Connected Graph Searching in Outerplanar Graphs

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1 Search games, connected search and planar graphs

Search games are a powerfull tool for studying various connectivity parameters of graphs. In the classical search game, we consider an undirected graph $G = (V, E)$ whose edges are initially *contaminated*. A set of searchers try to *clean* the graph. At the beginning the graph contains no searchers. At each step of the game, a searcher can be placed on an arbitrary vertex of the graph or, if the searcher is already on a vertex v it can slide through an edge e incident to v. In the former case the edge e is *cleaned* by the searcher. If, for some clean edge e, there is a path from e to a contaminated edge such that no searcher separates the two edges on the path, then e becomes *recontaminated*. The *search number* $\mathrm{s}(G)$ of G is the minimum number of searchers required to clean all the edges of G.

The search number differs by at most one from another well-known graph parameter, namely the pathwidth. The treewidth, the branchwidth and several parameters of the same flavour can be defined by versions of the search game.

In this paper we consider a variant of the search game introduced by Barrière et al. [1], called *connected search*. It requires that, at each step of the search game, the set of clean edges induces a connected subgraph of G. The minimum number of searchers that can clean G by a connected search is denoted by $\mathrm{cs}(G)$.

A *plane graph* is a particular drawing of a planar graph in the plane without crossings. An *outerplane graph* is a planar embedding of an outerplanar graph with every vertex on the exterior face. If G is a plane graph then G^* denotes its geometric dual. The *weak dual* T^* of an outerplane graph G is the graph obtained from the dual G^* by deleting the vertex corresponding to the exterior face of G. It is not hard to show that if G is connected, T^* is a tree.

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The search parameters of a plane graph G are often close to the ones of its dual. If G is 2-connected, the branchwidth of the two graphs is equal [6] and their treewidth differs by at most one unit [5, 4].

Bodlaender and Fomin [3] proved that the pathwidth of a 2-connected outerplane graph and the pathwidth of its weak dual differ by at most a factor of 2 and deduce an efficient 2-approximation algorithm for the pathwidth of outerplanar graphs. In this paper we prove that, for any outerplane graph G, $\mathrm{cs}(T^*)/2 \leq \mathrm{cs}(G) \leq 2\,\mathrm{cs}(T^*) + 1$ and provide a 4-approximation algorithm for the pathwidth of outerplanar graphs.

2 Medial graphs

For $X \subseteq E(G)$ let $\delta_G(X)$ be the set of all vertices incident to edges in X and $E(G) \setminus X$. For $Y \subseteq V(G)$ let $\partial_G(Y)$ be the set of all edges with an end in Y and an end in $V(G) \setminus Y$.

Take a drawing of a graph G in a sphere. Let M_G be a graph with vertex set $E(G)$ and let C_v ($v \in V(G)$) be circuits of M_G, with the following properties:

- The circuits C_v ($v \in V(G)$) are mutually edge-disjoint and have union M_G;
- For each $v \in V(G)$ let the adjacent to v edges $\{v, x_1\}, \{v, x_2\}, \ldots, \{v, x_t\}$ be enumerated according to the cyclic order in the drawing of G; then C_v has vertex set $\{v, x_1\}, \{v, x_2\}, \ldots,$ $\{v, x_t\}$ and in C_v vertex $\{v, x_i\}$ is adjacent to $\{v, x_{i+1}\}$ ($1 \leq i \leq t$), where $x_0 = x_t$.

M_G is called a *medial graph* of G. Notice that if G is 2-connected then M_G is isomorphic to M_{G^*}.

Lemma 1. *Let M_G be the medial graph of a plane graph G. Then for every $X \subseteq E(G)$*

$$|\delta_G(X)| \leq |\partial_{M_G}(X)|/2.$$

3 The results

Theorem 1. *Let G be a 2-connected outerplane graph and T^* be its weak dual. Then* $\mathrm{cs}(G) \leq 2\,\mathrm{cs}(T^*) + 1$.

Let k be the connected search of T^*. Barrière et al. [2] characterized the trees of connected search number k as being exactly the *k-caterpillars*. For two distinct vertices u, v of T, let $T_v[u]$ denote the subtree of T rooted in u and containing v. In a k-caterpillar we distinguish a path called the *spine* of the caterpillar. A 1-caterpillar is formed only by its spine. For any $k > 1$, a k-caterpillar T is such that for any vertex u on the spine P and for any neighbour v of u not on the spine, $T_v[u]$ is a k'-caterpillar, with $k' \leq k - 1$, and u is an endpoint of the spine of $T_v[u]$.

Based on this observation, we have to construct a connected search strategy on G using $2k + 1$ searchers. An expansion in an arbitrary graph H is a sequence $(X_0, X_2, \ldots, X_p)$ of subsets of $E(H)$ such that $X_0 = \emptyset$, $X_p = E(H)$ and, for each $1 \leq i \leq p$, $|X_i \setminus X_{i-1}| = 1$. The expansion corresponds to the sequence of cleaned edges of G.

Lemma 2. *Let v_{out} denote the vertex of G^* corresponding to the outerface. There exists an expansion $(X_0^*, X_2^*, \ldots, X_p^*)$ of G^* such that*

1. *For each $1 \leq i \leq p$, $|\delta_{G^*}(X_i^*)| \leq k$.*
2. *For each $1 \leq i \leq p$ and $u \in V(G^*) \setminus \{v_{\text{out}}\}$, $V(C_u) \cap X_i$ (as vertex subset of M_{G^*}) induces a connected subgraph in M_{G^*}.*
3. *For each $1 \leq i \leq p$, the subgraph of M_{G^*} induced by the vertex set $V(C_{v_{\text{out}}}) \cap X_i$ has at most k connected components.*
4. *For every edges $\{u, v_{\text{out}}\} \in X_j \backslash X_{j-1}$ and $\{u, w\} \in X_i \backslash X_{i-1}$, $w \neq v_{\text{out}}$ implies $i < j$, i.e. all edges $\{u, w\}$ adjacent to an edge $\{u, v_{\text{out}}\}$ with endpoint v_{out} appear before $\{u, v_{\text{out}}\}$ in the sequence.*

Proof. (Hints). We construct a monoton expansion, by adding a new edge at each step. Let $P = (u_1, u_2, \ldots, u_q)$ be the spine of T^*. The spine can be chosen such that u_1 (resp. u_q) has no other neighbours in T^* but u_2 (resp. u_{q-1}).

For each i, $2 \leq i \leq q-1$, we order the neighbours of u_i according to the clockwise cyclic order of the drawing of T^*. Let $v_1, v_2, \ldots v_r$ be this order with $u_{i-1} = v_1$ and $u_{i+1} = v_j$.

The edges of the spine will be added to the expansion respecting their order in P. Let $X_1^* = \{\{u_1, u_2\}\}$, $X_2^* = \{\{u_1, u_2\}, \{u_1, v_{\text{out}}\}\}$ (if $\{u_1, v_{\text{out}}\}$ is a multiple edge we add it and its copies consecutively; we also do the same for all the multiple edges incident to v_{out} that may appear). After adding to the expansion the edge $\{u_{i-1}u_i\}$, we clean recursively the subtrees $T^*_{v_s}[u_i]$ in the order $s = 2, 3, \ldots, j-1, r, r-1, \ldots, j+1$. Eventually, we clean the edge $\{u_i u_{i+1}\}$ and, if it exists, the edge $\{u_i v_{\text{out}}\}$. The subtree $T^*_{v_s}[u_i]$ is cleaned according to the same rules as T^*, starting with its spine rooted in u_i (recall that $T^*_{v_s}[u_i]$ is a k'-caterpillar for some $k' \leq k-1$).

Property 1 is due to the fact that T is a k-caterpillar. The expansion clearly satisfies property 4 of the lemma. Property 2 is due to our ordering in the cleaning of the subtrees $T^*_{v_s}[u_i]$. Property 3 also comes from this ordering and the fact that T^* is a k-caterpillar. □

Proof. (of Theorem 1). We show that, for any X_i^* of the expansion described in Lemma 2, $|\partial_{M_{G^*}}(X_i^*)| \leq 4k$. For each $u \in V(T^*)$ on the border $E(C_u) \cap \partial_{M_{G^*}}(X_i^*)$ is empty if $u \notin \delta_{G^*}(X_i^*)$, and by property 2 it contains at most two vertices if $u \in \delta_{G^*}(X_i^*)$. By property 1, there are at most k vertices u of the second type. By property 3, $E(C_{v_{\text{out}}}) \cap \partial_{M_{G^*}}(X_i^*)$ has at most $2k$ vertices. Altogether, we have $|\partial_{M_{G^*}}(X_i^*)| \leq 4k$.

Since G is 2-connected the medial graphs M_G and M_{G^*} are isomorphic (this isomorphism maps each edge to its dual). Consequently $\partial_{M_G}(X_i) = \partial_{M_{G^*}}(X_i^*)$, where $X_i \subseteq E(G)$ is the set of edges dual to X_i^*. By Lemma 1, $\delta_G(X_i) \leq 2k$.

Properties 2 and 4 imply that all the sets X_i, $1 \le i \le p$, form a connected subgraph of G. Now we can easily construct a connected search strategy cleaning G with at most $2k+1$ searchers. At each step we use $2k$ searchers to "guard" X_i and one aditional searcher to clear the edge of $X_{i+1} \setminus X_i$. $\square$

The next statement is the converse of Theorem 1.

Theorem 2. *For any 2-connected outerplanar graph G, $\mathrm{cs}(T^*) \le 2cs(G)$.*

Proof. For any 2-connected outerplanar graph G, we have $\mathrm{s}(T^*) \le s(G)$ (see [3]). Also, for any tree its connected search number is at most twice its search number [2]. $\square$

4 Conclusion

We proved that for any 2-connected outerplanar graph G, $\mathrm{cs}(T^*)/2 \le \mathrm{cs}(G) \le 2\,\mathrm{cs}(T^*) + 1$. Notice that the construction of the expansion described in the proof Lemma 2 can be done in $O(|V(G)|)$ steps and this provides a 4-approximation algorithm for the connected search number of 2-connected outerplanar graphs.

We leave as an open question whether the factors of Theorems 1 and 2 are tight. Actually, we think that for the latter inequality we have $\mathrm{cs}(T^*) \le cs(G)$, which would directly imply that our approximation algorithm for $\mathrm{cs}(G)$ is within a factor of 2 from the optimum.

References

[1] L. Barrière, P. Flocchini, P. Fraigniaud, and N. Santoro. Capture of an intruder by mobile agents. In *Proceedings 14th ACM Symposium on Parallel Algorithms and Architectures (SPAA)*, pages 200–209. ACM, 2002.

[2] L. Barrière, P. Fraigniaud, N. Santoro, and D. Thilikos. Searching is not jumping. In *Proceedings 29th Workshop on Graph-theoretic Concepts in Computer Science (WG 2003)*, volume 2880 of *Lecture Notes in Computer Science*, pages 34–45. Springer, 2003.

[3] H. Bodlaender and F. Fomin. Approximation of pathwidth of outerplanar graphs. *J. of Algorithms*, 43(2):190–200, 2002.

[4] V. Bouchitté, F. Mazoit, and I. Todinca. Chordal embeddings of planar graphs. *Discrete Mathematics*, 273(1-3):85–102, 2003. Special issue devoted to EuroComb'01.

[5] D. Lapoire. Treewidth and duality in planar hypergraphs. See web page: http://dept-info.labri.u-bordeaux.fr/~lapoire/papers/dual_planar_treewidth.ps.

[6] P.D. Seymour and R. Thomas. Call routing and the ratcatcher. *Combinatorica*, 14(2):217–241, 1994.