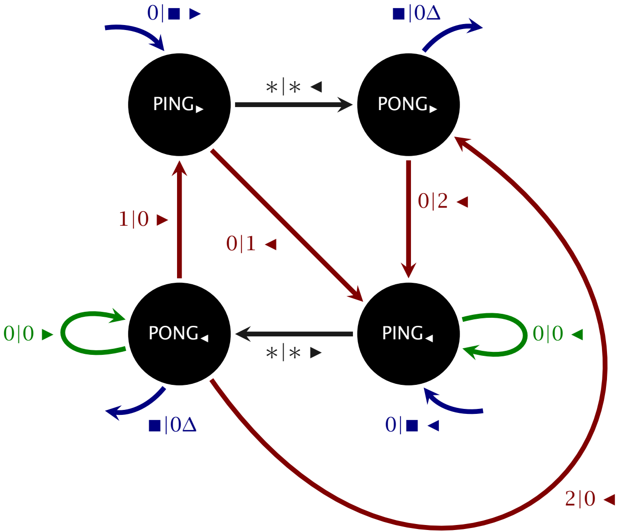


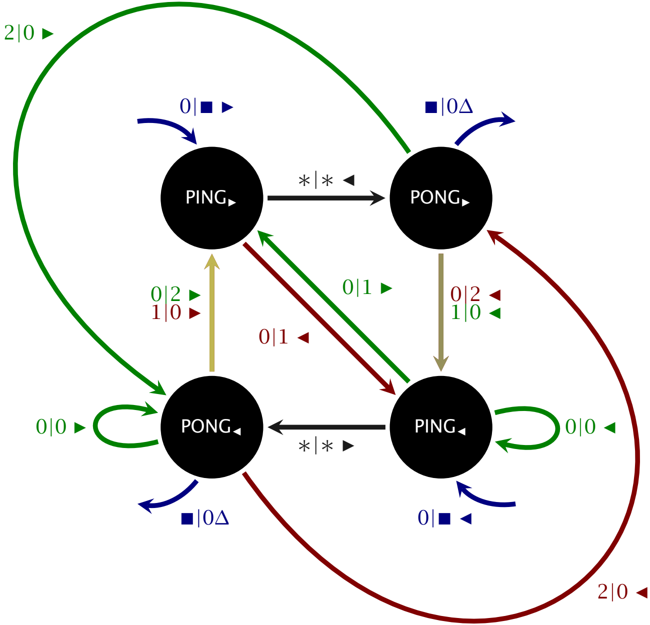
0|■ ▶
■|0Δ
0|0 ▶
PING▸
∗|∗ ◀
PONG▸
0|0 ◀
1|0 ▶
0|1 ◀
0|0 ▶
PONG◂
PING◂
0|0 ◀
∗|∗ ▶
■|0Δ
0|■ ◀

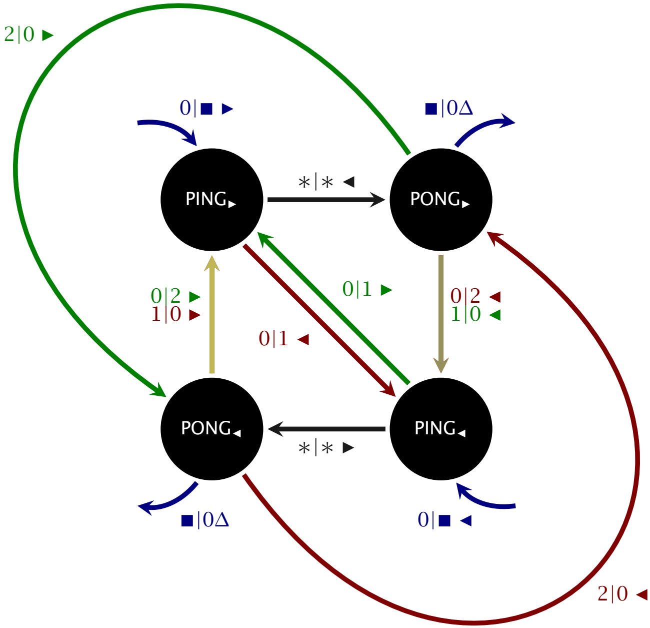


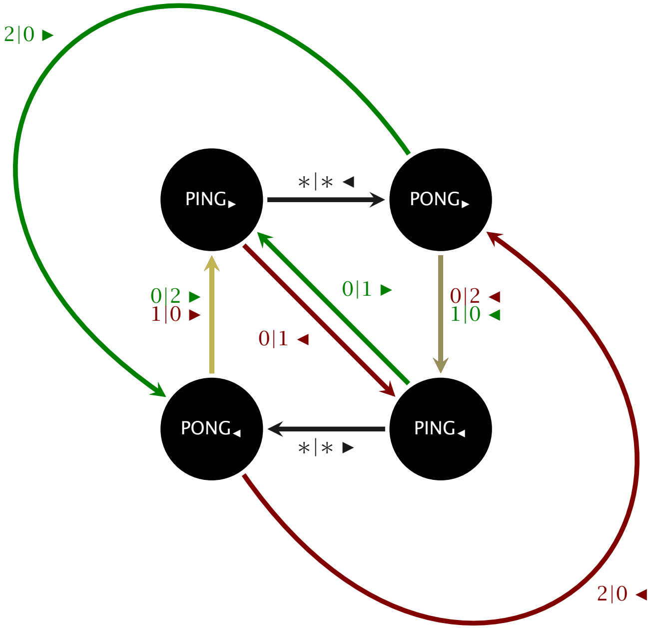
0|■ ▶
■|0Δ
| ◀
PING▸
PONG▸
0|0 ◀
1|0 ▶
0|1 ◀
0|0 ▶
PONG◂
PING◂
0|0 ◀
| ▶
■|0Δ
0|■ ◀


0|■ ▶
■|0Δ
PING▶
| ◀
PONG▶
0|0 ◀
0|2 ◀
1|0 ▶
0|1 ◀
0|0 ▶
PONG◀
PING◀
0|0 ◀
| ▶
■|0Δ
0|■ ◀
2|0 ◀

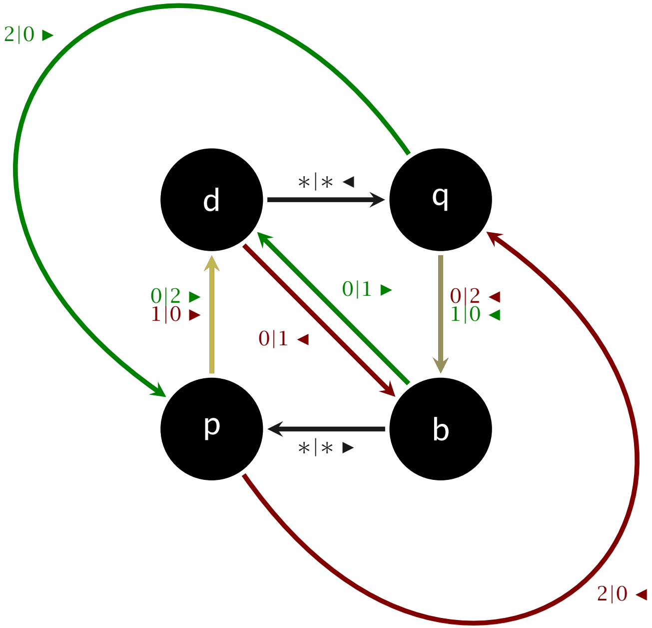

0|■ ▶
■|0Δ
PING▸
∗|∗ ◀
PONG▸
0|2 ◀
1|0 ▶
0|1 ◀
0|0 ▶
PONG◂
PING◂
0|0 ◀
∗|∗ ▶
■|0Δ
0|■ ◀
2|0 ◀


2|0 ▶
0|■ ▶
■|0Δ
∗|∗ ◀
PING▶
PONG▶
0|1 ▶
0|2 ▶
1|0 ▶
0|2 ◀
1|0 ◀
0|1 ◀
0|0 ▶
PONG◀
PING◀
0|0 ◀
∗|∗ ▶
■|0Δ
0|■ ◀
2|0 ◀


2|0 ▶
0|■ ▶
■|0Δ
PING▸
∗|∗ ◀
PONG▸
0|1 ▶
0|2 ▶
1|0 ▶
0|1 ◀
0|2 ◀
1|0 ◀
PONG◂
∗|∗ ▶
PING◂
■|0Δ
0|■ ◀
2|0 ◀



2|0 ▶
PING▶
∗|∗ ◀
PONG▶
0|2 ▶
1|0 ▶
0|1 ▶
0|1 ◀
0|2 ◀
1|0 ◀
PONG◀
PING◀
∗|∗ ▶
2|0 ◀


2|0 ▶
d
∗|∗ ◀
q
0|2 ▶
1|0 ▶
0|1 ▶
0|1 ◀
0|2 ◀
1|0 ◀
p
b
∗|∗ ▶
2|0 ◀

Recursive behavior

PING$_\blacktriangleright(n)$:
    for i=1 to n:
        d. 0|1, b ◀
        PING$_\blacktriangleleft(i-1)$
    d. $x|x$, q ◀
    for i=n downto 1:
        q. 0|2, b ◀
        PING$_\blacktriangleleft(i-1)$
    q. $y|0$, $\alpha(y)$ $\tau(y)$

PING$_\blacktriangleleft(n)$:
    for i=1 to n:
        b. 0|1, d ▶
        PING$_\blacktriangleright(i-1)$
    b. $x|x$, p ▶
    for i=n downto 1:
        p. 0|2, d ▶
        PING$_\blacktriangleright(i-1)$
    p. $y|0$, $\alpha'(y)$ $\tau'(y)$

$$\begin{cases} f(0) & = 2 \\ f(n+1) & = 3f(n) \end{cases}$$

Substitutive trace subshift

$$\varphi\begin{pmatrix}0\\ \mathbf{b}\end{pmatrix} = \begin{matrix}0 & 0 & 1 & 1\\ \mathbf{b} & \mathbf{d} & \mathbf{b} & \mathbf{p}\end{matrix} \qquad \varphi\begin{pmatrix}0\\ \mathbf{d}\end{pmatrix} = \begin{matrix}0 & 0 & 1 & 1\\ \mathbf{d} & \mathbf{b} & \mathbf{d} & \mathbf{q}\end{matrix}$$

$$\varphi\begin{pmatrix}\mathrm{x}\\ \mathbf{b}\end{pmatrix} = \begin{matrix}\mathrm{x}\\ \mathbf{b}\end{matrix} \qquad \varphi\begin{pmatrix}\mathrm{x}\\ \mathbf{d}\end{pmatrix} = \begin{matrix}\mathrm{x}\\ \mathbf{d}\end{matrix}$$

$$\varphi\begin{pmatrix}0\\ \mathbf{p}\end{pmatrix} = \begin{matrix}0 & 0 & 2 & 1\\ \mathbf{p} & \mathbf{d} & \mathbf{b} & \mathbf{p}\end{matrix} \qquad \varphi\begin{pmatrix}0\\ \mathbf{q}\end{pmatrix} = \begin{matrix}0 & 0 & 2 & 1\\ \mathbf{q} & \mathbf{b} & \mathbf{d} & \mathbf{q}\end{matrix}$$

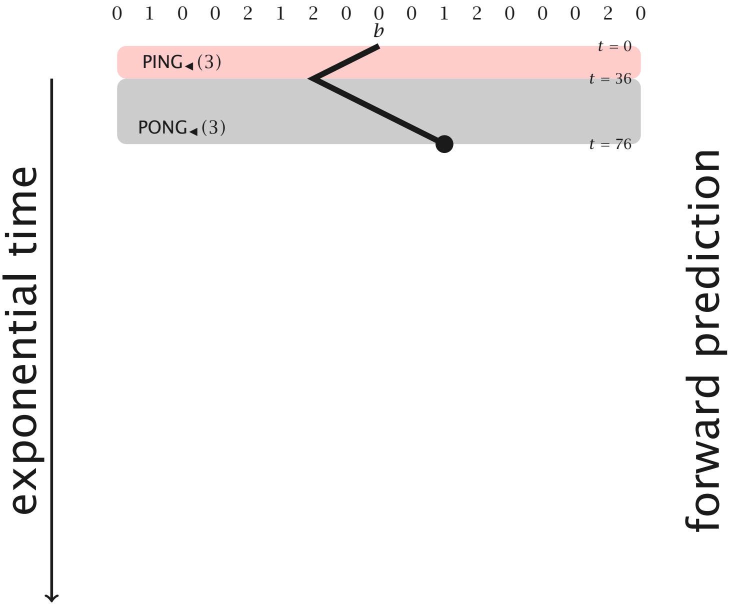
$$\varphi\begin{pmatrix}\mathrm{x}\\ \mathbf{p}\end{pmatrix} = \begin{matrix}0 & \mathrm{x} & 2 & \mathrm{x}\\ \mathbf{p} & \mathbf{d} & \mathbf{q} & \mathbf{p}\end{matrix} \qquad \varphi\begin{pmatrix}\mathrm{x}\\ \mathbf{q}\end{pmatrix} = \begin{matrix}0 & \mathrm{x} & 2 & \mathrm{x}\\ \mathbf{q} & \mathbf{b} & \mathbf{p} & \mathbf{q}\end{matrix}$$

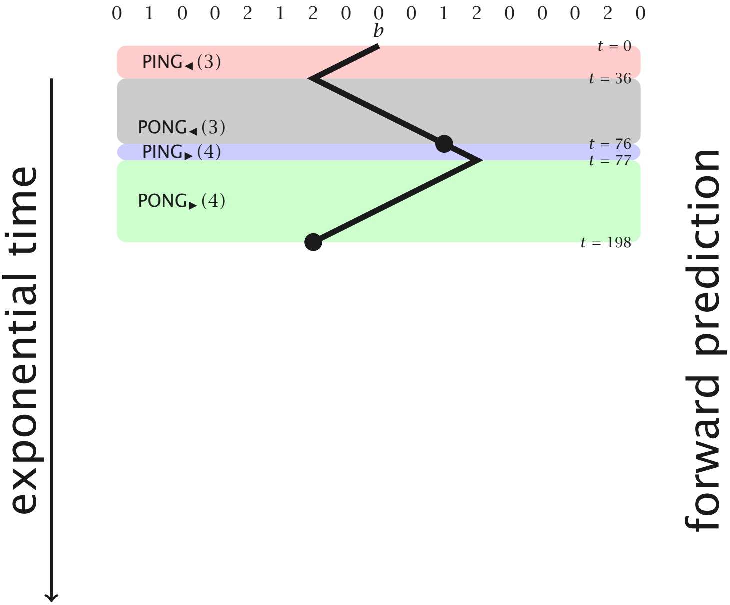
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0

b

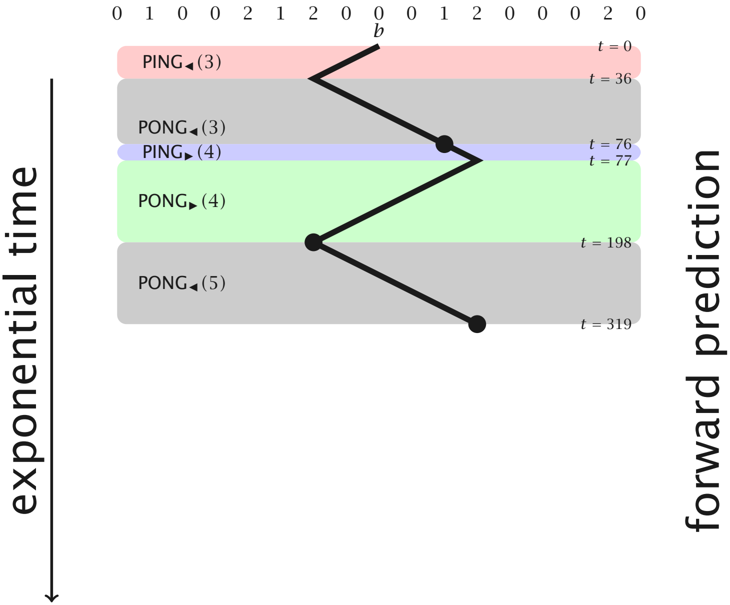
exponential time

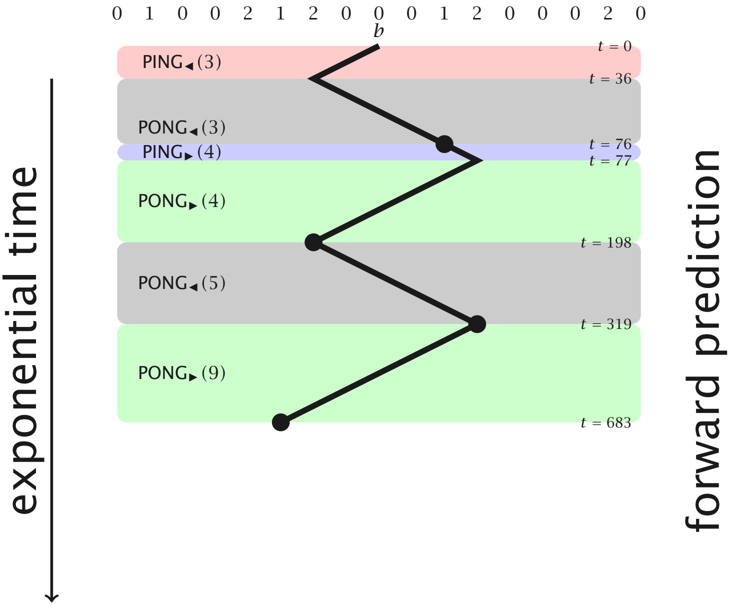
forward prediction

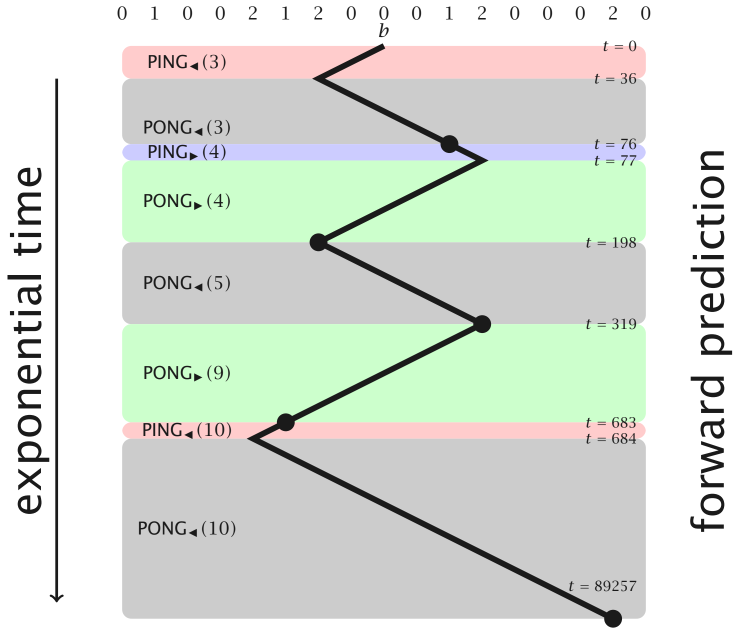

0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
t = 0
PING◀(3)
t = 36
PONG◀(3)
t = 76
exponential time
forward prediction

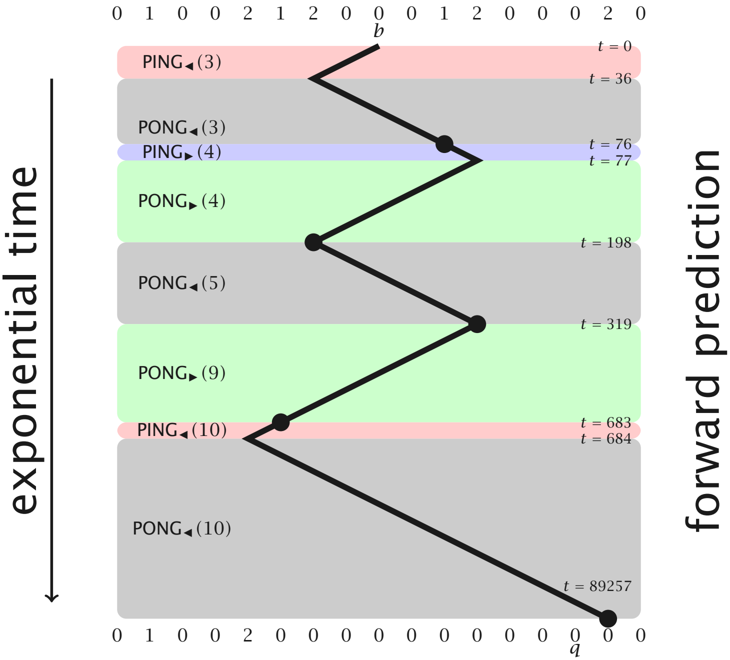


0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
t = 0
PING◀(3)
t = 36
PONG◀(3)
t = 76
PING▶(4)
t = 77
PONG▶(4)
t = 198
exponential time
forward prediction

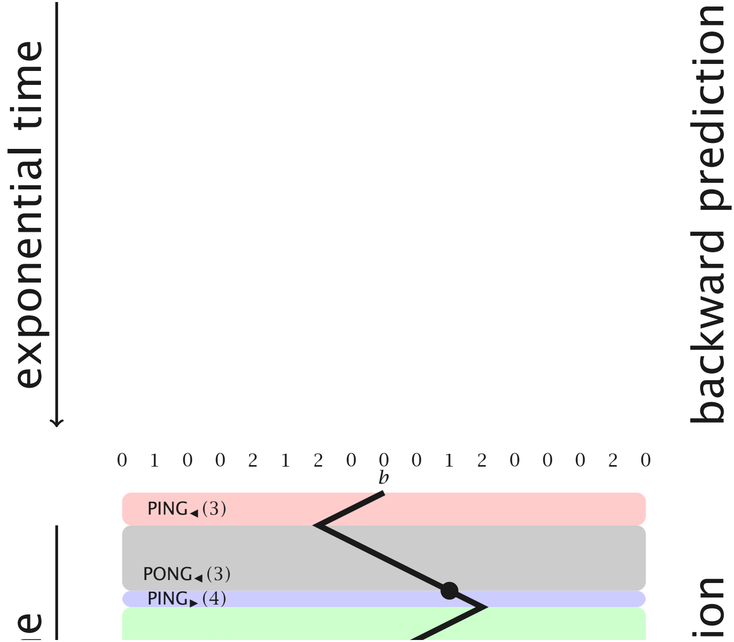

0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
t = 0
PING◀(3)
t = 36
PONG◀(3)
t = 76
PING▶(4)
t = 77
PONG▶(4)
t = 198
PONG◀(5)
t = 319
exponential time
forward prediction


0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
t = 0
PING◀(3)
t = 36
PONG◀(3)
t = 76
PING▶(4)
t = 77
PONG▶(4)
t = 198
PONG◀(5)
t = 319
PONG▶(9)
t = 683
exponential time
forward prediction

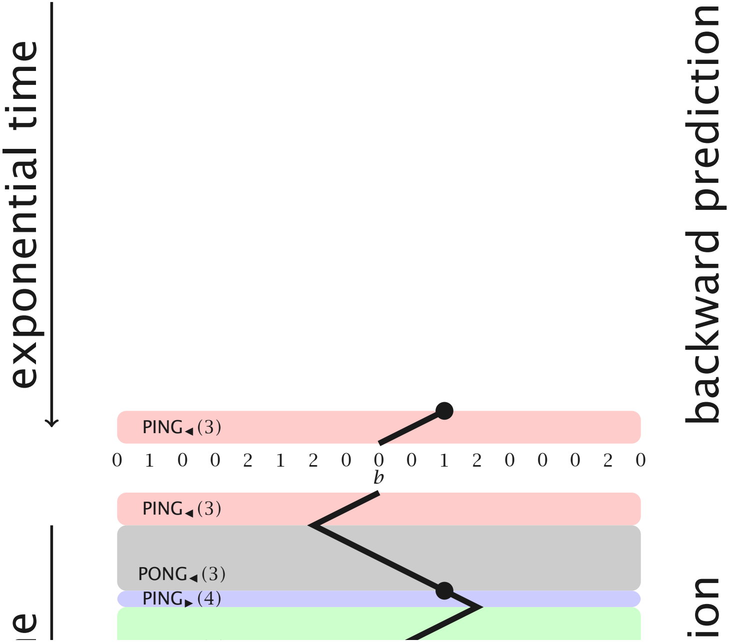

0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
t = 0
PING◀(3)
t = 36
PONG◀(3)
t = 76
PING▶(4)
t = 77
PONG▶(4)
t = 198
PONG◀(5)
t = 319
PONG▶(9)
t = 683
PING◀(10)
t = 684
PONG◀(10)
t = 89257
exponential time
forward prediction

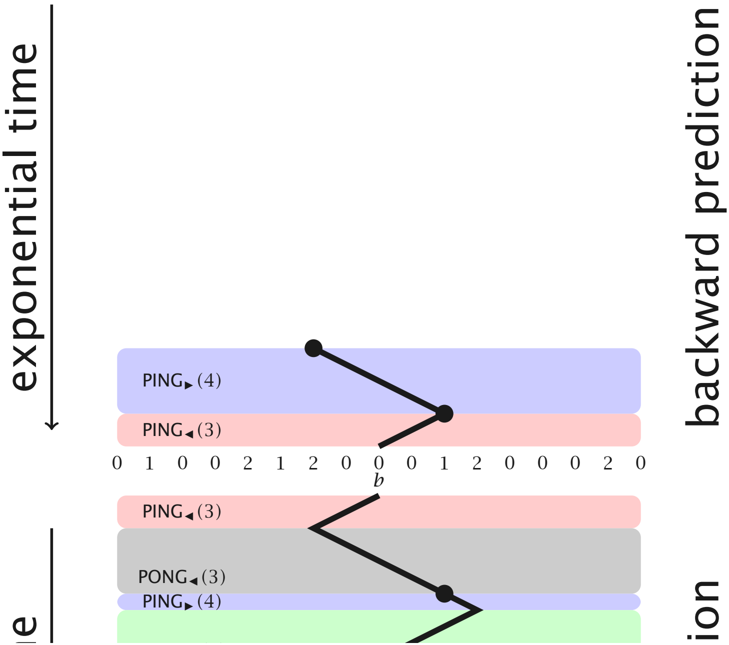


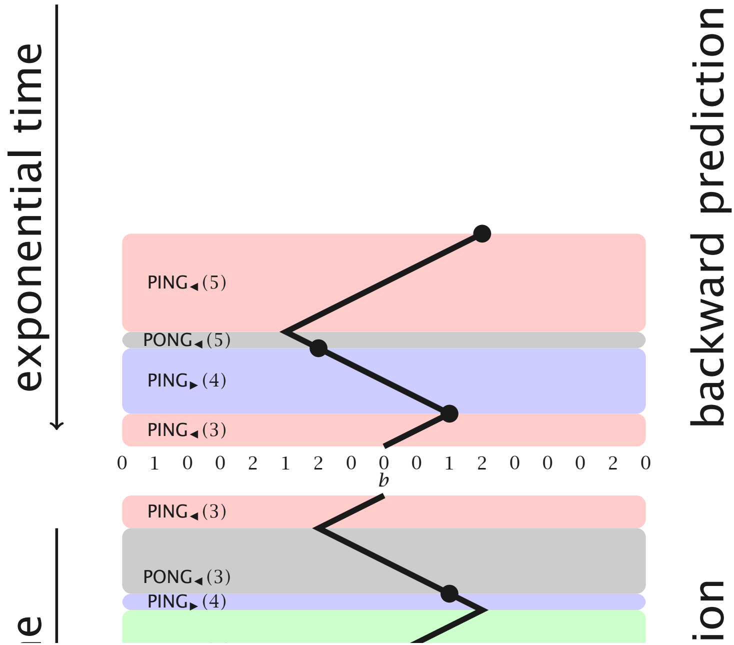
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
t = 0
PING◂(3)
t = 36
PONG◂(3)
t = 76
PING▸(4)
t = 77
PONG▸(4)
t = 198
PONG◂(5)
t = 319
PONG▸(9)
t = 683
PING◂(10)
t = 684
PONG◂(10)
t = 89257
0 1 0 0 2 0 0 0 0 0 0 0 0 0 0 0 0
q
exponential time
forward prediction

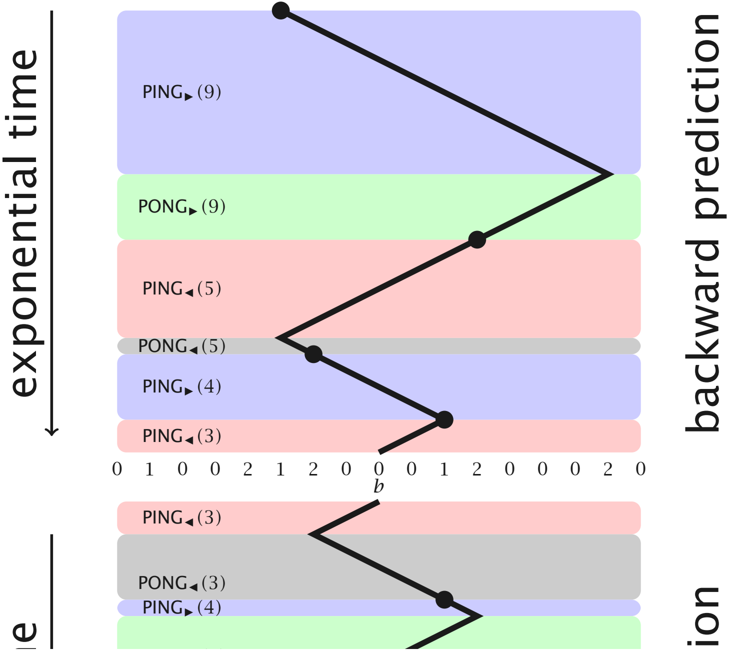


exponential time
backward prediction
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◀(3)
PONG◀(3)
PING▶(4)

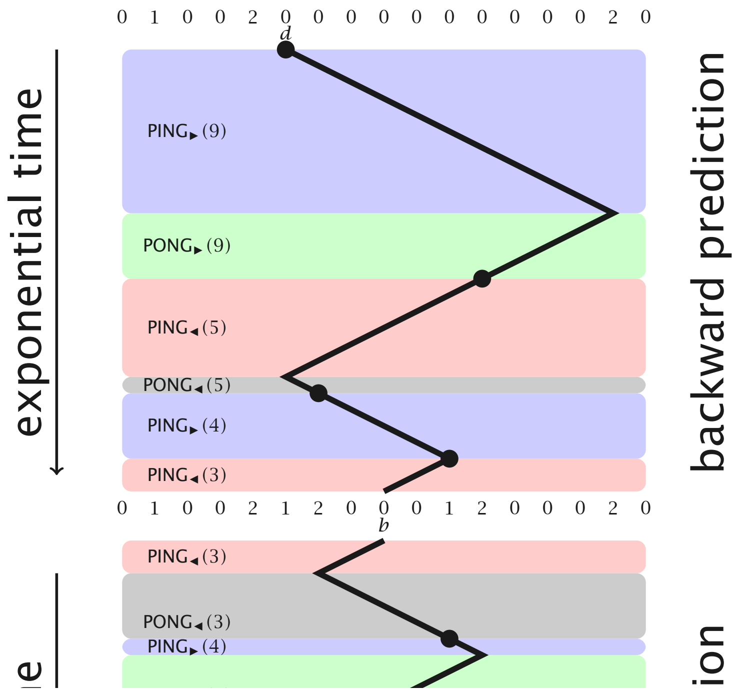

exponential time
backward prediction
PING◂(3)
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◂(3)
PONG◂(3)
PING▸(4)
ion


exponential time
backward prediction
PING▸(4)
PING◂(3)
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◂(3)
PONG◂(3)
PING▸(4)


exponential time
backward prediction
PING◂(5)
PONG◂(5)
PING▸(4)
PING◂(3)
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◂(3)
PONG◂(3)
PING▸(4)
ion



exponential time
backward prediction
PING▸ (9)
PONG▸ (9)
PING◂ (5)
PONG◂ (5)
PING▸ (4)
PING◂ (3)
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◂ (3)
PONG◂ (3)
PING▸ (4)



0 1 0 0 2 0 0 0 0 0 0 0 0 0 0 2 0
d
exponential time
backward prediction
PING▸ (9)
PONG▸ (9)
PING◂ (5)
PONG◂ (5)
PING▸ (4)
PING◂ (3)
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◂ (3)
PONG◂ (3)
PING▸ (4)
ion

SMART is (TMH-)transitive

Proposition $\left({}^{\omega}2 \cdot \underset{p}{2}\ 2^{\omega} \right)$ is a **transitive point**.

Proof

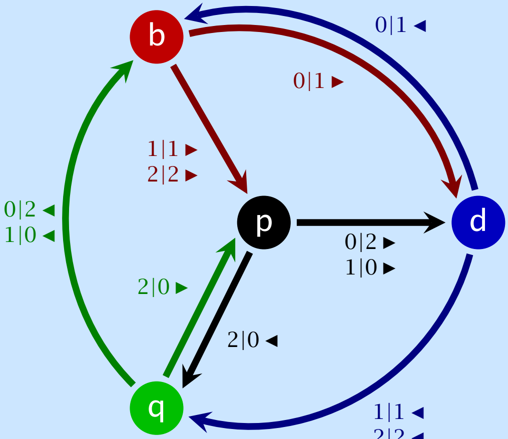
(Forward) For all $k \geqslant 0$:

$$\left({}^{\omega}2 \cdot \underset{p}{2}\ 2^{\omega} \right) \vdash^{*} \left({}^{\omega}2\ \underset{q}{2}\ 0^{k} \cdot 0\ 0^{k}\ 2^{\omega} \right) \quad .$$

(Backward) For every partial configuration $\left(\underset{\leftarrow}{u}\ \underset{\alpha}{\cdot}\ \underset{\rightarrow}{v} \right)$, there exist $w, w' \in \{0,1,2\}^{*}$ and $k > 0$ big enough such that

$$\left({}^{\omega}2\ \underset{q}{2}\ 0^{k} \cdot 0\ 0^{k}\ 2^{\omega} \right) \vdash^{*} \left({}^{\omega}2\ w\ \underset{\leftarrow}{u}\ \underset{\alpha}{\cdot}\ \underset{\rightarrow}{v}\ w'\ 2^{\omega} \right) \quad .$$

■



6. The embedding technique

Searching for a reduction

If we want to prove the following:

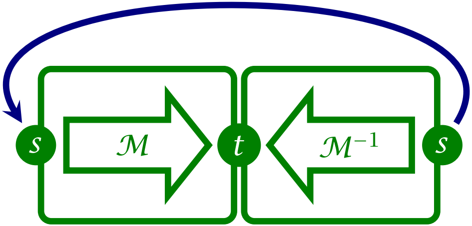
Theorem To find if a given **complete reversible Turing machine** admits a **periodic orbit** is Σ_1-complete.

In the partial case we use the following tool:

Prop[KO08] To find if a given **(aperiodic) RTM** can reach a given state t from a given state s is Σ_1-complete.

The partial case

Principle of the reduction Associate to an (aperiodic) RTM $\mathcal{M}$ with given s and t a new machine with a periodic orbit if and only if t is reachable from s.



We need to find a way to **complete** the constructed machine.
We will **embed** it into a **complete aperiodic** RTM.

Reversing time

Combine Turing machines to construct bigger ones.

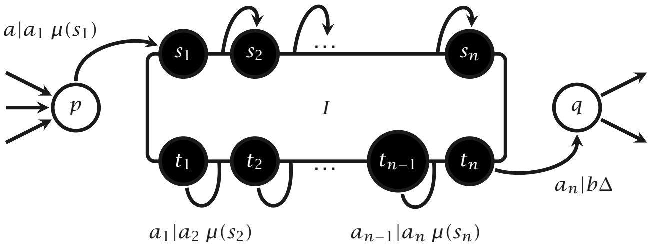
Reversing the time Given a reversible TM $M = (Q, \Sigma, \delta)$, construct $M_+ = (Q \times \{+\}, \Sigma, \delta^+)$ and $M_- = (Q \times \{-\}, \Sigma, \delta^-)$ where $(s, +)$ encodes M in state s running **forward** and $(s, -)$ running **backward**.

A typical use connects halting pairs from one machine to the corresponding starting pair of the other.

Embedding technique

A TM I with starting pairs $(s_1, a_1), \ldots, (s_n, a_n)$ and halting pairs $(t_1, a_1), \ldots, (t_n, a_n)$ is **innocuous** if starting from $(s_i, c, p + \mu(s_i))$ where $c(p) = a_i$ the machine might only halt in (t_i, c, p).

The **embedding** H^I of an **invited** innocuous TM I inside a **host** TM H is the TM containing a copy of both I and H where one transition $\delta(p, a) = (q, b, \Delta)$ from H is replaced by



Undecidability of transitivity

BRA Reachability Problem[Σ_1^0-comp. too] Given a binary reversible aperiodic TM, a starting pair (s, a) and a halting pair (t, b), decide if (t, b) is reachable from (s, a).

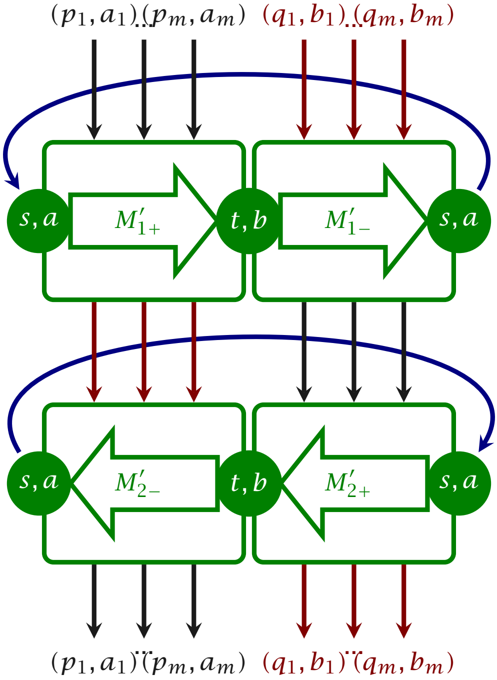
Theorem $\overline{\text{BRA Reachability Problem}} \leqslant_m$ Transitivity Problem

Proof

Let M, (s, a), (t, b) be an instance of the BRA Reachability Problem and M' be a copy of M with a third symbol $\$$.

Apply *Reversing time* to 2 copies of M' to construct an innocuous TM I as follows.

SMART^I is transitive iff (t, b) is not reachable from (s, a). ■



$(p_1, a_1) \dots (p_m, a_m)$
$(q_1, b_1) \dots (q_m, b_m)$
s, a
M'_{1+}
t, b
M'_{1-}
s, a
s, a
M'_{2-}
t, b
M'_{2+}
s, a
$(p_1, a_1) \dots (p_m, a_m)$
$(q_1, b_1) \dots (q_m, b_m)$

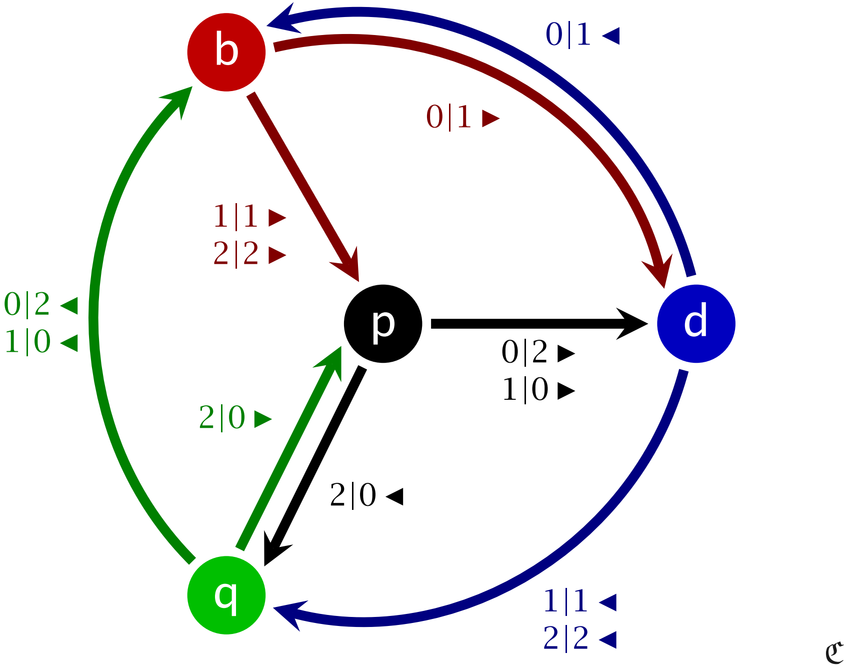
Conclusion

The embedding technique can be used to prove several undecidability results on TM.

Theorem **Transitivity** is Π_1^0-hard in TMH, TMT and ST.

Theorem **Minimality** is Σ_1^0-hard in TMT and ST.

Theorem To find if a given **complete reversible Turing machine** admits a **periodic orbit** is Σ_1-complete.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