# A Small Minimal Aperiodic Reversible Turing Machine

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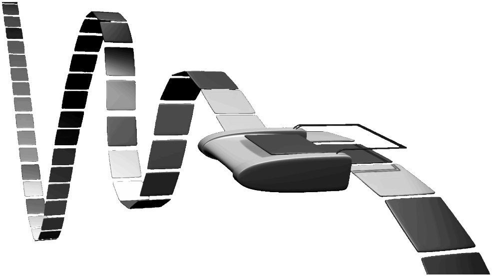
# Turing machines

The classical **Turing machine**: finitely many **states**, a (bi-)infinite **tape**, a mobile i/o **head** pointing on a cell (optionally: blank symbol, starting and halting states).



**Halting Problem[$\Sigma_1^0$-comp.]** Given a TM and a finite starting configuration, decide if a halting state is eventually reached.

# Reachability and similar questions

**Reachability Problem[$\Sigma_1^0$-comp.]** Given a TM and two states $s$ and $t$, decide if state $t$ is reachable from state $s$.

**Totality Problem[$\Pi_2^0$-comp.]** Given a TM, decide if it eventually halts starting from any **finite configuration**.

**Mortality Problem[$\Sigma_1^0$-comp.]** Given a TM, decide if it eventually halts starting from any configuration.

**Periodicity Problem[$\Sigma_1^0$-comp.]** Given a TM, decide if every configuration eventually loops by reaching itself again.

# The Transitivity Problem

**Transitivity Problem[$\Pi_1^0$-hard]** Given a TM, decide if every **partial configuration** is reachable from every **partial configuration** (completed into a proper configuration).

A **partial configuration** specifies only a finite segment of the tape plus the state and head position.

$$???????ab.ba\underset{\mathbf{q}}{b}aa???????$$

**Question** How do we prove the undecidability of the Transitivity Problem?

# The Transitivity Problem

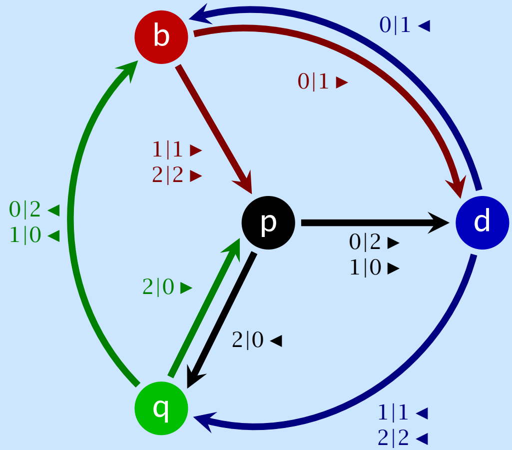
**Transitivity Problem[$\Pi_1^0$-hard]** Given a TM, decide if every **partial configuration** is reachable from every **partial configuration** (completed into a proper configuration).

A **partial configuration** specifies only a finite segment of the tape plus the state and head position.

$$???????ab.ba\underset{\mathbf{q}}{b}aa???????$$

**Question** How do we prove the undecidability of the Transitivity Problem?

**Question** ...and first, how do you build a transitive TM?




# 1. Dynamics of Turing machines

# Turing machines

A **Turing machine** is a triple $(Q, \Sigma, \delta)$ where $Q$ is the finite set of states, $\Sigma$ is the finite set of tape symbols and $\delta : Q \times \Sigma \to Q \times \Sigma \times \{-1, 0, 1\}$ is the partial transition function.

A transition $\delta(s, a) = (t, b, d)$ means:
*"in state $s$, when reading the symbol $a$ on the tape, replace it by $b$ move the head in direction $d$ and enter state $t$."*

**Configurations** are triples $(s, c, p) \in Q \times \Sigma^{\mathbb{Z}} \times \mathbb{Z}$.

A **transition** transforms $(s, c, p)$ into $(t, c', p + d)$ where $\delta(s, c(p)) = (t, b, d)$ and $c' = c$ everywhere but $c'(p) = b$.

**Notation** $(s, c, p) \vdash (t, c', p + d)$ and closures $\vdash^+$ and $\vdash^*$

# Definitions

A configuration $(s, c, p)$ is:

- **halting** if $\delta(s, c(p))$ is undefined, $(s, c(p))$ is a **halting pair**
- **periodic** if $(s, c, p) \vdash^{+} (s, c, p)$

A TM $(Q, \Sigma, \delta)$ is:

- **complete** if $\delta$ is complete
- **aperiodic** if it has no periodic configuration
- **surjective** if every configuration has a preimage
- **injective** if every configuration has at most one preimage

For complete machines surjective is equivalent to injective.

# Reversibility

Injective TM are in fact reversible TM.

**Definition** A **reversible** TM $M = (Q, \Sigma, \delta)$ is characterized by a partial injective map $\rho$ and a map $\mu$ such that $\delta(s, a) = (t, b, \mu(t))$ where $\rho(s, a) = (t, b)$.

The **reverse** of $M$ is $M^{-1}$ where $\delta^{-1}(t, b) = (s, a, -\mu(s))$.

$$(s, c, p) \vdash_M (t, c', p + \mu(t)) \Longrightarrow (t, c', p) \vdash_{M^{-1}} (s, c, p - \mu(s))$$

A **starting pair** is a halting pair of the reverse.

A **starting configuration** is a halting config of the reverse.

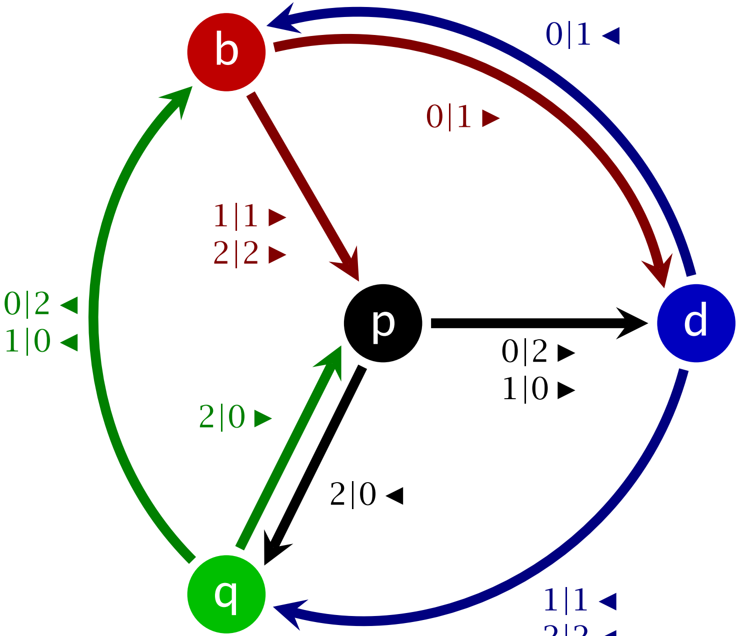
# Naive dynamics

A **topological dynamical system** is a pair $(X, T)$ where the topological space $X$ is the **phase space** and the continuous function $T : X \to X$ is the **global transition function**.

The **orbit** of $x \in X$ is $\mathcal{O}(x) = (T^n(x))_{n \in \mathbb{N}}$.

Using the **product topology** one obtains a **topological dynamical system** $(X, T)$ for a TM where the phase space is $X = Q \times \Sigma^{\mathbb{Z}} \times \mathbb{Z}$ and the transition function $T$ is continuous.

Unfortunately, $X$ is not **compact**, we follow Kůrka's alternative compact dynamical models TMH and TMT.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# Moving head vs moving tape dynamics

**TMH**

$$X_h \subset (Q \cup \Sigma)^{\mathbb{Z}}$$
$$T_h : X_h \to X_h$$

$\cdots$000000**b**000000000$\cdots$
$\cdots$0000001**d**00000000$\cdots$
$\cdots$000000**b**110000000$\cdots$
$\cdots$0000001**p**10000000$\cdots$
$\cdots$00000010**d**0000000$\cdots$
$\cdots$0000001**b**01000000$\cdots$
$\cdots$00000011**d**1000000$\cdots$
$\cdots$0000001**q**11000000$\cdots$
$\cdots$000000**b**101000000$\cdots$
$\cdots$0000001**p**01000000$\cdots$
$\vdots$

**TMT**

$$X_t = {}^{\omega}\Sigma \times Q \times \Sigma^{\omega}$$
$$T_t : X_t \to X_t$$

$\cdots$0000000**b**00000000$\cdots$
$\cdots$0000001**d**00000000$\cdots$
$\cdots$0000000**b**11000000$\cdots$
$\cdots$0000001**p**10000000$\cdots$
$\cdots$0000010**d**00000000$\cdots$
$\cdots$0000001**b**01000000$\cdots$
$\cdots$0000011**d**10000000$\cdots$
$\cdots$0000001**q**11000000$\cdots$
$\cdots$0000000**b**10100000$\cdots$
$\cdots$0000001**p**01000000$\cdots$
$\vdots$

# Trace-shift dynamics

**ST**

$$S_T \subseteq (Q \times \Sigma)^{\omega}$$

$$\sigma : S_T \to S_T$$

**TMT**

$$X_t = {}^{\omega}\Sigma \times Q \times \Sigma^{\omega}$$

$$T_t : X_t \to X_t$$

The column shift of TMT

0 0 1 1 0 0 1 1 1 0 . . .
**b d b p d b d q b p**

$\cdots$ 0000000**b**00000000 $\cdots$
$\cdots$ 0000001**d**00000000 $\cdots$
$\cdots$ 0000000**b**11000000 $\cdots$
$\cdots$ 0000001**p**10000000 $\cdots$
$\cdots$ 0000010**d**00000000 $\cdots$
$\cdots$ 0000001**b**01000000 $\cdots$
$\cdots$ 0000011**d**10000000 $\cdots$
$\cdots$ 0000001**q**11000000 $\cdots$
$\cdots$ 0000000**b**10100000 $\cdots$
$\cdots$ 0000001**p**01000000 $\cdots$
$\vdots$

# Topological transitivity

**Definition** A dynamical system $(X,T)$ is **transitive** is it admits a **transitive point** $x$ such that $\overline{\mathcal{O}(x)} = X$.

**Proposition** $(X,T)$ is **transitive** iff for every pair of open sets $U,V \subseteq X$, there exists $t$ such that $T^t(U) \cap V \neq \emptyset$.

**TMH** $\forall u,v,u',v' \; \exists w,z,w',z',n \quad T_h^n(wu.vz) = w'u'.v'z'$

**TMT** $\forall u,v,\alpha,u',v',\beta \; \exists w,z,w',z',n \; T_t^n(wu,\alpha,vz) = (w'u',\beta,v'z')$

**ST** $\forall u,v \in S_T \;\; \exists w \in S_T \quad uwv \in S_T$

TMH transitive $\implies$ TMT transitive $\implies$ ST transitive.

TMH transitive implies **complete**, **reversible** and **aperiodic**

# Transitivities

**Definition** A point $x \in X$ is **periodic** if it admits a **period** $p > 0$ such that $T^p(x) = x$.

**Proposition** A TM with a **periodic** point is **not ST transitive**.

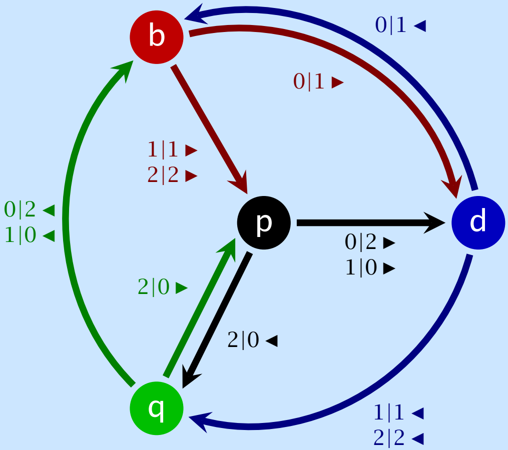
The single-state **shift** TM is **TMT transitive** but **not TMH**.

$$\delta(q, x) = (q, x, \blacktriangleright)$$

The single-state **eraser** TM is **ST transitive** but **not TMT**.

$$\delta(q, x) = (q, 0, \blacktriangleright)$$

**Question** How do we construct a complete reversible aperiodic TM?




# 2. a SMART machine

# The SMART machine $\mathfrak{C}$

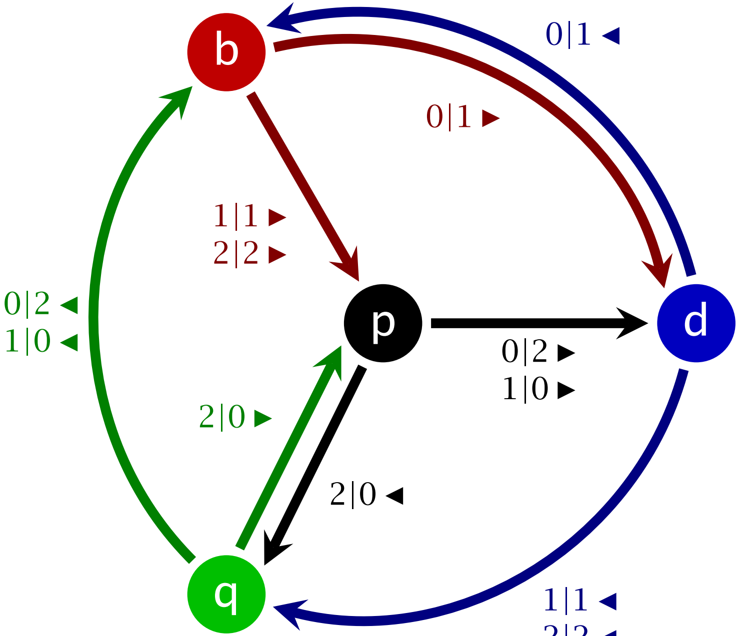
**Conj[Kůrka97]** Every **complete** TM has a **periodic** point.

**Thm[BCN02]** No, here is an **aperiodic** complete TM.

**Rk** It relies on the **bounded search** technique [Hooper66].

In 2008, I asked **J. Cassaigne** if he had a reversible version of the BCN construction...

...he answered with a small machine $\mathfrak{C}$ which is a reversible and (drastic) simplification of the BCN machine.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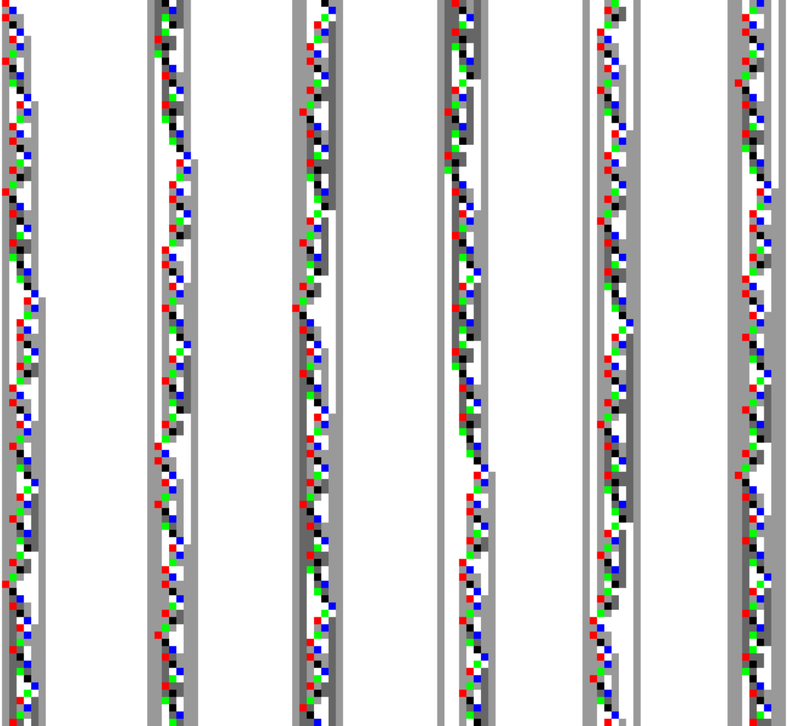
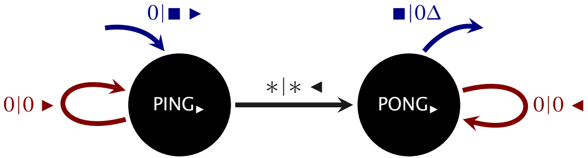


# The SMART machine $\mathfrak{C}$

A 4-state 3-symbols TM with nice properties:

- **complete** no halting configuration
- **reversible** reversed by a TM...
- **time-symetric** ...essentially itself (up to details)
- **aperiodic** no time periodic orbit
- **substitutive** substitution-generated trace-shift language
- **TMH-transitive** dense orbits with moving head
- **TMT-minimal** every orbit is dense with moving tape

How does it work?



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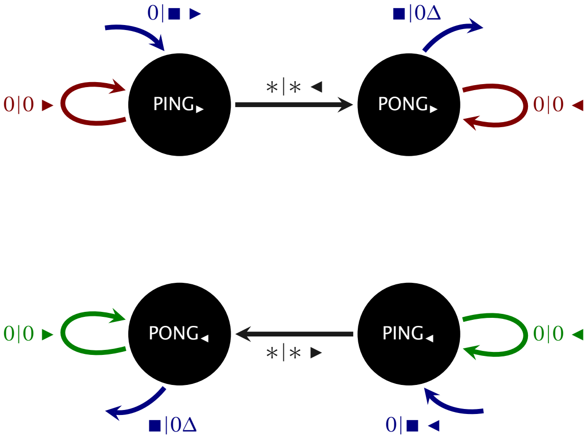

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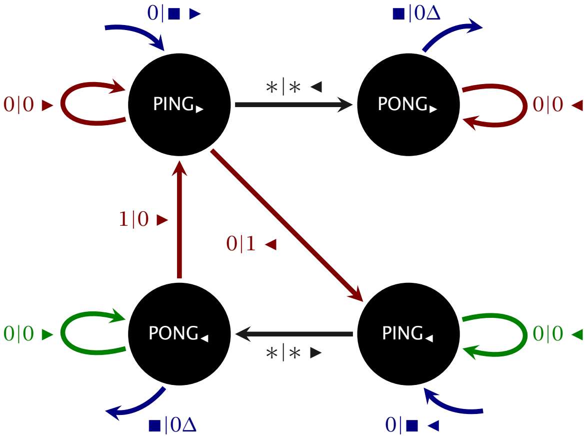

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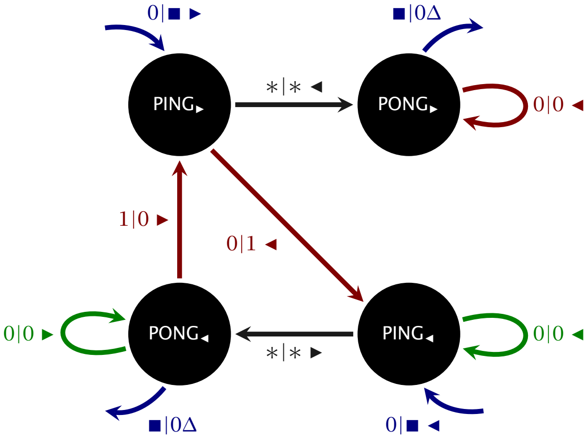

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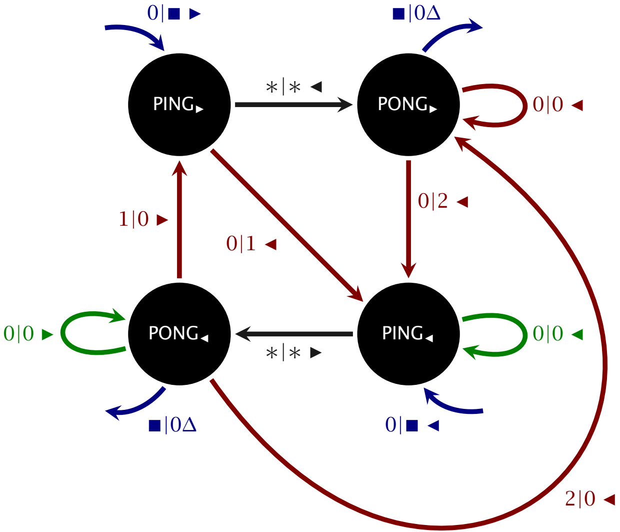

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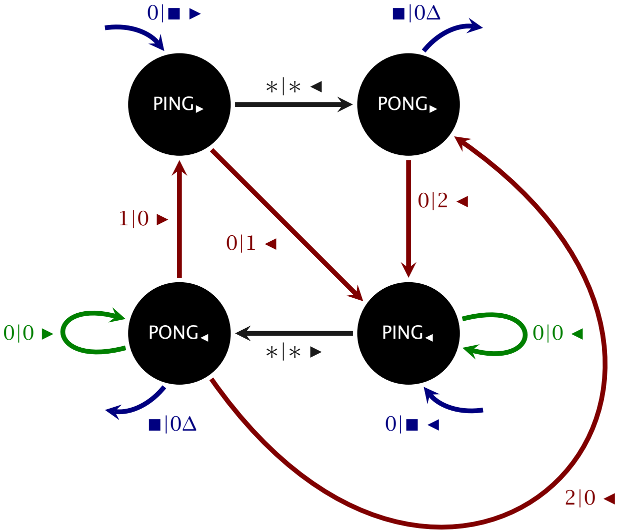

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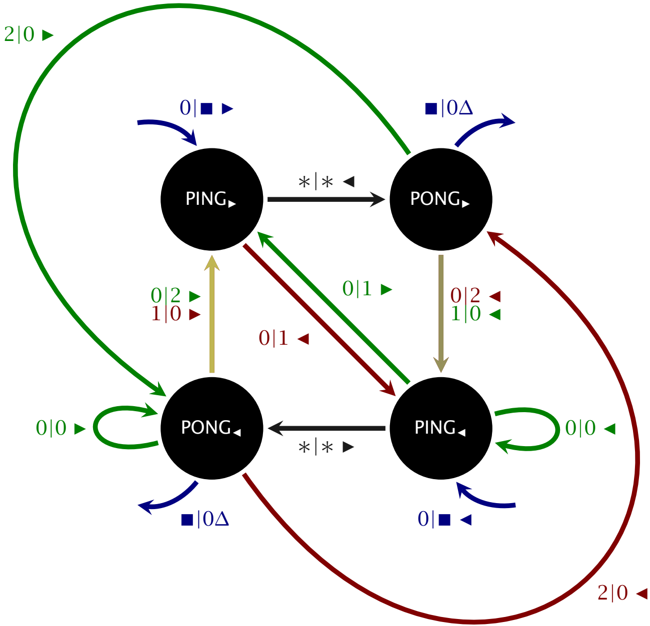

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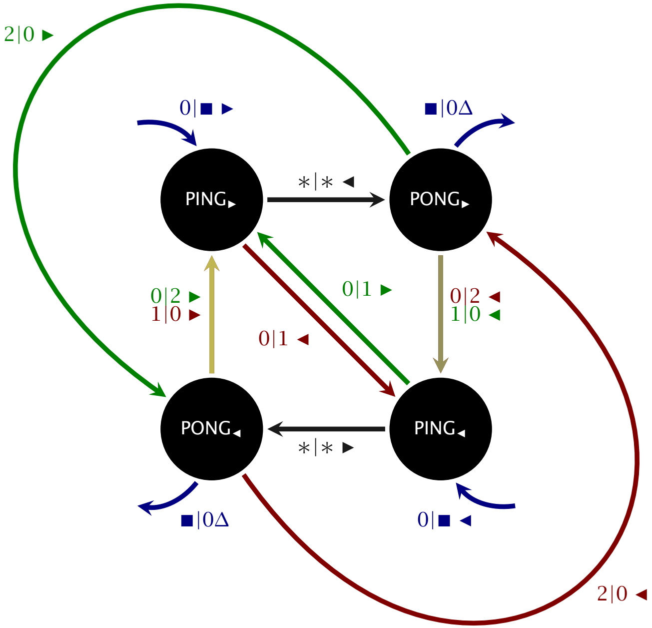

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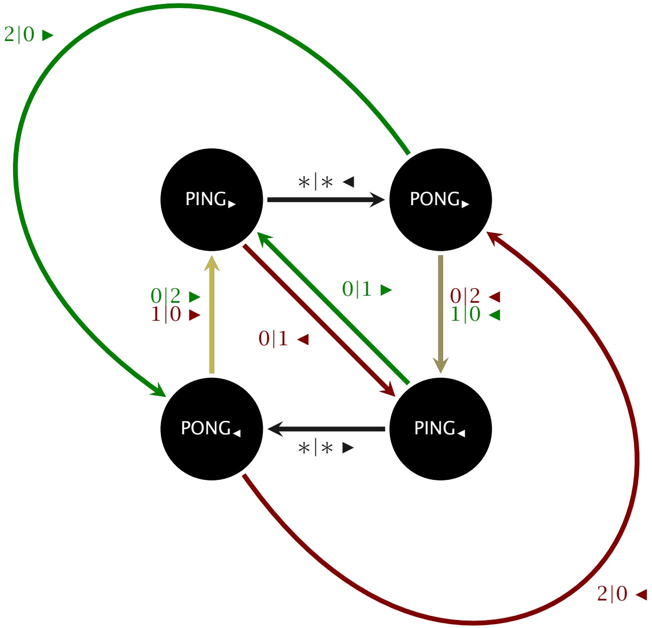

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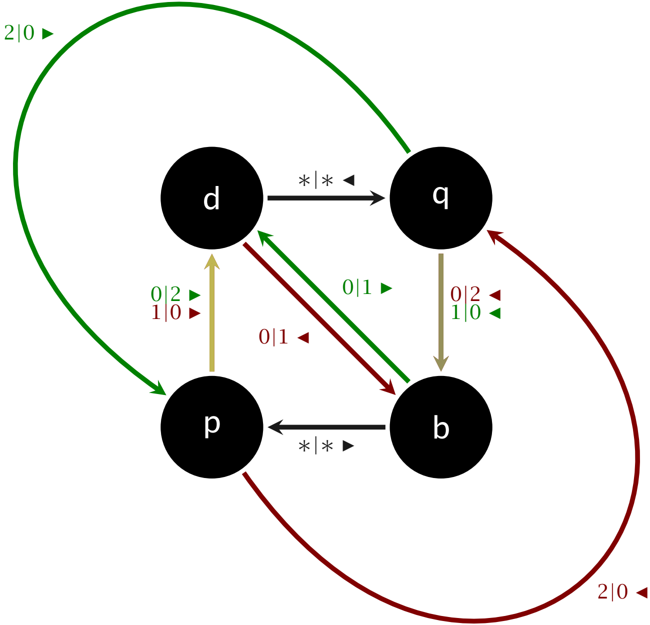
2|0 ◀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0|1 ◀
0|2 ◀
1|0 ◀
∗|∗ ▶
2|0 ◀

# Recursive behavior

PING$_{\blacktriangleright}(n)$:
    for i=1 to n:
        d. 0|1, b ◀
        PING$_{\blacktriangleleft}(i-1)$
    d. $x|x$, q ◀
    for i=n downto 1:
        q. 0|2, b ◀
        PING$_{\blacktriangleleft}(i-1)$
    q. $y|0$, $\alpha(y)$ $\tau(y)$

PING$_{\blacktriangleleft}(n)$:
    for i=1 to n:
        b. 0|1, d ▶
        PING$_{\blacktriangleright}(i-1)$
    b. $x|x$, p ▶
    for i=n downto 1:
        p. 0|2, d ▶
        PING$_{\blacktriangleright}(i-1)$
    p. $y|0$, $\alpha'(y)$ $\tau'(y)$

$$\begin{cases} f(0) & = 2 \\ f(n+1) & = 3f(n) + 2 \end{cases}$$

$$f(n) = 3^{n+1} - 1$$

# Substitutive trace subshift

$$\varphi\begin{pmatrix}0\\ \mathbf{b}\end{pmatrix} = \begin{matrix}0&0&1&1\\ \mathbf{b}&\mathbf{d}&\mathbf{b}&\mathbf{p}\end{matrix} \qquad \varphi\begin{pmatrix}0\\ \mathbf{d}\end{pmatrix} = \begin{matrix}0&0&1&1\\ \mathbf{d}&\mathbf{b}&\mathbf{d}&\mathbf{q}\end{matrix}$$

$$\varphi\begin{pmatrix}x\\ \mathbf{b}\end{pmatrix} = \begin{matrix}x\\ \mathbf{b}\end{matrix} \qquad \varphi\begin{pmatrix}x\\ \mathbf{d}\end{pmatrix} = \begin{matrix}x\\ \mathbf{d}\end{matrix}$$

$$\varphi\begin{pmatrix}0\\ \mathbf{p}\end{pmatrix} = \begin{matrix}0&0&2&1\\ \mathbf{p}&\mathbf{d}&\mathbf{b}&\mathbf{p}\end{matrix} \qquad \varphi\begin{pmatrix}0\\ \mathbf{q}\end{pmatrix} = \begin{matrix}0&0&2&1\\ \mathbf{q}&\mathbf{b}&\mathbf{d}&\mathbf{q}\end{matrix}$$

$$\varphi\begin{pmatrix}x\\ \mathbf{p}\end{pmatrix} = \begin{matrix}0&x&2&x\\ \mathbf{p}&\mathbf{d}&\mathbf{q}&\mathbf{p}\end{matrix} \qquad \varphi\begin{pmatrix}x\\ \mathbf{q}\end{pmatrix} = \begin{matrix}0&x&2&x\\ \mathbf{q}&\mathbf{b}&\mathbf{p}&\mathbf{q}\end{matrix}$$

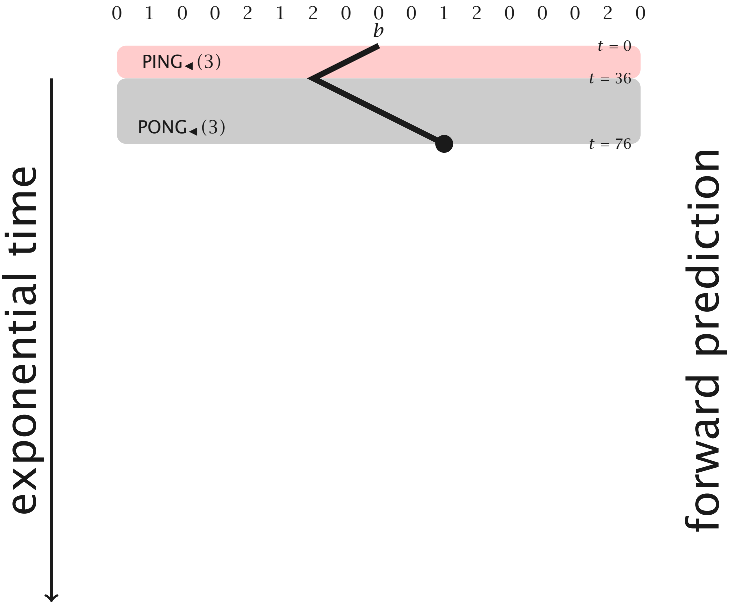
$$\left|\varphi^n\begin{pmatrix}0\\ \mathbf{b}\end{pmatrix}\right| = \frac{3^{n+1}-1}{2}$$

0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0

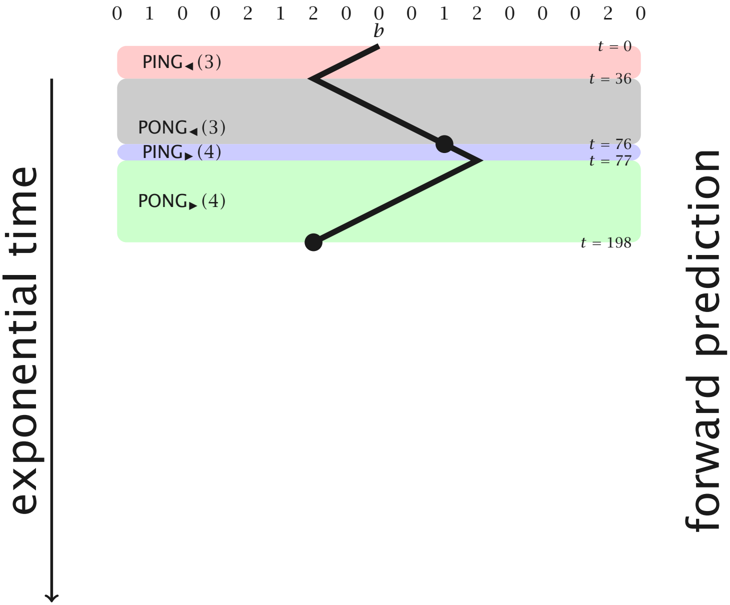
*b*

exponential time

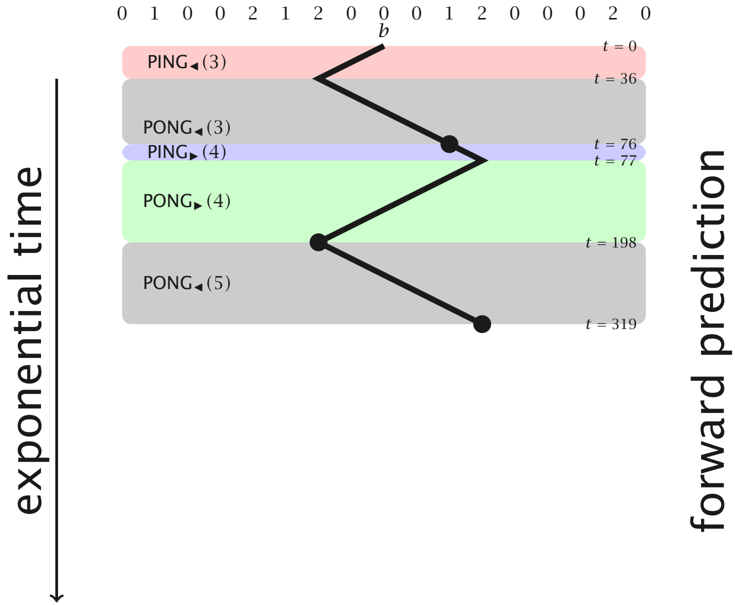
forward prediction



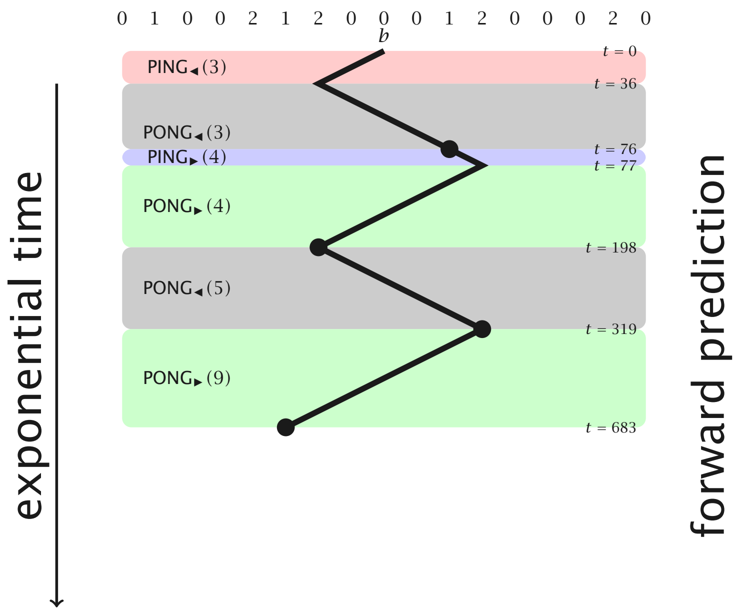
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◂(3)
t = 0
t = 36
PONG◂(3)
t = 76
exponential time
forward prediction



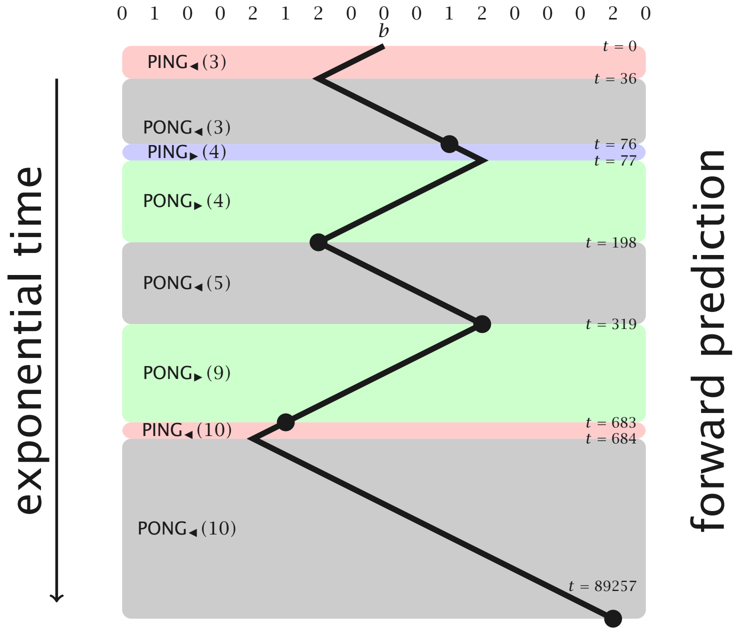
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
t = 0
PING◀(3)
t = 36
PONG◀(3)
t = 76
PING▶(4)
t = 77
PONG▶(4)
t = 198
exponential time
forward prediction



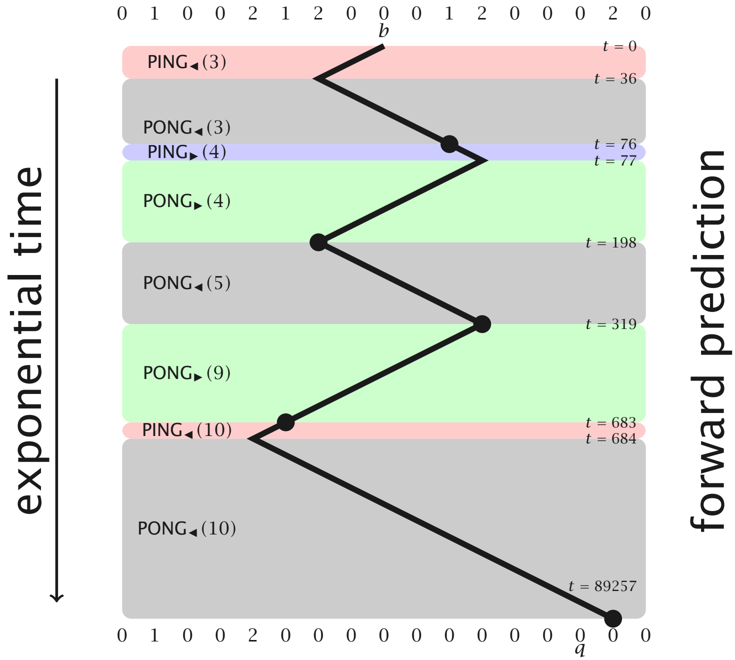
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
t = 0
PING◀(3)
t = 36
PONG◀(3)
t = 76
PING▶(4)
t = 77
PONG▶(4)
t = 198
PONG◀(5)
t = 319
exponential time
forward prediction



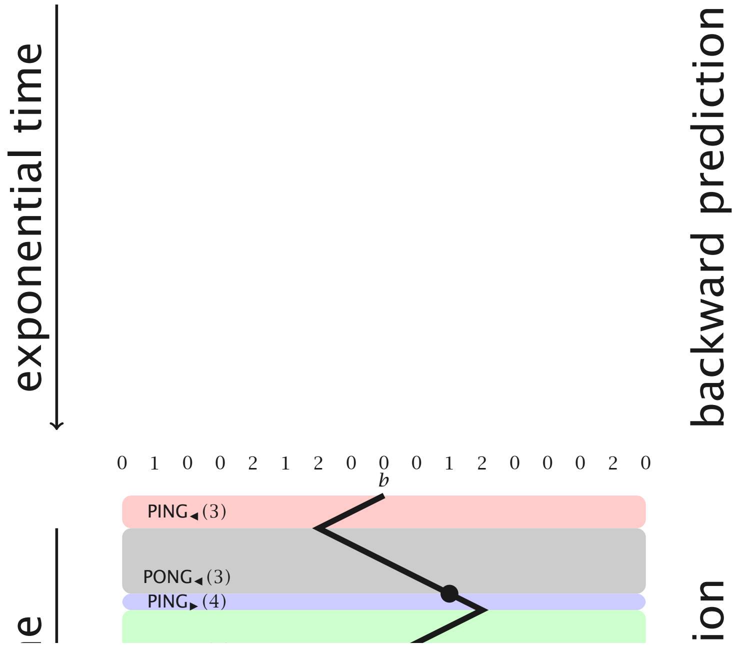
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
t = 0
PING◀(3)
t = 36
PONG◀(3)
t = 76
PING▶(4)
t = 77
PONG▶(4)
t = 198
PONG◀(5)
t = 319
PONG▶(9)
t = 683
exponential time
forward prediction

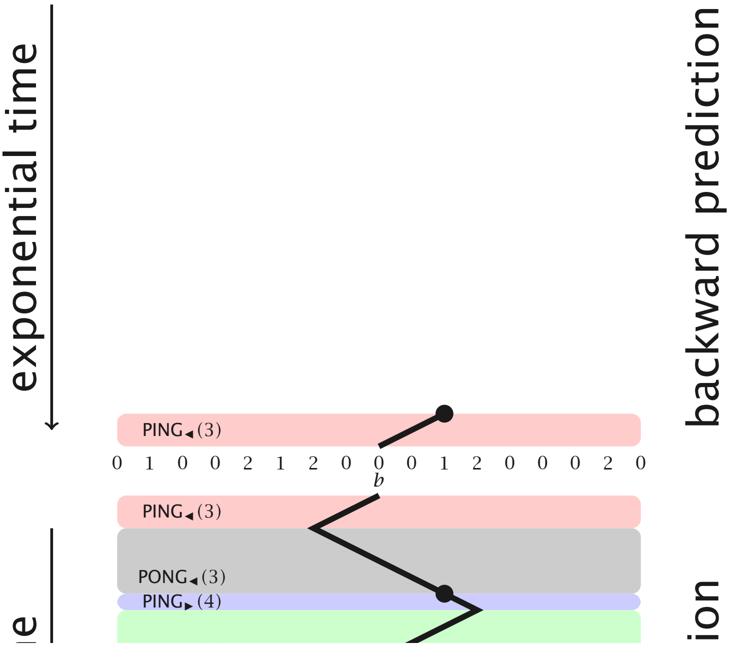


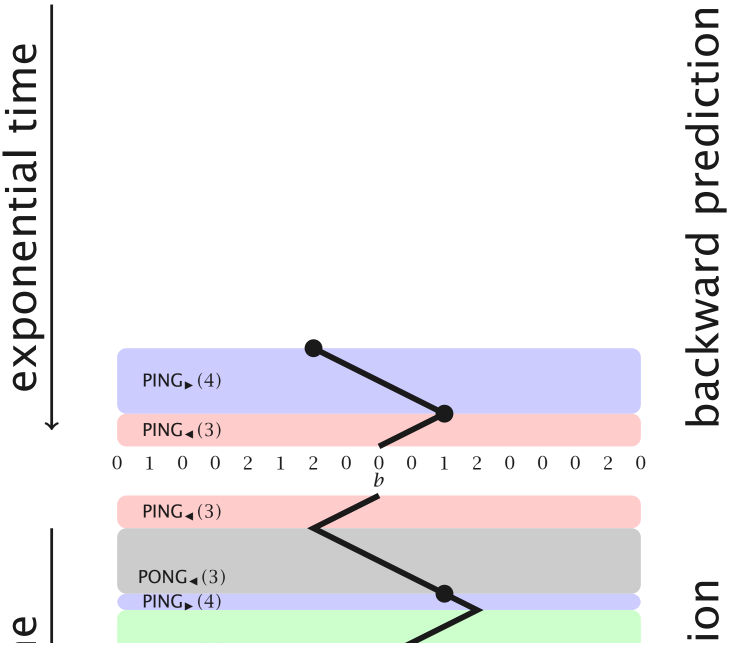
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
t = 0
PING◂(3)
t = 36
PONG◂(3)
t = 76
PING▸(4)
t = 77
PONG▸(4)
t = 198
PONG◂(5)
t = 319
PONG▸(9)
t = 683
PING◂(10)
t = 684
PONG◂(10)
t = 89257
exponential time
forward prediction



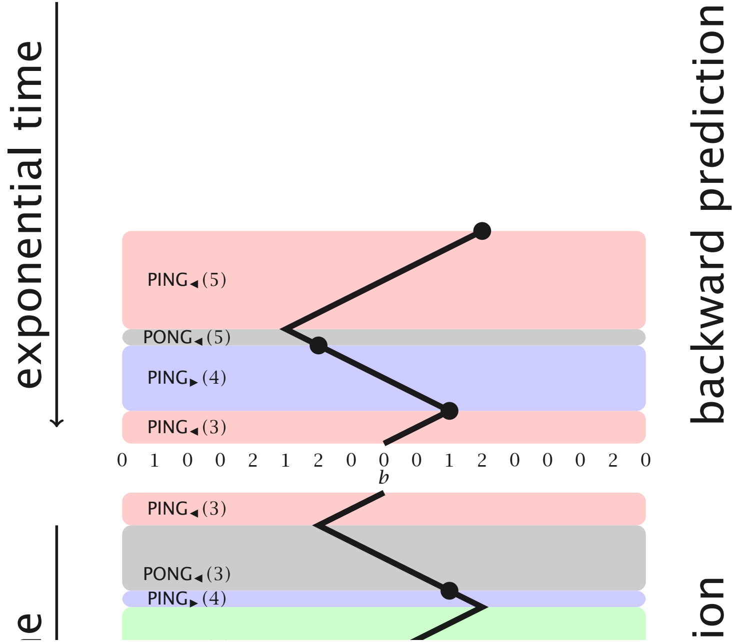
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
t = 0
PING◂(3)
t = 36
PONG◂(3)
t = 76
PING▸(4)
t = 77
PONG▸(4)
t = 198
PONG◂(5)
t = 319
PONG▸(9)
t = 683
PING◂(10)
t = 684
PONG◂(10)
t = 89257
0 1 0 0 2 0 0 0 0 0 0 0 0 0 0 0 0
q
exponential time
forward prediction

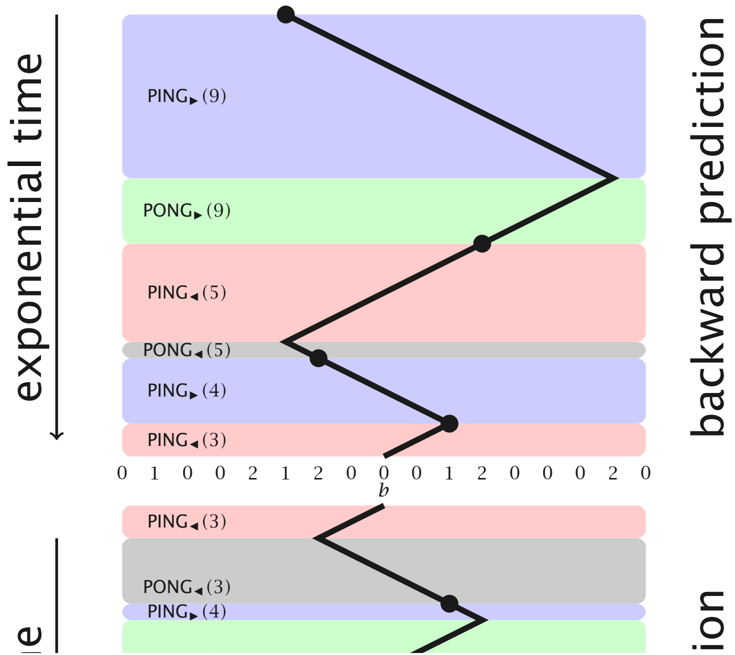

exponential time
backward prediction
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◂(3)
PONG◂(3)
PING▸(4)


exponential time
backward prediction
PING◂(3)
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◂(3)
PONG◂(3)
PING▸(4)

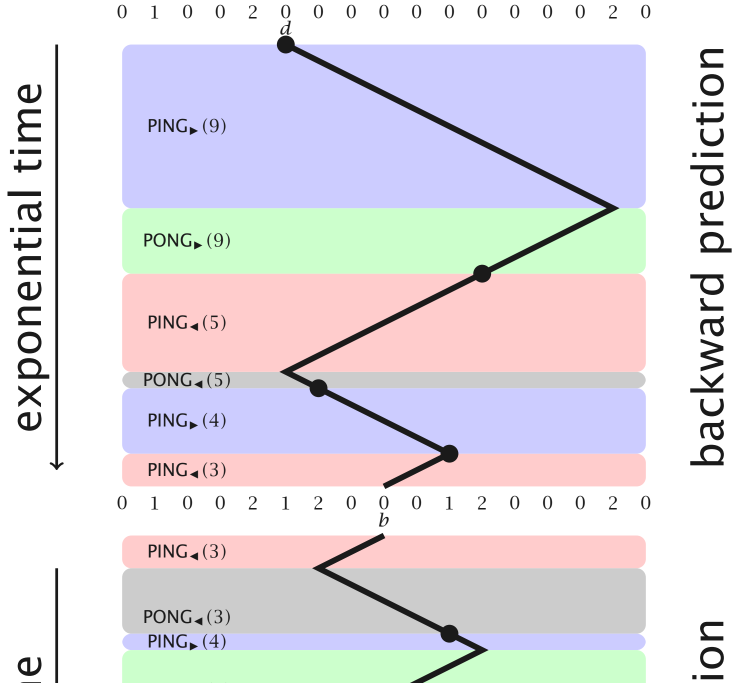


exponential time
backward prediction
PING▸(4)
PING◂(3)
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◂(3)
PONG◂(3)
PING▸(4)


exponential time
backward prediction
PING◂(5)
PONG◂(5)
PING▸(4)
PING◂(3)
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◂(3)
PONG◂(3)
PING▸(4)



exponential time
backward prediction
PING▸ (9)
PONG▸ (9)
PING◂ (5)
PONG◂ (5)
PING▸ (4)
PING◂ (3)
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◂ (3)
PONG◂ (3)
PING▸ (4)
ion



0 1 0 0 2 0 0 0 0 0 0 0 0 0 0 2 0
d
PING▸ (9)
PONG▸ (9)
PING◂ (5)
PONG◂ (5)
PING▸ (4)
PING◂ (3)
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◂ (3)
PONG◂ (3)
PING▸ (4)
exponential time
backward prediction

# SMART is (TMH-)transitive

**Proposition** $\left( {}^{\omega}2 \cdot \underset{p}{2}\ 2^{\omega} \right)$ is a **transitive point**.

**Proof**

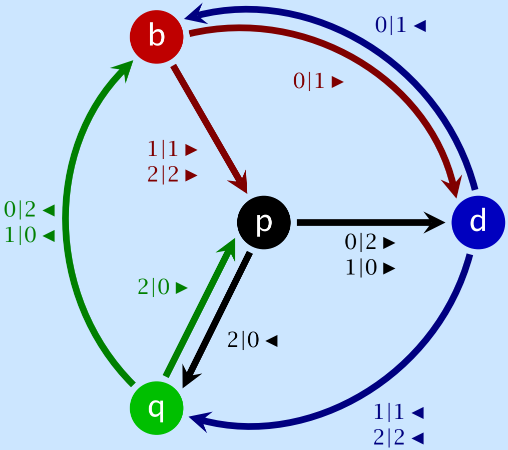
*(Forward)* For all $k \geqslant 0$:

$$\left( {}^{\omega}2 \cdot \underset{p}{2}\ 2^{\omega} \right) \vdash^{*} \left( {}^{\omega}2\ \underset{q}{2}\ 0^{k} \cdot 0\ 0^{k}\ 2^{\omega} \right) \quad .$$

*(Backward)* For every partial configuration $\left( \underset{\leftarrow}{u}\ \underset{\alpha}{\cdot}\ \underset{\rightarrow}{v} \right)$, there exist $w, w' \in \{0,1,2\}^{*}$ and $k > 0$ big enough such that

$$\left( {}^{\omega}2\ \underset{q}{2}\ 0^{k} \cdot 0\ 0^{k}\ 2^{\omega} \right) \vdash^{*} \left( {}^{\omega}2\ w\ \underset{\leftarrow}{u}\ \underset{\alpha}{\cdot}\ \underset{\rightarrow}{v}\ w'\ 2^{\omega} \right) \quad .$$

■




# 3. The embedding technique

# Reversing time

Combine Turing machines to construct bigger ones.

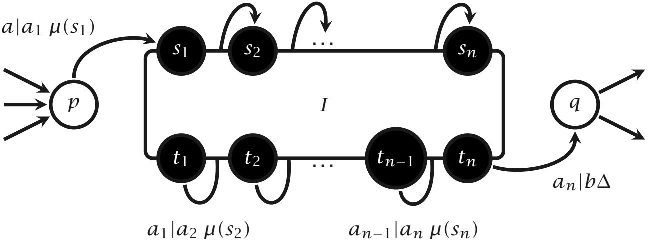
**Reversing the time** Given a reversible TM $M = (Q, \Sigma, \delta)$, construct $M_+ = (Q \times \{+\}, \Sigma, \delta^+)$ and $M_- = (Q \times \{-\}, \Sigma, \delta^-)$ where $(s, +)$ encodes $M$ in state $s$ running **forward** and $(s, -)$ running **backward**.

A typical use connects halting pairs from one machine to the corresponding starting pair of the other.

# Embedding technique

A TM $I$ with starting pairs $(s_1, a_1), \ldots, (s_n, a_n)$ and halting pairs $(t_1, a_1), \ldots, (t_n, a_n)$ is **innocuous** if starting from $(s_i, c, p + \mu(s_i))$ where $c(p) = a_i$ the machine might only halt in $(t_i, c, p)$.

The **embedding** $H^I$ of an **invited** innocuous TM $I$ inside a **host** TM $H$ is the TM containing a copy of both $I$ and $H$ where one transition $\delta(p, a) = (q, b, \Delta)$ from $H$ is replaced by



# Undecidability of transitivity

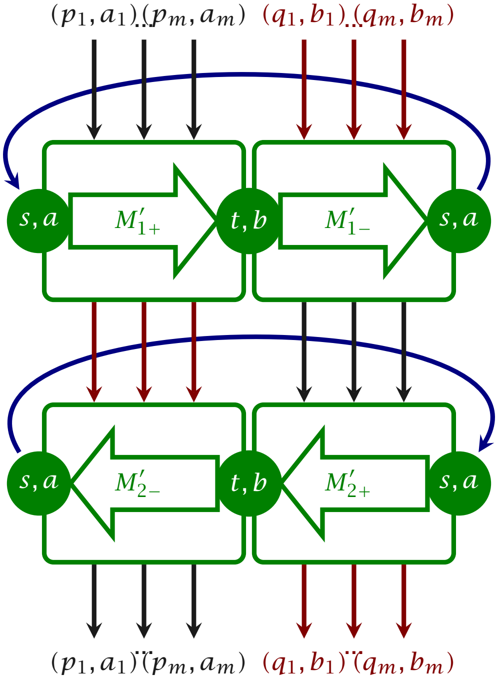
**BRA Reachability Problem[$\Sigma_1^0$-comp. too]** Given a binary reversible aperiodic TM, a starting pair $(s, a)$ and a halting pair $(t, b)$, decide if $(t, b)$ is reachable from $(s, a)$.

**Theorem** $\overline{\text{BRA Reachability Problem}} \leqslant_m$ Transitivity Problem

**Proof**
Let $M$, $(s, a)$, $(t, b)$ be an instance of the BRA Reachability Problem and $M'$ be a copy of $M$ with a third symbol $\$$.

Apply *Reversing time* to 2 copies of $M'$ to construct an innocuous TM $I$ as follows.

SMART$^I$ is transitive iff $(t, b)$ is not reachable from $(s, a)$. ■


$(p_1, a_1) \ldots (p_m, a_m)$
$(q_1, b_1) \ldots (q_m, b_m)$
$s, a$
$M'_{1+}$
$t, b$
$M'_{1-}$
$s, a$
$s, a$
$M'_{2-}$
$t, b$
$M'_{2+}$
$s, a$
$(p_1, a_1) \ldots (p_m, a_m)$
$(q_1, b_1) \ldots (q_m, b_m)$

# Conclusion

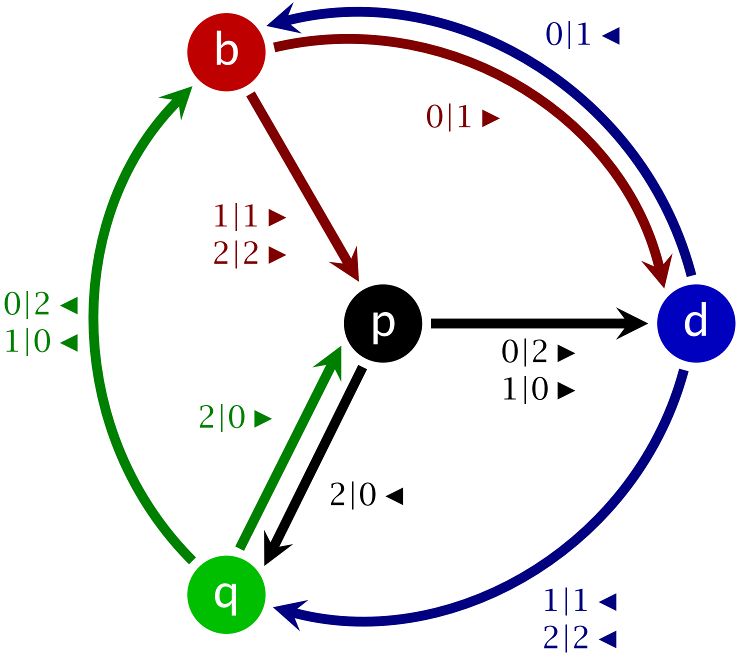
The embedding technique can be used to prove several undecidability results on TM.

**Theorem** **Transitivity** is $\Pi^0_1$-hard in TMH, TMT and ST.

**Theorem** **Minimality** is $\Sigma^0_1$-hard in TMT and ST.

What is the exact complexity of both these properties?

Is there some kind of Rice theorem for dynamical properties?



b
p
d
q
0|1 ◀
0|1 ▶
1|1 ▶
2|2 ▶
0|2 ◀
1|0 ◀
0|2 ▶
1|0 ▶
2|0 ▶
2|0 ◀
1|1 ◀
2|2 ◀
𝔈

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