Treewidth of planar graphs: connections with duality

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1 Preliminaries

A graph is said to be *chordal* if each cycle with at least four vertices has a chord, that is an edge between two non-consecutive vertices of the cycle. Given an arbitrary graph $G = (V, E)$, a *triangulation* of G is a chordal graph $H = (V, F)$ such that $E \subseteq F$. We say that H is a *minimal triangulation* of G if no proper subgraph of H is a triangulation of G. The **treewidth** $\mathrm{tw}(H)$ of a chordal graph is its maximum cliquesize minus one. The treewidth of an arbitrary graph G is the minimum, over all triangulations H of G, of $\mathrm{tw}(H)$. When computing the treewidth of G we can clearly restrict to minimal triangulations. Treewidth was introduced by Robertson and Seymour in connection with graph minors [5], but it has wide algorithmic applications since many NP-hard problems become polynomial when restricted to graphs of bounded treewidth.

Robertson and Seymour conjectured in [5] that the treewidth of a planar graph G and its dual G^* differ by at most one. This conjecture was recently proved by Lapoire [3], who actually gives a more general result, on hypergraphs of bounded genus. Nevertheless, the proof of Lapoire is rather long and technical. Here we show that any minimal triangulation H of a planar graph G can be easily transformed into a triangulation H^* of G^* such that $\mathrm{tw}(H^*) \leq \mathrm{tw}(H) + 1$.

The *minimal separators* play a crucial role in the characterization of the minimal triangulations of a graph. A subset $S \subseteq V$ separates two nonadjacent vertices $a, b \in V$ if a and b are in different connected components of $G \backslash S$. S is a *minimal a, b-separator* if it separates a and b and no proper subset of S separates a and b. We say that S is a *minimal separator* of G if there are two vertices a and b such that S is a minimal a, b-separator. Notice that a minimal separator can be strictly included into another. We denote by Δ_G the set of all minimal separators of G. Two minimal separators S and T *cross* if T intersects at least two components of $G \backslash S$. Otherwise, S and T are *parallel*. Both relations are symmetric.
Let $S \in \Delta_G$ be a minimal separator. We denote by G_S the graph obtained from G by *completing* S, i.e. by adding an edge between every pair of non-adjacent vertices of S. If $\Gamma \subseteq \Delta_G$ is a set of separators of G, G_Γ is the graph obtained by completing all the separators of Γ. The results of [2], concluded in [4], establish a strong relation between the minimal triangulations of a graph and its minimal separators.

Theorem 1 *H is a minimal triangulation of G if and only if there is a maximal set of pairwise parallel separators $\Gamma \subseteq \Delta_G$ such that $H = G_\Gamma$.*

Since it is easy to extend our results to simply connected or disconnected graphs, we will restrict ourselves to 2-connected graphs.

2 Minimal separators in planar graphs

Consider a 2-connected planar graph $G = (V, E)$. We fix an embedding of G in the plane $\mathbb{R}^2$. Let F be the set of faces of this embedding. The *intermediate graph* G_I has vertex set $V \cup F$. We place an edge in G_I between an original vertex $v \in V$ and a face-vertex $f \in F$ whenever the corresponding vertex and face are incident in G. Notice that $(G^*)_I = G_I$.

Let ν be a cycle of G_I (by "cycle" we will always mean a cycle which does not get through a same vertex twice). The drawing of ν forms a Jordan curve in the plane $\mathbb{R}^2$, denoted $\tilde{\nu}$. It is easy to see that if $\tilde{\nu}$ separates two original vertices x and y in the plane (i.e. x and y are in different regions of $\mathbb{R}^2 \backslash \tilde{\nu}$), then $\nu \cap V$ separates x and y in G. Therefore, the original vertices of ν form a separator in G. Conversely, to each minimal separator S of G we can associate a cycle ν of G_I (see [1]).

Proposition 1 *Let S be a minimal separator of the planar graph G. Consider two connected components C and D of $G \backslash S$. There is a cycle ν_S of G_I such that $\tilde{\nu}_S$ separates C and D in the plane.*

This cycle is usually not unique. In the case of 3-connected planar graphs, notice that if S is a minimal separator, then $G \backslash S$ has exactly two connected components C and D. For each couple of original vertices x and y incident to a same face, fix a unique face $f(x, y)$ containing both x and y. We say that a cycle ν of G_I is *well-formed* if, for any two consecutive original vertices $x, y \in \nu$, the face-vertex between them is $f(x, y)$. If G is a 3-connected planar graph, for any minimal separator S there is a unique well-formed cycle of G_I separating C and D in the plane.

In what follows, G denotes a 3-connected planar graph. However, our main results can be easily extended to arbitrary planar graphs.

We say that two Jordan curves $\tilde{\nu}_1$ and $\tilde{\nu}_2$ *cross* if $\tilde{\nu}_1$ intersects the two regions defined by $\tilde{\nu}_2$ in the plane. Otherwise, they are *parallel*. Two cycles ν_1 and ν_2 of G_I *cross* if and only if $\tilde{\nu}_1$ and $\tilde{\nu}_2$ cross. Notice that the parallel and crossing relations between curves and cycles are symmetric.

Proposition 2 *Two minimal separators S and T are parallel of G if and only if the corresponding cycles ν_S and ν_T of G_I are parallel.*

Let $\tilde{\nu}$ be a Jordan curve in the plane. Let R be one of the regions of $\mathbb{R}^2 \backslash \tilde{\nu}$. We say that $(\tilde{\nu}, R) = \tilde{\nu} \cup R$ is a one-block region of the plane, bordered by $\tilde{\nu}$. Let $\tilde{\mathcal{C}}$ be a set of curves such that for each $\tilde{\nu} \in \tilde{\mathcal{C}}$, there is a one-block region $(\tilde{\nu}, R(\tilde{\nu}))$ containing all the curves of $\tilde{\mathcal{C}}$. We define the *region between* the elements of $\mathcal{C}$ as $RB(\mathcal{C}) = \bigcap_{\tilde{\nu} \in \tilde{\mathcal{C}}} (\tilde{\nu}, R(\tilde{\nu}))$. We say that the region between the curves of $\mathcal{C}$ is *bordered* by $\mathcal{C}$. A subset $BR \subseteq \mathbb{R}^2$ of the plane is a *block region* if $BR = \mathbb{R}^2$, BR is a one-block region $(\tilde{\nu}, R)$ or BR is the region between some set of curves $\tilde{\mathcal{C}}$.

3 Minimal triangulations of G and G^*

Let G be a 3-connected planar graph and let H be a minimal triangulation of G. According to theorem 1, there is a maximal set of pairwise parallel separators $\Gamma \subseteq \Delta_G$ such that $H = G_\Gamma$. Let $\mathcal{C}(\Gamma) = \{\nu_S | S \in \Gamma\}$ be the cycles of G_I associated to the minimal separators of Γ and let $\tilde{\mathcal{C}}(\Gamma) = \{\tilde{\nu}_S | S \in \Gamma\}$ be the curves associated to these cycles. According to proposition 2, the cycles of $\mathcal{C}(\Gamma)$ are pairwise parallel. Thus, the curves of $\tilde{\mathcal{C}}(\Gamma)$ split the plane into block regions. Consider the set of all the block regions bordered by some elements of $\tilde{\mathcal{C}}$. The inclusion-minimal elements of this set are called minimal block regions defined by $\tilde{\mathcal{C}}$. We show that any maximal clique Ω of H corresponds to the original vertices contained in a minimal block region defined by $\tilde{\mathcal{C}}(\Gamma)$.

Theorem 2 *Let G be a 3-connected planar graph and let $H = G_\Gamma$ be a minimal triangulation of G. $\Omega \subseteq V$ is a maximal clique of H if and only if there is a minimal block region BR defined by $\tilde{\mathcal{C}}(\Gamma)$ such that $\Omega = BR \cap V$.*

Let now $\mathcal{C}$ be an *arbitrary* set of pairwise parallel cycles of G_I. This family $\tilde{\mathcal{C}}$ of curves associated to these cycles splits the plane into block regions. Let G^* be the dual of G. The graph $H^*(\mathcal{C}) = (F, E_H)$ has vertex set F. We place an edge between two face-vertices f and f' of H if and only if f and f' are in same a minimal block region defined by $\tilde{\mathcal{C}}$. Equivalently, f and f' are non-adjacent in $H^*(\mathcal{C})$ if and only if there is a $\tilde{\nu} \in \tilde{\mathcal{C}}$ separating f and f' in the plane.

Theorem 3 *$H^*(\mathcal{C})$ is a triangulation of G^*. Moreover, any clique Ω^* of H^* is contained in some minimal block region BR defined by $\tilde{\mathcal{C}}$.*

Let $H = G_\Gamma$ be a minimal triangulation of G. Consider the cycles $\mathcal{C}(\Gamma)$ associated to the minimal separators of Γ and the corresponding curves $\tilde{\mathcal{C}}(\Gamma)$. We could try to considerate the triangulation $H^*(\mathcal{C}(\Gamma))$ of G^*, but unfortunately it does not satisfy $\mathrm{tw}(H^*) \leq \mathrm{tw}(H) + 1$.

Thus, we consider a maximal set of pairwise parallel cycles $\mathcal{C}'$ of G_I such that $\mathcal{C}(\Gamma) \subseteq \mathcal{C}'$. Clearly, each minimal block region defined by $\tilde{\mathcal{C}}'$ is contained in a minimal block region defined $\tilde{\mathcal{C}}(\Gamma)$.

Theorem 4 *Let $\mathcal{C}'$ be a maximal set of pairwise parallel cycles of G_I. Let BR be a minimal block region defined by $\tilde{\mathcal{C}}'$. Then $BR \cap G_I$ is either formed by a cycle $\tilde{\nu}$ and a path $\tilde{\mu}$ joining two vertices of $\tilde{\nu}$ or BR is a one-block region $(\tilde{\nu}, R)$ and $BR \cap G_I = \nu$ where ν is the cycle of G_I associated to $\tilde{\nu}$. In particular, $|BR \cap V^*| \leq |BR \cap V| + 1$.*

According to theorem 3, each maximal clique Ω^* of H^* is contained in some minimal block region BR, and by the previous theorem it has at most one more vertex than $\Omega = BR \cap V$. By theorem 2, Ω is a clique of H. Hence, $|\Omega^*| \leq |\Omega| + 1$ and thus $\mathrm{tw}(H^*) \leq \mathrm{tw}(H) + 1$. By considering an optimal triangulation H of G, we obtain a triangulation H^* of G^* of width at most $\mathrm{tw}(G) + 1$. We conclude that $\mathrm{tw}(G^*) \leq \mathrm{tw}(G) + 1$.

So we can state:

Theorem 5 (Main theorem) *Let $G = (V, E)$ be a planar graph.*

$$|\,\mathrm{tw}(G) - \mathrm{tw}(G^*)| \leq 1.$$

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