Treewidth: using the potential maximal cliques

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Abstract

We show that the treewidth is polynomially tractable for all classes of graphs with polynomial number of minimal separators.

1 Introduction

The *treewidth* of graphs, introduced by Robertson and Seymour [10], has been intensively studied in the last years, mainly because many NP-hard problems become solvable in polynomial and even linear time when restricted to graphs with small treewidth. These algorithms use a *tree-decomposition* (or *triangulation*) of small width of the graph.

A graph is *chordal* or *triangulated* if every cycle of length greater or equal to four has a chord, i.e. an edge between two non-consecutive vertices of the cycle. A triangulation of a graph is a chordal embedding, that is a supergraph, on the same vertex set, which is triangulated. The treewidth problem is to find a triangulation such that the size of the largest clique is as small as possible. In particular, we can restrict ourselves to triangulations minimal by inclusion, that we call *minimal triangulations*.

Unfortunately the computation of the treewidth of a graph is NP-hard. There exist several classes of graphs for which the treewidth problem is polynomially tractable. Among them there are the chordal bipartite graphs, circle and circular-arc graphs, AT-free graphs with polynomial number of minimal separators. Most of these algorithms use the fact that these classes of graphs have a polynomial number of *minimal separators*. It was conjectured in [6, 7] that the treewidth and the minimum fill-in should be tractable in polynomial time for all the graphs having a polynomial number of minimal separators (the minimal separators of a graph can be enumerated in polynomial time in their number [1]).

This ESA'93 conjecture was solved in [2, 3, 4]. We gather here the main results of these papers. The core of this work is the notion of *potential maximal clique*. A vertex set Ω is a potential maximal clique of a graph G if it appears as a maximal clique in some minimal triangulation of G. We give an algorithmic characterization of a potential maximal clique and we prove that the potential maximal cliques are sufficient for computing the treewidth of graphs. Finally, we show how to enumerate all the potential maximal cliques of a graph, in polynomial time in the number of its minimal separators.

2 Preliminaries

Throughout this paper we consider finite, simple, undirected and connected graphs. Let $G = (V, E)$ be a graph. We will denote by n and m the number of vertices, respectively the number of edges of G. For a vertex set $V' \subseteq V$ of G, we denote by $N_G(V')$ the neighborhood of V' in $G \backslash V'$ – so $N_G(V') \subseteq V \backslash V'$.

A subset $S \subseteq V$ is an *a, b-separator* for two nonadjacent vertices $a, b \in V$ if the removal of S from the graph separates a and b in different connected components. S is a *minimal a, b-separator* if no proper subset of S separates a and b. We say that S is a *minimal separator* of G if there are two vertices a and b such that S is a minimal a, b-separator. We denote by Δ_G the set of all minimal separators of G.

Let G be a graph and S be a minimal separator of G. We note $\mathcal{C}_G(S)$ the set of connected components of $G \backslash S$. A component $C \in \mathcal{C}_G(S)$ is a *full component associated to* S if every vertex of S is adjacent to some vertex of C, i.e. $N_G(C) = S$. If $C \in \mathcal{C}(S)$, we say that $(S, C) = S \cup C$ is a *block* associated to S. A block (S, C) is called *full* if C is a full component associated to S.

The results of [8], concluded in [9], establish a strong relation between the minimal triangulations of a graph and its minimal separators. They use the notion of *parallel* separators. Let S and T be two minimal separators of G. If S separates two vertices of T we say that S *crosses* T (denoted by $S\#T$). Otherwise, S and T are *parallel*, denoted by $S \| T$. The crossing and parallel relations are symmetric.

Consider a graph $G = (V, E)$ and a vertex set $X \subseteq V$. We denote by G_X the graph obtained from G by *completing* X, i.e. by adding an edge between every pair of non-adjacent vertices of X. If $\mathcal{X} = \{X_1, \ldots, X_p\}$ is a set of subsets of V, $G_{\mathcal{X}}$ is the graph obtained by completing all the elements of $\mathcal{X}$.

Theorem 1 ([8, 9]) *Let $\Gamma \in \Delta_G$ be a maximal set of pairwise parallel separators of G. Then $H = G_\Gamma$ is a minimal triangulation of G and $\Delta_H = \Gamma$.*

Let H be a minimal triangulation of a graph G. Then Δ_H is a maximal set of pairwise parallel separators of G and $H = G_{\Delta_H}$.

In other terms, every minimal triangulation of a graph G is obtained by considering a maximal set Γ of pairwise parallel separators of G and completing the separators of Γ. The minimal separators of the triangulation are exactly the elements of Γ.

It is important to know that the elements of Γ, who become the separators of H, have strictly the same behavior in H as in G. Indeed, the connected components of $H - S$ are exactly the same as the connected components of $G - S$, for every $S \in \Gamma$. Moreover, the full components associated to S are the same in the two graphs.

3 Potential maximal cliques

The *potential maximal cliques* are the central object of this paper. We present in this section some known results about the potential maximal cliques of a graph (see also [2, 3]).

Definition 1 *A vertex set Ω of a graph G is called a* potential maximal clique *if there is a minimal triangulation H of G such that Ω is a maximal clique of H.*

If K is a vertex set of G, we denote by $\Delta_G(K)$ the minimal separators of G included in K. A potential maximal clique Ω is strongly related to the minimal separators contained in Ω.

Definition 2 *Let G be a graph and $\mathcal{S} \subseteq \Delta_G$ a set of pairwise parallel separators such that for any $S \in \mathcal{S}$, there is a block $(S, C(S))$ containing all the separators of $\mathcal{S}$. Suppose that $\mathcal{S}$, ordered by inclusion, has no greatest element. We define the* piece between *the elements of $\mathcal{S}$ by $P(\mathcal{S}) = \bigcap_{S \in \mathcal{S}}(S, C(S))$*

Consider a chordal graph H and a maximal clique Ω of H. It can be shown that either Ω is some block (S, C) of H, or Ω is the piece between the minimal separators of $\Delta_H(\Omega)$.

This property extends to the potential maximal cliques of an arbitrary graph. Consider an arbitrary graph G and a potential maximal clique Ω of G. If H is a minimal triangulation of G such that Ω is a maximal clique of H, one can prove that $\Delta_H(\Omega) = \Delta_G(\Omega)$. Since the minimal separators of H have the same behaviour in H and in G (see theorem 1 and the following remarks), the relation between Ω and $\Delta_G(\Omega)$ is the same in G as in H. This leads to a first characterization of a potential maximal clique.

Theorem 2 *Let K be a vertex set of G.*

Suppose that $\Delta_G(K)$ has a maximum element S, i.e. every T in $\Delta_G(K)$ is included in S. Then K is a potential maximal clique if and only if K is some block (S, C) and $G_S[K]$ is a clique.

Suppose now that $\Delta_G(K)$ ordered by inclusion has no greatest element. Then K is a potential maximal clique if and only if $K = P(\Delta_G(K))$ and $G_{\Delta_G(K)}[K]$ is a clique.

Let us give now a simpler characterization of a potential maximal clique. For any vertex set K, we denote by $C_1(K), \ldots, C_p(K)$ the connected components of $G \backslash K$. We denote by $S_i(K)$ the vertices of K adjacent to at least one vertex of $C_i(K)$. Finally, we denote by $\mathcal{S}_G(K)$ the set of all $S_i(K)$ in the graph G, i.e. $\mathcal{S}_G(K)$ is formed by the neighborhoods, in the graph G, of the connected components of $G \backslash K$.

Theorem 3 *Let $K \subseteq V$ be a set of vertices. K is a potential maximal clique if and only if:*

1. Each $S_i(K)$ is strictly *contained in K.*

2. $G_{\mathcal{S}_G(K)}[K]$ is a clique.

Moreover, if K is a potential maximal clique, then $\mathcal{S}_G(K) = \Delta_G(K)$.

Theorem 3 leads to an algorithm that, given a vertex set K, verifies in $\mathcal{O}(nm)$ if K is a potential maximal clique of G.

4 Treewidth and potential maximal cliques

We prove here that the potential maximal cliques are sufficient for computing the treewidth of graphs [2, 3]. Given a graph G and the set Π_G of all its potential maximal cliques, we want to check if $tw(G) \leq k$. The complexity of the decision algorithm will not depend on k. Since $tw(G) < n$, we can easily obtain an algorithm computing the treewidth of G. The complexity of this algorithm is polynomial in n and $|\Pi_G|$.

Elimination step We will eliminate those potential maximal cliques that cannot appear in *any* minimal triangulation of cliquesize $\leq k+1$. We eliminate first all the potential maximal cliques Ω with $|\Omega| > k+1$, so we obtain a set $\Pi'_G \subseteq \Pi_G$.

Notice that, for any chordal graph H and for any full block (S, C) of H, there is a maximal clique Ω of H such that $S \subset \Omega \subseteq (S, C)$. We use this observation to eliminate all the potential maximal cliques of G that cannot appear in any minimal triangulation of width $\leq k$.

While there is some $\Omega \in \Pi'_G$ and a full block (S, C) with $S \subset \Omega$ such that no potential maximal clique $\Omega' \in \Pi'_G$ satisfies $S \subset \Omega' \subseteq (S, C)$, we eliminate Ω from Π'_G. At the end of this elimination process, if Π'_G is empty we can conclude that $tw(G) > k$.

Extraction step Suppose now that Π'_G is not empty. We will construct a triangulation of width at most k. More exactly, we will choose a set $\mathcal{K}$ of potential maximal cliques of Π'_G that will form a *clique tree* (see [5]).

Let Ω be any potential maximal clique of Π'_G. We put $\mathcal{K} = \{\Omega\}$. For all the minimal separators $S \subset \Omega$ and for all full blocks (S, C) with $\Omega \cap C = \emptyset$, let $\Omega' \in \Pi'_G$ be a potential maximal clique such that $S \subset \Omega' \subseteq (S, C)$. We add the potential maximal cliques Ω' to $\mathcal{K}$. We reiterate the process with the potential maximal cliques Ω' instead of Ω and with the full blocs (S', C') with $S' \subset \Omega'$ and $C' \cap \Omega' = \emptyset$. The algorithm stops when all the vertices of G are covered by elements of $\mathcal{K}$.

We can prove that $H = G_{\mathcal{K}}$ is a minimal triangulation of G and the potential maximal cliques of H are exactly the elements of $\mathcal{K}$. Therefore, H is a minimal triangulation of width at most k.

Both the elimination and the extraction step are polynomial in n and in Π_G.

5 Listing the potential maximal cliques

We give in this section an algorithm enumerating the potential maximal cliques of an arbitrary graph. The algorithm is polynomial in the number of minimal separators of the input graph (see full version in [4]).

Theorem 3 tells us that if Ω is a potential maximal clique of a graph G, then Ω is a clique in $G_{\Delta_G(\Omega)}$. We will divide the minimal separators of $\Delta_G(\Omega)$ into two classes: those which create edges in $G_{\Delta_G(\Omega)}$, which are called *actives*, and the others, which are called *inactives*. More precisely:

Definition 3 *Let Ω be a potential maximal clique of a graph G and let $S \subset \Omega$ be a minimal separator of G. We say that S is an* active separator for Ω *if Ω is not a clique in the graph $G_{\Delta_G(\Omega)\setminus\{S\}}$, obtained from G by completing all the minimal separators contained in Ω, except S. Otherwise, S is called* inactive for Ω.

Let Ω be a potential maximal clique of G and $S \subset \Omega$ a minimal separator, active for Ω. Let (S, C_Ω) be the block associated to S containing Ω and let $x, y \in \Omega$ be two non-adjacent vertices of $G_{\Delta_G(\Omega)\setminus\{S\}}$. We can prove that $\Omega\setminus S$ is an minimal x, y-separator in $G[C_\Omega \cup \{x, y\}]$. The set $T' = \Omega\setminus S$ is a minimal x, y-separator of the subgraph of G induced by $C_\Omega \cup \{x, y\}$. Thus, there is a minimal x, y-separator T of G such that $T' \subseteq T$ and $T \cap C_\Omega = T'$. We deduce:

Theorem 4 *Let Ω be a potential maximal clique and S be a minimal separator, active for Ω. Let (S, C_Ω) be the block associated to S containing Ω. There is a minimal separator T of G such that $\Omega = S \cup (T \cap C_\Omega)$.*

It follows easily that the number of potential maximal cliques containing at least one active separator is at most $n|\Delta_G|^2$. Clearly, these potential maximal cliques can be computed directly from the minimal separators of the graph.

Nevertheless, a potential maximal clique may not have active separators, so the previous remark does not allow us to compute all the potential maximal cliques of G. We consider now a graph G' obtained from G by removing some vertex a. We can prove that, if we have all the potential maximal cliques of G', then we can easily compute all the potential maximal cliques of G.

Theorem 5 *Let Ω be a potential maximal clique of G and let $G' = G\setminus\{a\}$. Then one of the following holds:*

1. *$\Omega = \Omega' \cup \{a\}$, where Ω' is a potential maximal clique of G'.*
2. *$\Omega = \Omega'$, where Ω' is a potential maximal clique of G'.*
3. *$\Omega = S \cup \{a\}$, where S is a minimal separator of G.*
4. *$\Omega = S \cup (C \cap T)$, where S and T are minimal separators of G and C is a connected component of $G\setminus S$.*

Moreover, the cases 1 and 2 are mutually exclusive.

In particular, G has at most $n|\Delta_G|^2 + |\Delta_G|$ potential maximal cliques more than G'.

Notice that G has at least as many minimal separators as G': for any minimal separator S of G', S is a minimal separator of G or $S \cup \{a\}$ is a minimal separator of G. Let now $a_1, a_2, \ldots, a_n$ be an arbitrary ordering of the vertices of G. We denote by G_i the graph $G[\{a_1, \ldots, a_i\}]$, so $G_n = G$ and G_1 has a single vertex. We have that $|\Delta_{G_i}| \leq |\Delta_{G_{i+1}}|$, so each graph G_i has at most $|\Delta_G|$ minimal separators. Clearly, the graph G_1 has a unique potential maximal clique. It follows that G has at most $\mathcal{O}(n^2|\Delta_G|^2)$ potential maximal cliques. Actually, we can improve this result and show that any graph G has at most $n|\Delta_G|^2 + n|\Delta_G| + 1$ potential maximal cliques.

Theorem 6 *The number of the potential maximal cliques of a graph is polynomially bounded in the number of its minimal separators and in the size of the graph. Moreover, all the potential maximal cliques can be listed in polynomial time in the number of minimal separators of the graph.*

We conclude that the treewidth is polynomially tractable for all classes of graphs having a polynomial number of minimal separators.

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