On treewidth approximations

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1 Introduction

A graph is said to be *chordal* if each cycle with at least four vertices has a chord, that is an edge between two non-consecutive vertices of the cycle. Given an arbitrary graph $G = (V, E)$, a *triangulation* of G is a chordal graph $H = (V, F)$ such that $E \subseteq F$. We say that H is a *minimal triangulation* of G if no proper subgraph of H is a triangulation of G. The **treewidth** $\mathrm{tw}(H)$ of a chordal graph is its maximum cliquesize minus one. The treewidth of an arbitrary graph G is the minimum, over all triangulations H of G, of $\mathrm{tw}(H)$. When computing the treewidth of G we can clearly restrict to minimal triangulations. Many NP-hard problems become polynomial when restricted to graphs of bounded treewidth.
An *asteroidal set* of a graph G is a set of pairwise non adjacent vertices A such that for each $x \in A$, there is one connected component of $G \backslash N(x)$ containing all the vertices of $A \backslash \{x\}$. The asteroidal number $\mathrm{an}(G)$ of the graph G is the maximal cardinality of an asteroidal set of G.

We give in this paper some approximation results on the treewidth of graphs. For graphs of bounded asteroidal number, we obtain a constant factor approximation. For arbitrary graphs, we give an approximation which is no more than $\mathcal{O}(\log k)$ times the optimal value, where k is the treewidth of the graph.

2 The minimum cardinality separator strategy

The *minimal separators* play a crucial role in the characterization of the minimal triangulations of a graph. A subset $S \subseteq V$ separates two nonadjacent vertices $a, b \in V$ if a and b are in different connected components of $G \backslash S$. S is a *minimal a, b-separator* if it separates a and b and no proper subset of S separates a and b. We say that S is a *minimal separator* of G if there are two vertices a and b such that S is a minimal a, b-separator. Notice that a minimal separator can be strictly included into another. We denote by Δ_G the set of all minimal separators of G. Two minimal separators S and T *cross* if T intersects at least two components of $G \backslash S$. Otherwise, S and T are *parallel*. Both relations are symmetric.

Let $S \in \Delta_G$ be a minimal separator. We denote by G_S the graph obtained from G by *completing* S, i.e. by adding an edge between every pair of non-adjacent vertices of S. If $\Gamma \subseteq \Delta_G$ is a set of separators of G, G_Γ is the graph obtained by completing all the separators of Γ. The results of [5], concluded in [6], establish a strong relation between the minimal triangulations of a graph and its minimal separators.

Theorem 1 *H is a minimal triangulation of G if and only if there is a maximal set of pairwise parallel separators $\Gamma \subseteq \Delta_G$ such that $H = G_\Gamma$.*

Using this theorem and some other results of [6] we obtain an algorithm for computing minimal triangulations of graphs.

$H \leftarrow G$
while there is a $S \in \Delta_H$ such that $H[S]$ is not a clique
  $H \leftarrow H_S$
return H

Our aim is to obtain a triangulation H of minimum cliquesize. Since the minimal separators S chosen in the **while** loop become cliques, a natural idea is to choose a minimal separator S of minimum cardinality. Finding a minimum cardinality separator of a graph can be done in polynomial time by standard flow techniques [1]. This heuristic can be implemented to have polynomial running time in the following way.

MCSep
  $lp \leftarrow \{G\}$ /∗ lp is the list of pieces ∗/
  while at least one element P of lp is not a clique **do**
    $S \leftarrow$ a minimum cardinality separator of P
    compute the connected components $C_1, \ldots C_p$ of $P \backslash S$
    in lp, replace P by the smaller pieces $P_S[S \cup C_i]$, $1 \leq i \leq p$
    /∗ N.B. we have completed S in each of the pieces $P_S[S \cup C_i]$ ∗/
  end_while
  return $H = \bigcup_{P \in lp} P$

We prove that the heuristic does not give a constant factor approximation for the treewidth of arbitrary graphs. Nevertheless, this holds for special classes of graphs. Let us characterize the pieces obtained during the algorithm.
The following definitions are strongly related with the *blocking sets* and the *blocks* introduced in [3]. Let S be a minimal separator of G and C be a connected component of $G \backslash S$. The set of vertices $S \cup C$ is called a one-block, denoted (S, C).

Definition 2 *Let G be a graph and $\mathcal{S} \subseteq \Delta_G$ a set of pairwise parallel separators such that for any $S \in \mathcal{S}$, there is a one-block $(S, C(S))$ containing all the elements of $\mathcal{S}$. Suppose that $\mathcal{S}$, ordered by inclusion, has no greatest element. We define the* piece between *the elements of $\mathcal{S}$ by $P(\mathcal{S}) = \bigcap_{S \in \mathcal{S}} (S, C(S))$.*

Definition 3 *Let B be a set of vertices of a graph G. We denote by $C_1, \ldots, C_p$ the connected components of $G \backslash B$ and by S_i the neighborhood of C_i. We say that B is a* block *of G if the sets S_i are minimal separators of G and $B = G$, there is an $i \in [1, p]$ such that B is a one-block (S_i, C) or $B = P(S_1, \ldots, S_p)$. If B is a block, we say that the minimal separators $S_1, \ldots, S_p$* border *B.*

The *realization* $R(B)$ of a block B is the graph $G[B]_{S_1,\ldots,S_p}$ induced by B and where each minimal separator bordering B is completed into a clique.

Theorem 4 ([5]) *Every piece obtained during the execution of the* **MCSep** *algorithm is the realization of some block of G.*

3 Graphs of bounded asteroidal number

Let B be a block of the graph G and $\mathcal{S}$ be the set of minimal separators bordering B. The *blocking set* of B is formed by the inclusion-maximal elements of $\mathcal{S}$. The cardinality of the blocking set of B is at most $\mathrm{an}(G)$[3]. We prove:

Theorem 5 *Let G be a graph of asteroidal number at most a and let k be the treewidth of G. Consider a block B of G such that all the minimal separators bordering B are of size at most $4k$. If $|B| > 8ak$, there is a minimal separator $S \subseteq B$ of G such that $|S| \leq 4k$. Moreover S is a separator of $R(B)$.*

The idea of the proof is that at most $4ak$ vertices of B are on the *border* of B, that is on the minimal separators bordering B. So more than half of the vertices are on the *interior* $Int(B)$ of B. All the vertices of $Int(B)$ cannot have more than $4k$ neighbors in B (this would imply that $G[B]$ has more than $k|B|$ edges, contradicting the fact that $G[B]$ is of treewidth at most k). Thus, there is some $x \in Int(B)$ whose neighborhood in B has $|N_B(x)| \leq 4k$ vertices. We will find then a separator $S \subseteq N_B(x)$, separating two vertices of $R(B)$.

We conclude that the algorithm **MCSep** applied to the graph G will split the graph G into pieces of size at most $8ak$. Given an arbitrary graph G, the **MCSep** strategy outputs a triangulation of width most $8\,\mathrm{an}(G) \times \mathrm{tw}(G)$. If we fix the parameter a, we can modify the algorithm in order to output "$\mathrm{an}(G) > a$" or a triangulation of width at most $8a$ times the optimal value.

4 Treewidth approximations for arbitrary graphs

Let G be a graph and a set of vertices $W \subseteq V$. A *ρ-separator* of W is a subset of vertices $S \subseteq V$ such that any connected component of $G \backslash S$ contains at most $\rho|W|$ vertices of W. Any set of vertices W has a $1/2$ separator of size $\mathrm{tw}(G)$ [2]. Even et al. proved the following result:

Theorem 6 ([4]) *Let W be a set of vertices of G. Let τ be the minimum size of a ρ-separator of W, $1/2 \leq \rho < 1$. Given ρ', $\rho < \rho' \leq 1$, there is a polynomial algorithm computing a ρ'-separator of W of size $\frac{4.04\rho}{\rho'-\rho}(5 + \ln(\frac{\rho}{\rho'-\rho}\tau))\tau$.*

In their approximation algorithm for the treewidth of arbitrary graphs Bodlaender et al. [2] used a similar result, in which $\rho' = 2/3$ and the $\ln \tau$ factor is replaced by a $\log n$ factor. They obtain an $\mathcal{O}(\log n)$ approximation for the treewidth problem.

By using theorem 6, we produce a triangulation of G of width $\mathcal{O}(k \ln k)$, where k is the treewidth of G.

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