Pathwidth of circular-arc graphs

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Abstract. The pathwidth of a graph G is the minimum clique number of H minus one, over all interval supergraphs H of G. Although pathwidth is a well-known and well-studied graph parameter, there are extremely few graph classes for which pathwidh is known to be tractable in polynomial time. We give in this paper an $\mathcal{O}(n^2)$-time algorithm computing the pathwidth of circular-arc graphs.

1 Introduction

A graph is an *interval graph* if it is the intersection graph of a finite set of intervals on a line. The *pathwidth* of an arbitrary graph G is the minimum clique number of H, over all interval supergraphs H of G, on the same vertex set. Pathwidth has been introduced in the first article of Robertson and Seymour's Graph Minor series [12]. It is a well-known and well studied graph parameter, and not surprisingly NP-hard to compute. Note that the pathwidth has been redefined and studied in different contexts under different names. The parameter is also equal to the vertex separation number, and to the interval thickness or node search number of the graph, minus one (see [1] for a survey).

The pathwidth problem is fixed parameter tractable. Indeed, since the class of graphs of pathwidth at most k is minor-closed, using Roberston and Seymour's results on graph minors there exists an $\mathcal{O}(n^2)$ algorithm for recognizing graphs of pathwidth at most k, for any constant k; unfortunately this technique is not constructive. Bodlaender and Kloks [3] give a constructive linear-time algorithm deciding, for any fixed k, if the pathwidth of an input graph is at most k. Their algorithm first computes a tree decomposition of the input graph of width at most k, using the fact that the treewidth of a graph is at most its pathwidth (see Section 3 for definitions). Then this tree decomposition is used to decide if the pathwidth is at most k. The best approximation algorithms for pathwidth are also based on approximation algorithms for treewidth, combined with the fact that $\mathrm{pwd}(G) \leq \mathrm{twd}(G) \cdot \log n$.

By [3], the pathwidth is polynomially tractable for all classes of graphs of bounded treewidth. For trees and forests there exist several algorithms solving the problem in $\mathcal{O}(n \log n)$ time [11, 5]); only recently, Skodinis [13] gave a linear time algorithm. Even the unicyclic graphs, obtained from a tree by adding one edge, require more complicated algorithms in order to obtain an $\mathcal{O}(n \log n)$ time bound [4]. There exist some other graph classes for which the pathwidth problem is polynomial, e.g. permutation graphs, but this is mainly because for these classes the pathwidth equals the treewidth. Roughly speaking, almost everything that we know about computing pathwidth comes from its relationship with treewidth. More surprisingly, even for classes of graphs of very small treewidth, like outerplanar or Halin graphs, there are interesting approximation algorithms for pathwidth [2, 6]. Although in these cases the parameter is polynomially tractable by the algorithm of [3], the running time is huge and the dynamic programming technique of [3] could not be translated into simple, combinatorial algorithms.

In this paper we give the first polynomial time algorithm computing the pathwidth of circular-arc graphs. A graph is a *circular-arc* graph if it is the intersection graph of a finite set of arcs on a circle. The pathwidth of these graphs can be easily approximated within a factor of 2. Nevertheless, for circular-arc graphs the pathwidth is not necessarily equal to the treewidth, and clearly this class is not of bounded treewidth, therefore we cannot use the classical techniques for computing pathwidth. Our algorithm is based on a study of interval completions of circular-arc graphs. An interval completion of a graph G is an interval supergraph H, on the same vertex set. If no interval completion H' of G is a strict subgraph of H, we say that H is a minimal interval completion of G. We study the minimal interval completions of circular-arc graphs and we characterize a subclass of these minimal interval completions, containing optimal solutions to the pathwidth problem. Based on this combinatorial result, we give an $O(n^2)$ algorithm computing the pathwidth of circular-arc graphs.

2 Definitions and basic results

Let $G = (V, E)$ be a finite, undirected and simple graph. Moreover we only consider connected graphs — in the non connected case each connected component can be treated separately. Denote $n = |V|$, $m = |E|$. If $G = (V, E)$ is a subgraph of $G' = (V', E')$ (i.e. $V \subseteq V'$ and $E \subseteq E'$) we write $G \subseteq G'$. The *neighborhood* of a vertex v in G is $N_G(v) = \{u \mid \{u, v\} \in E\}$. Similarly, for a set $A \subseteq V$, $N_G(A) = \bigcup_{v \in A} N_G(v) \setminus A$. As usual, the subscript is sometimes omitted. For a set of vertices $A \subset V$, $G[A]$ is the *subgraph* of G *induced* by A, that is the graph (A, E_A), where $E_A = \{\{x, y\} \mid \{x, y\} \subseteq A$ and $\{x, y\} \in E\}$.

The *intersection graph* of a family V of n sets is the graph $G = (V, E)$, where the vertices are the sets and the edges are the pairs of sets that intersect. Every graph is the intersection graph of some family of sets [14]. A graph is an *interval graph* if it is the intersection graph of a finite set of intervals on a line. A graph is a *circular-arc* graph if it is the intersection graph of a finite set of arcs on a circle. A *model* of an interval graph or a circular-arc graph G is a set of intervals or circular arcs that represent G in this way. Without loss of generality we may assume that no two circular arcs of the model share a common end point.

Given a model M_G of a circular-arc $G = (V, E)$, we introduce some vocabulary concerning the arcs. We identify a vertex $v \in V$ with the corresponding arc in M_G. We call the clockwise endpoint of an arc v the *left* endpoint, denoted by $l(v)$, and the counterclockwise endpoint the *right* endpoint, denoted by $r(v)$. Note that an interval graph is a special case of circular-arc graphs - it is a circular-arc graph that can be modelled with arcs that do not cover the entire circle. If G is a circular-arc graph with a model M_G, and X is a subset of the vertices, then $M_G[X]$ is the *restriction* of M_G to X, namely, the result of removing from M_G the arcs corresponding to vertices not in M_G. Clearly, $M_G[X]$ is a model of $G[X]$.

Given a graph G, there exists a linear-time algorithm recognizing whether G is a circular-arc graph [10]. The algorithm produces a circular-arc model if such a model exists. Therefore, from now on we assume that our input is a circular arc-graph together with a model M_G.

To each point p on the circle in M_G we assign the set of arcs $V(p)$ that intersect it. Clearly, $V(p)$ is a clique in G. In particular, we are interested in the ones maximal with respect to inclusion. To analyze them, we need not consider all points. In fact, it is enough to take the set of right endpoints. Using only the right endpoints that have the corresponding set of arcs maximal with respect to inclusion, we define the following structure describing G.

Definition 1. *Given a circular-arc model M_G of $G = (V, E)$, a* clique-cycle *is the cycle $CC_{G,M} = (\mathcal{X}, C)$. The vertices are the right endpoints in M_G with the corresponding sets of circular-arcs maximal by inclusion among all endpoints in M_G. The edges form a cycle, with every vertex $q \in \mathcal{X}$ adjacent to $X_l, X_r \in \mathcal{X}$, where X_l (X_r) is the element of $\mathcal{X}$ clockwise (counterclockwise) closest to X in M_G.*

3 Connected decompositions

Definition 2. *Let $\mathcal{X} = \{X_1, ..., X_k\}$ be a set of subsets of V such that X_i, $1 \leq i \leq k$, is a clique in $G = (V, E)$. If, for every $\{v_i, v_j\} \in E$, there is some X_p such that $\{v_i, v_j\} \subseteq X_p$, then $\mathcal{X}$ is called an* edge clique cover *of G.*

Definition 3. *A* connected decomposition *of an arbitrary graph $G = (V, E)$ is a graph $D = (\mathcal{X}, A)$, where $\mathcal{X}$ is a family of subsets of V called* bags *and A is any set of edges on $\mathcal{X}$, such that the following three conditions are satisfied.*

1. *Each vertex $v \in V$ appears in some bag.*
2. *For every edge $v_iv_j \in E$ there is a bag containing both v_i and v_j.*
3. *For every vertex $v \in V$, the bags containing v induce a connected subgraph of D.*

$D = (\mathcal{X}, A)$, a connected decomposition of G, is a clique connected decomposition *of G if $\mathcal{X}$ is an edge clique cover of G.*

A path decomposition *(*tree decomposition*) is a connected decomposition with D being a path (tree). The* width *of a decomposition is the size of a largest bag, minus one. The* pathwidth *(*treewidth*) of a graph G, denoted by* $\mathrm{pwd}(G)$ *(*$\mathrm{twd}(G)$*) is the minimum width over all path decompositions (tree decompositions) of G.*

Lemma 1 (see, e.g., [7]). *A graph G is a chordal (interval) graph iff it has a clique tree (path) decomposition.*

Similarly, we can characterize circular-arc graphs through clique cycle decompositions. A *cycle decomposition* is a connected decomposition with D being a cycle, and a *clique cycle decomposition* is a cycle decomposition with all bags being cliques.

By Definition 1, we have:

Lemma 2. *Given a circular-arc model M_G of $G = (V, E)$, $CC_{G,M} = (\mathcal{X}, C)$ is a clique cycle decomposition of G.*

We easily deduce:

Theorem 1. *A graph G is a circular-arc graph if and only if it has a clique cycle decomposition.*

Given a clique path decomposition of an interval graph, the intersection of two consecutive cliques form a separator. Given a circular model M_G of $G = (V, E)$ and a clique cycle decomposition $CC_{G,M} = (\mathcal{X}, C)$, we say that the intersection of two consecutive cliques of the cycle is a *semi-separator*.

4 Folding

Given a path decomposition P of G, let $\texttt{PathFill}(G, P)$ be the graph obtained by adding edges to G so that each bag of P becomes a clique. It is straight forward to verify that $\texttt{PathFill}(G, P)$ is an interval supergraph of G, for every path decomposition P. Moreover P is a clique path decomposition of $\texttt{PathFill}(G, P)$.

Definition 4 (see also [8]). *Let $\mathcal{X}$ be an edge clique cover of an arbitrary graph G and let $\mathcal{Q} = (Q_1, ..., Q_k)$ be a permutation of $\mathcal{X}$. We say that $(G, \mathcal{Q})$ is a* folding *of G by $\mathcal{Q}$.*

To any folding of G by an ordered edge clique cover $\mathcal{Q}$ we can naturally associate, by Algorithm FillFolding of Figure 1, an interval supergraph $H = \texttt{FillFolding}(G, \mathcal{Q})$ of G. The algorithm also constructs a clique path decomposition of H.

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Algorithm FillFolding
Input:   Graph G = (V, E) and Q = (Q_1, ..., Q_k), a sequence of subsets of V;
Output:  A supergraph H of G;

P = Q;
for each vertex v of G do
    s = min{i | x ∈ Q_i};
    t = max{i | x ∈ Q_i};
    for j = s + 1 to t − 1 do
        P_j = P_j ∪ {v};
end-for
H = PathFill(G, P);
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Fig. 1. The FillFolding algorithm.

Lemma 3. *Given a folding $(G, \mathcal{Q})$ of G, the graph $H = \texttt{FillFolding}(G, \mathcal{Q})$ is an interval completion of G.*

Proof. Observe that after the **for** loops, P is a path decomposition of H, since every edge is contained in some bag, and for every vertex the bags containing it induce a subpath of P. Hence, since $H = \texttt{PathFill}(G, P)$, it is an interval completion of G. □

We shall also say that the graph $H = \texttt{FillFolding}(G, \mathcal{Q})$ is *defined* by the folding $(G, \mathcal{Q})$. The graph defined by a folding is not necessarily a minimal interval completion of G. Nevertheless, we prove in Theorem 2 that every minimal interval completion of G is defined by some folding.

Theorem 2. *Let H be a minimal interval completion of a graph G with an edge clique cover $\mathcal{X}$. Then there exists a folding $(G, \mathcal{Q})$, where $\mathcal{Q}$ is a permutation of $\mathcal{X}$, such that $H = \texttt{FillFolding}(G, \mathcal{Q})$.*

Proof. Let $\mathcal{X} = \{X_i \mid 1 \leq i \leq p\}$ and $\mathcal{K} = \{\Omega_i \mid 1 \leq i \leq k\}$ denote an enumeration of $\mathcal{X}$ and the set of maximal cliques of H, respectively. Let $P = (\Omega_1, \ldots, \Omega_k)$ be a clique-path of H. It defines a linear order on the set $\mathcal{K}$. Let us use it to construct a linear order on $\mathcal{X}$.

In a natural way, P defines a linear pre-order on $\mathcal{X}$ by

$$X_a \leq X_b \text{ if } \exists i, j \text{ such that } X_a \subseteq \Omega_i, X_b \subseteq \Omega_j, \text{ where } 1 \leq i \leq j \leq k, 1 \leq a, b \leq p,$$

where for a clique X_i that is contained in several maximal cliques of H, consider just the first occurrence. Transform it into a linear order (sequence) $\mathcal{Q}$ by fixing any permutation inside the equivalence classes.

Let us define $H' = \texttt{FillFolding}(G, \mathcal{Q})$, and prove that $H' = H$. By Lemma 3, H' is an interval completion of G. Moreover, $E(H') \subseteq E(H)$, since $xy \in E(H')$ only if the interval between the first and the last element in $\mathcal{Q}$ that contains x intersects the one corresponding to y. In this case, the corresponding intervals in P intersect as well, so there is $xy \in E(H)$. By minimality of H, there is $H = H'$. □

5 Folding of circular-arc graphs

Let $(\mathcal{X}, C)$ be the clique cycle of the circular-arc graph $G = (V, E)$, obtained like in Definition 1. Consider a permutation $\mathcal{Q}$ of the set of bags $\mathcal{X}$. In the case of the circular-arc graphs, we study the permutation $\mathcal{Q}$ with respect to the circular ordering of the cliques on the cycle of the decomposition. Therefore it is more convenient to think of a folding as a *triple* $(\mathcal{X}, C, \mathcal{Q})$.

A folding $(\mathcal{X}, C, \mathcal{Q})$ naturally defines an upper part and a lower part of the cycle $(\mathcal{X}, C)$. Let Q_L, Q_R be the leftmost and rightmost element of the permutation $\mathcal{Q}$. Let $\mathcal{X}^{down}$ ($\mathcal{X}^{down}$) denote the cliques counterclockwise (clockwise) between Q_L and Q_R on the cycle. Let $\mathcal{Q}^{down} = (Q_L = Q_{l_1}, Q_{l_2}, \ldots, Q_{l_r} = Q_R)$ denote the restriction of $\mathcal{Q}$ to $\mathcal{X}^{down}$. Similarly let $\mathcal{Q}^{up} = (Q_L = Q_{u_1}, Q_{u_2}, \ldots, Q_{u_t} = Q_R)$ denote the restriction of $\mathcal{Q}$ to $\mathcal{X}^{up}$.

Definition 5. *Given a clique cycle decomposition $(\mathcal{X}, C)$ of G and a permutation $\mathcal{Q}$ of $\mathcal{X}$, we say that a clique $X \in \mathcal{X}$ is a* pivot *of the folding $(\mathcal{X}, C, \mathcal{Q})$ if its neighbors on the cycle appear on the same side of X in Q. We extend this definition to any subset $\mathcal{X}'$ of $\mathcal{X}$: $X \in \mathcal{X}'$ is a pivot w.r.t. $\mathcal{X}'$ if $X_L, X_R \in \mathcal{X}'$, its closest neighbors on the cycle among the elements of $\mathcal{X}'$, are on the same side of X in $\mathcal{Q}$.*

Remark 1. Let $(\mathcal{X}, C, \mathcal{Q})$ be a folding of the circular-arc graph G. Consider the clique path decomposition P produced by the algorithm $\texttt{FillFolding}(G, \mathcal{Q})$. Observe that each bag of P is the union of the clique $Q \in \mathcal{Q}$ which corresponds to the bag at the initialization step, and some semi-separators of type $Q' \cap Q''$, where Q', Q'' are two cliques consecutive on the cycle, but separated by Q in $\mathcal{Q}$. We say that the clique Q and the semi-separators have been *merged* by the folding.

Definition 6. *Let $(\mathcal{X}, C)$ be a clique cycle decomposition of $G = (V, E)$. A permutation $\mathcal{Q}$ of a subset $X' \subseteq X$, is* k-monotone *if it contains exactly k pivots. The* monotonicity *of $\mathcal{Q}$ is the minimum k such that $\mathcal{Q}$ is k-monotone.*

The main combinatorial result of the paper consists in proving that there exists a 2-monotone folding $(\mathcal{X}, C, \mathcal{Q})$ such that $H = \texttt{FillFolding}(G, \mathcal{Q})$ is an interval completion of G satisfying $\mathrm{pwd}(H) = \mathrm{pwd}(G)$. Therefore, the optimum interval completion for the pathwidth problem can be found among the completions defined by 2-monotone foldings. In a two-monotone folding, the only pivots are the first and last element of $\mathcal{Q}$. Moreover, $\mathcal{Q}^{up}$ ($\mathcal{Q}^{down}$) is clockwise (counterclockwise) consecutive on the cycle $(\mathcal{X}, C)$.

The following lemma is straightforward.

Lemma 4. *Let $(\mathcal{X}, C, \mathcal{Q})$ be a 2-monotone folding and let P be the clique path decomposition produced by $\texttt{FillFolding}(G, \mathcal{Q})$. Every bag of P is the union of a clique $Q \in \mathcal{Q}$ and of a unique semi-separator corresponding to the edge $\{Q', Q''\}$ of the cycle, such that Q separates Q' and Q'' in the permutation $\mathcal{Q}$.*

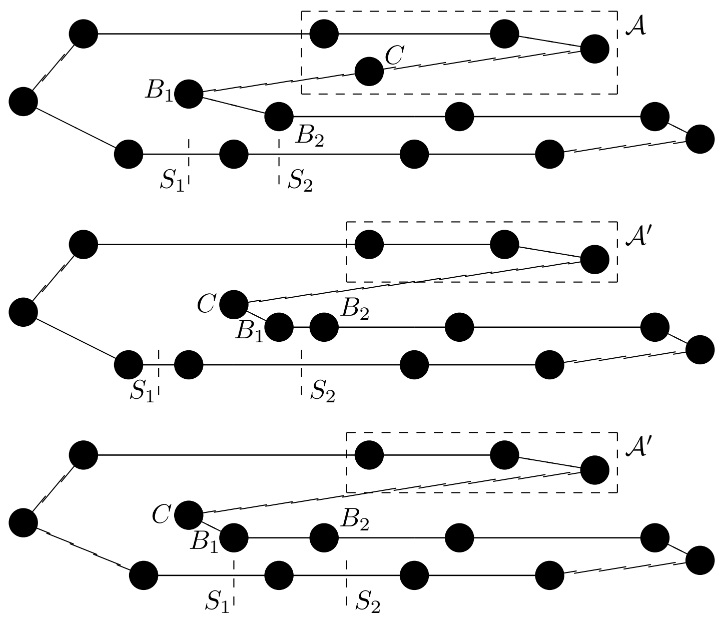

Fig. 2. Reduction of $\mathcal{A}$ (top) to $\mathcal{A}'$: one way (middle) or the other (bottom)

Definition 7. *Let $(\mathcal{X}, C)$ be a clique cycle decomposition of $G = (V, E)$ and let $(\mathcal{X}, C, \mathcal{Q})$ be a 4-monotone folding. Let Q_L, Q_R be the end cliques (pivots) of $\mathcal{Q}$. Let B_1, P be the other pivots, ordered as in $\mathcal{Q}$. Assume w.l.o.g. that B_1, P belong to $\mathcal{Q}^{up}$. The consecutive part of the cycle that appears counterclockwise, starting right after B_1, passing through P, continuing as long as it stays after B_1 in $\mathcal{Q}$ is called the* anomaly *(see the top part of Figure 2).*

Notice that for a 4-monotone folding $\mathcal{Q}$, the restriction of $\mathcal{Q}$ to $\mathcal{X} \setminus \mathcal{A}$ is 2-monotone.

One of our main tools (Theorem 3) shows that if $(\mathcal{X}, C, \mathcal{Q})$ is a 4-monotone folding which defines $H = \texttt{FillFolding}(G, \mathcal{Q})$, then there exists a 2-monotone folding $(\mathcal{X}, C, \mathcal{Q}')$ defining an interval graph $H' = \texttt{FillFolding}(G, \mathcal{Q}')$ of pathwidth smaller or equal the pathwidth of H. Here is an informal sketch of the main idea. Consider an anomaly $\mathcal{A}$ of the 4-monotone folding $(\mathcal{X}, C, \mathcal{Q})$. Suppose that the anomaly is in the upper part of the cycle, as on Figure 2.

Notice that when we restrict $(\mathcal{X}, C)$ to the cliques that are not in the anomaly (just remove every $X \in \mathcal{A}$, making the former neighbors of X adjacent), then we obtain a clique cycle of $\overline{G} = G[\bigcup(X \setminus \mathcal{A})]$, an induced subgraph of G. Moreover, we obtain $(\overline{\mathcal{X}}, \overline{C}, \overline{\mathcal{Q}})$, a folding of $\overline{G}$, where $\overline{\mathcal{Q}}$ is the restriction of sequence $\mathcal{Q}$ to the cliques not in the anomaly $\overline{\mathcal{X}} = \mathcal{X} \setminus \mathcal{A}$. It defines an interval completion $\overline{H} = \texttt{FillFolding}(\overline{G}, \overline{\mathcal{Q}})$.

Definition 8. *Let $(\mathcal{X}, C)$ be a clique cycle decomposition of $G = (V, E)$ and let $(\mathcal{X}, C, \mathcal{Q})$ be a 4-monotone folding with the anomaly $\mathcal{A}$. The $\mathcal{A}$*-width *of $(\mathcal{X}, C, \mathcal{Q})$ is the pathwidth of $\overline{H} = \texttt{FillFolding}(\overline{G}, \overline{\mathcal{Q}})$, where $\overline{G}$ is the intersection graph defined by the clique cycle $(\mathcal{X}, C)$ restricted to $\overline{\mathcal{X}} = \mathcal{X} \setminus \mathcal{A}$ and $\overline{\mathcal{Q}}$ is the sequence $\mathcal{Q}$ restricted to $\overline{\mathcal{X}}$.*

One step of the procedure is to slightly modify the folding $(\mathcal{X}, C, \mathcal{Q})$ to obtain $(\mathcal{X}, C, \mathcal{Q}')$ with a strictly smaller anomaly $\mathcal{A}'$, ensuring that the $\mathcal{A}'$-width of $(\mathcal{X}, C, \mathcal{Q}')$ is not bigger than the pathwidth of H. Continue until the anomaly is empty. Eventually, this yields a folding

$(\mathcal{X}, C, \mathcal{Q}'')$ which is 2-monotone. Its anomaly $\mathcal{A}''$ being empty, the $\mathcal{A}''$-width of this folding is equal to the pathwidth of $H'' = \texttt{FillFolding}(G, \mathcal{Q}'')$, and it is not bigger than the pathwidth of H.

Let us give in more detail the construction of $(\mathcal{X}, C, \mathcal{Q}')$ based on $(\mathcal{X}, C, \mathcal{Q})$. Consider the pivots of $(\mathcal{X}, C, \mathcal{Q})$ that are not end-cliques of $\mathcal{Q}$. Let P be the one that belongs to the anomaly $\mathcal{A}$, and let B_1 be the other one. Let B_{k+1} be the neighbor of B_1 on the cycle that belongs to the anomaly. Let $B_2, \ldots, B_k$ be the cliques, which do not belong to the anomaly, that follow B_1 clockwise on the cycle and appear before B_{k+1} in $\mathcal{Q}$. Let $S_1, \ldots, S_{k+1}$ be the semi-separators on the lower part of the cycle, such that S_i is merged with the corresponding B_i in $\texttt{FillFolding}(G, \mathcal{Q})$. In this setting, we choose a semi-separator S_l and permute $\mathcal{Q}$ in order to put all $B_{k+1}, B_1, \ldots, B_{l-1}$ (in this order) between Q and Q', where Q, Q' are the consecutive cliques in the lower part of the cycle such that $S_l = Q \cap Q'$. We choose S_l such that the new folding $(\mathcal{X}, C, \mathcal{Q}')$ has the desired property. We say that in such a situation we *put* $B_{k+1}, B_1, \ldots, B_{l-1}$ *on* the semi-separator S_l. This construction is illustrated in Figure 2. Informally, the bags $B_{k+1}, B_1, \ldots, B_{l-1}$ slide (without jumping) along the cycle, in the clockwise sense, and they stop above the edge of the cycle corresponding to S_l.

Theorem 3. *Let $(\mathcal{X}, C)$ be a clique cycle decomposition of $G = (V, E)$. Let $(\mathcal{X}, C, \mathcal{Q})$ be a 4-monotone folding and $H = \texttt{FillFolding}(G, \mathcal{Q})$. Then there is a 2-monotone folding $(\mathcal{X}, C, \mathcal{Q}')$ such that the pathwidth of $H' = \texttt{FillFolding}(G, \mathcal{Q}')$ is not bigger than the pathwidth of H. Moreover, we can assume that $(\mathcal{X}, C, \mathcal{Q}')$ is such that $\mathcal{X}'^{up} = \mathcal{X}^{up}$ and $\mathcal{X}'^{down} = \mathcal{X}^{down}$.*

Proof. Let $(\mathcal{X}, C, \mathcal{Q}')$ be a $\leq$ 4-monotone folding with the anomaly $\mathcal{A}'$, such that $\mathcal{X}'^{up} = \mathcal{X}^{up}$ and $\mathcal{X}'^{down} = \mathcal{X}^{down}$, which satisfies the following Properties:

1. $\mathcal{A}' \subseteq \mathcal{A}$;
2. for any clique Q of $\mathcal{A}'$, the semi-separator S of the lower part of the cycle that is merged to Q in H' is the same as in H;
3. the $\mathcal{A}'$-width of $\mathcal{Q}'$ is not bigger than the pathwidth of H,
4. the anomaly $\mathcal{A}'$ is inclusion minimal among all such foldings.

Let us show that $\mathcal{A}'$ is in fact empty, thus the pathwidth of H' is not bigger than the pathwidth of H. Suppose $\mathcal{A}'$ is not empty.

We use the notations introduced in the informal description above. Let us use Y and S_Y as shorthands for B_{k+1} and S_{k+1}. By Property 2, we have

$$|Y \cup S_Y| \leq \mathrm{pwd}(H) + 1. \tag{1}$$

The semi-separators S_i, $1 \leq i \leq k+1$, can be partitioned as follows:

$$\begin{gathered} S_i = N_i^j \cup B_i^j \cup Y_i^j \cup I_i^j \text{ , for any } 1 \leq j \leq k \text{ , where} \\ N_i^j = S_i \setminus (B_j \cup Y), B_i^j = S_i \cap B_j \setminus Y, Y_i^j = S_i \cap Y \setminus B_j, I_i^j = S_i \cap B_j \cap Y. \end{gathered} \tag{2}$$

Claim. For any $1 \leq i \leq k$, $1 \leq p \leq q \leq k+1$, one of the following holds:

$$|B_i \cup S_p| \geq |B_i \cup S_q| \text{ or } |Y \cup S_q| \geq |Y \cup S_p| \tag{3}$$

Proof. Suppose it is not true. By Equation 2, we have:

$$|B_i \cup N_p^i \cup Y_p^i| < |B_i \cup N_q^i \cup Y_q^i| \text{ and } |Y \cup N_q^i \cup B_q^i| < |Y \cup N_p^i \cup B_p^i|,$$

which yields a contradiction: $|N_p^i| < |N_q^i|$ and $|N_q^i| < |N_p^i|$, since for any j, p, q, $1 \leq j \leq k$, $1 \leq p \leq q \leq k+1$ there is $Y_q^j \subseteq Y_p^j$ and $B_p^j \subseteq B_q^j$, by properties of the clique cycle. □

Claim. Let l be the biggest integer such that $|Y \cup S_Y| < |Y \cup S_i|$, for $1 \leq i \leq l-1$. Then

$$|Y \cup S_Y| \geq |Y \cup S_l|, \tag{4}$$

$$|B_i \cup S_i| \geq |B_i \cup S_l|, \text{ for any } 1 \leq i \leq l, \tag{5}$$

so we can put Y and all B_i, for $1 \leq i \leq l-1$, on the semi-separator S_l without augmenting the $\mathcal{A}'$-width.

Proof. The first equation is clear from the construction. Since $|Y \cup S_l| \leq |Y \cup S_Y|$ and $|Y \cup S_Y| < |Y \cup S_i|$, for any $1 \leq i \leq l-1$, there is $|Y \cup S_l| < |Y \cup S_i|$, for any $1 \leq i \leq l-1$. Now, by Equation 3, for any $1 \leq i \leq l-1$, we get $|B_i \cup S_i| \geq |B_i \cup S_l|$, for any $1 \leq i \leq l-1$. □

Therefore, by putting Y and all B_i, for $1 \leq i \leq l-1$, on S_l, we create a new folding $(\mathcal{X}, C, \mathcal{Q}'')$ with a strictly smaller anomaly $\mathcal{A}''$. Indeed, there is $Y \in \mathcal{A}' \setminus \mathcal{A}''$. Notice there may be other cliques in $\mathcal{A}' \setminus \mathcal{A}''$ as well.

Let us check that the $\mathcal{A}''$-width of the folding $(\mathcal{X}, C, \mathcal{Q}'')$ is at most $\mathrm{pwd}(H)$. For each clique X of $\mathcal{Q}'' \setminus \mathcal{A}''$, let S_X be the (unique) semi-separator of the lower part of the cycle to which X is merged in `FillFolding`(G, Q''). The $\mathcal{A}''$-width of $(\mathcal{X}, C, \mathcal{Q}'')$ is the maximum, over all cliques X, of $|X \cup S_X| - 1$. If X is also in $\mathcal{Q}' \setminus \mathcal{A}'$, this quantity is upper bounded by the $\mathcal{A}'$-width of $(\mathcal{X}, C, \mathcal{Q}')$ and the conclusion follows by Property 3. If $X = Y$, then $S_X = S_l$ and the conclusion follows from Equations 4 and 1. If X is one of the B_i's, $1 \leq i \leq l-1$, it follows from Equation 5 and Property 3. Finally, if X is one of the cliques of $\mathcal{A}' \setminus \mathcal{A}''$, different from Y, the conclusion follows from Property 2.

The new folding $(\mathcal{X}, C, \mathcal{Q}'')$ also respects Property 2, since in the permutation $\mathcal{Q}''$ the cliques of $\mathcal{A}''$ have the same position w.r.t. the lower part of the cycle as before.

The construction of $\mathcal{Q}''$ contradicts Property 4 of $\mathcal{Q}'$. So $\mathcal{A}'$ must be empty. □

Theorem 4. *Let $(\mathcal{X}, C)$ be a clique cycle decomposition of $G = (V, E)$. There is a folding $(\mathcal{X}, C, \mathcal{Q})$, with $\mathcal{Q}$ being a permutation of $\mathcal{X}$, such that $\mathcal{Q}$ is 2-monotone and $H =$ `FillFolding`$(G, \mathcal{Q})$ is an interval completion of G of pathwidth equal the pathwidth of G.*

Proof. Let $(\mathcal{X}, C, \mathcal{Q})$ be a folding of minimum monotonicity such that the pathwidth of $H =$ `FillFolding`$(G, \mathcal{Q})$ is not bigger that the pathwidth of G. We will prove that it is 2-monotone.

Suppose it is not. Assume w.l.o.g. that $\mathcal{X}^{up}$ contains some pivots other than Q_L, Q_R. Let B_1 be the leftmost pivot in $\mathcal{Q}^{up}$. Let P be the rightmost in $\mathcal{Q}^{up}$ among the pivots which are between Q_L and B_1 clockwise on the cycle $(\mathcal{X}, C)$. Let $\mathcal{Q}_L^{up}$ denote the subsequence of $\mathcal{Q}^{up}$ induced by cliques clockwise between Q_L and P (included) on the cycle. Let $\mathcal{Q}_C^{up}$ denote the subsequence of $\mathcal{Q}^{up}$ induced by cliques between P and B_1 (included), and $\mathcal{Q}_R^{up}$ denote the subsequence of $\mathcal{Q}^{up}$ induced by cliques between B_1 and Q_R (included). Let G_L^{up} be the graph defined by the folding $\mathcal{Q}_L^{up}$, restricted to the corresponding set of bags: $G_L^{up} =$ `FillFolding`$(G[\bigcup \mathcal{Q}_L^{up}], \mathcal{Q}_L^{up})$. We denote by P_L^{up} the clique path decomposition produced by the folding algorithm. Similarly, we define G_C^{up}, G_R^{up} and G^{down}, with the corresponding clique path decompositions. Let $\tilde{G}$ be the union of these four graphs. Note that $\tilde{G}$ is a circular-arc graph. A clique cycle decomposition $(\tilde{\mathcal{X}}, \tilde{C})$ of $\tilde{G}$ is obtained by gluing into a cycle the paths P_L^{up}, the reverse of P_C^{up}, P_R^{up}, and to the reverse of P^{down}. The gluing is performed by identifying the bags P, then B_1, Q_R and finally Q_L.

Moreover, this procedure yields a folding $(\tilde{\mathcal{X}}, \tilde{C}, \tilde{\mathcal{Q}})$ of $\tilde{G}$. The bags of $\tilde{\mathcal{X}}$ are in one-to-one correspondence to the bags of $\mathcal{X}$, so the permutation $\mathcal{Q}$ of $\mathcal{X}$ is translated into a permutation $\tilde{\mathcal{Q}}$ of $\tilde{\mathcal{X}}$. Notice that $(\tilde{\mathcal{X}}, \tilde{C}, \tilde{\mathcal{Q}})$ is a 4-monotone folding, since Q_L, B_1, P, Q_R are the only pivots left. Also $\texttt{FillFolding}(\tilde{G}, \tilde{\mathcal{Q}}) = H = \texttt{FillFolding}(G, \mathcal{Q})$.

By Theorem 3 on $(\tilde{\mathcal{X}}, \tilde{C}, \tilde{\mathcal{Q}})$ there is a 2-folding $(\tilde{\mathcal{X}}, \tilde{C}, \tilde{\mathcal{Q}}')$ such that the pathwidth of $H' = \texttt{FillFolding}(\tilde{G}, \tilde{\mathcal{Q}}')$ is not bigger than the pathwidth of H, thus not bigger than the pathwidth of G.

Since $\tilde{\mathcal{Q}}'$ is 2-monotone and $\tilde{\mathcal{X}}'^{up} = \tilde{\mathcal{X}}^{up}$, the only pivots of $\tilde{\mathcal{Q}}'$ are Q_L and Q_R. Notice that there is : $\tilde{\mathcal{Q}}'^{up}$ equals P_L^{up} glued to the reverse of P_C^{up} glued to P_R^{up}, and $\tilde{\mathcal{Q}}'^{down}$ equals P^{down}.

Because of the one-to-one correspondence between the elements of $\mathcal{X}$ and $\tilde{\mathcal{X}}$, we construct the folding $(\mathcal{X}, C, \mathcal{Q}')$ directly from $(\tilde{\mathcal{X}}, \tilde{C}, \tilde{\mathcal{Q}}')$, by just replacing the elements of $\tilde{\mathcal{X}}$ with the corresponding elements of $\mathcal{X}$. Clearly, B_1 and P are not pivots of $\mathcal{Q}'$, whereas all the other pivots of $\mathcal{Q}'$ are also pivots of $\mathcal{Q}$. Moreover, it is easy to verify that $\texttt{FillFolding}(G, \mathcal{Q}') = \texttt{FillFolding}(\tilde{G}, \tilde{\mathcal{Q}}')$. Therefore, $(\mathcal{X}, C, \mathcal{Q}')$ is a folding of strictly smaller monotonicity than $(\mathcal{X}, C, \mathcal{Q}')$, which also defines a completion of pathwidth not bigger that the pathwidth of G. A contradiction.

□

6 The algorithm

The algorithm for computing the pathwidth of circular-arc graphs is very similar to the algorithm computing the minimum fill-in for the same class of graphs [9].

Consider a clique cycle $CC_{G,M} = (\mathcal{X}, C)$ of the input graph G, obtained from a circular-arc model like in Definition 1. Subdivide each edge of the cycle by adding a new bag containing the semi-separator corresponding to the edge. We obtain a *clique-semi-separator* cycle alternating original clique bags and semi-separator bags. We should also see this cycle as a (regular) *polygon of scanpoints* $\mathcal{P}_G$, following the terminology of [9]; the scanpoints are the clique and semi-separator bags of the cycle. Therefore we associate to each scanpoint s the set of vertices $V(s)$, corresponding to the clique or semi-separator represented by the scanpoint. For each triangle T formed by three scanpoints s_1, s_2, s_3 , define the width $w(T)$ of the triangle as the cardinality of the union $V(s_1) \cup V(s_2) \cup V(s_3)$.

Definition 9. *A* linear (planar) triangulation LP *of the polygon of scanpoints* $\mathcal{P}_G$ *is a planar triangulation such that every triangle contains at most two diagonals. The width* $w(LT)$ *of the linear triangulation is the maximum width of its faces (triangles).*

First, we show that the pathwidth of G equals the minimum width of a linear planar triangulation, minus one. Eventually we give an algorithm computing a linear triangulation of minimum width.

Lemma 5. *Let* LT *be a linear planar triangulation of the polygon of scanpoints* $\mathcal{P}_G$*. There is a path decomposition of* G*, of width* $w(LT) - 1$*.*

Proof. Consider a graph whose vertex set is the set of triangles of LT, and such that two vertices are adjacent if and only if the corresponding triangles share a diagonal. This graph is called the inner dual of LT. Since LT is a linear triangulation, the maximum degree of the inner dual is 2. Clearly this graph is connected and contains no cycle, so it is a path P. The path decomposition of G is constructed using the path P. For each node T of P, let s_1, s_2, s_3 be the three corresponding scanpoints. The bag associated to T is $V(s_1) \cup V(s_2) \cup V(s_3)$. It remains to show that P and its bags form indeed a path decomposition of G. By contradiction, suppose that a vertex x of G appears in the bags corresponding to some nodes T, T' of P but not in the bag of T'', on the T, T'-subpath of P. Let D be an edge of T'', corresponding to a diagonal of $\mathcal{P}_G$. Since x is in the bags T and T', the circular arc corresponding to x intersects both sides of the cycle, with respect to the diagonal D. Hence at least one end-point of the diagonal D is on the circular-arc x – a contradiction. Clearly, the width of this path decomposition is $w(LT) - 1$. □

Lemma 6. *Let $(G, \mathcal{Q})$ be a 2-monotone folding of G. There exists a linear planar triangulation $LT(\mathcal{Q})$ of $\mathcal{P}_G$ such that the width of $LT(\mathcal{Q})$ is equal to the pathwidth of* $\texttt{FillFolding}(G, \mathcal{Q})$*, plus one.*

Proof. The folding being 2-monotone, both $\mathcal{Q}^{up}$ and $\mathcal{Q}^{down}$ are increasing subsequences of Q. For every couple (Q', Q'') of consecutive cliques of $\mathcal{Q}^{down}$, let $(Q_{u_i}, Q_{u_{i+1}}, \ldots, Q_{u_j})$ be the subsequence of $\mathcal{Q}^{up}$ appearing strictly between Q' and Q'' in Q. Add, in $\mathcal{P}_G$, a diagonal between

- The scanpoint s corresponding to the semi-separator of the edge $\{Q', Q''\}$ and every scanpoint corresponding to a clique of $(Q_{u_i}, Q_{u_{i+1}}, \ldots, Q_{u_j})$.
- The scan-point s and every scanpoint on some edge of the cycle incident to a clique of $(Q_{u_i}, Q_{u_{i+1}}, \ldots, Q_{u_j})$, including the edge preceding Q_{u_i} and the edge after Q_{u_j}.

The symmetric operation is performed by permuting the role of $\mathcal{Q}^{up}$ and $\mathcal{Q}^{down}$.

It is not hard to check that by adding this set of diagonals we obtain indeed a linear triangulation $LT(\mathcal{Q})$ of the polygon of scanpoints. Each triangle T of $LT(\mathcal{Q})$ has exactly two semi-separator scanpoints and a clique-scanpoint. More precisely, for each triangle T the clique scan-point Q_T is incident to one of the semi-separator scanpoints. The other scanpoint is either incident to Q_T (if Q_T is the first or last clique of $\mathcal{Q}$) or it corresponds to an edge Q_aQ_b of the clique cycle, such that Q_T is between Q_a and Q_b in Q. Note that $Q_T \cup (Q_a \cap Q_b)$ is also a clique of the graph $H = \texttt{FillFolding}(G, Q)$, by the construction of the graph H. It follows that $w(LT(\mathcal{Q})) \leq \text{pwd}(H) + 1$. Conversely, consider any three cliques Q_a, Q_b, Q_T such that Q_T is between Q_a and Q_b in Q and $\{Q_a, Q_b\}$ is an edge of the clique cycle. Then $LT(\mathcal{Q})$ contains a diagonal D such that one of its end-points corresponds to Q_T and the other is the semi-separator scanpoint corresponding to the edge $\{Q_a, Q_b\}$. It follows that $Q_T \cup (Q_a \cap Q_b) \leq w(LT(\mathcal{Q}))$. By Lemma 4, every bag of the path decomposition of H, obtained by the folding algorithm, is of type $Q_T \cup (Q_a \cap Q_b)$ – and the conclusion follows. □

Theorem 5. *For any circular-arc graph G, its pathwidth is the minimum, over all linear planar triangulations LT of $\mathcal{P}_G$, of $w(LT) - 1$.*

Proof. By Lemma 5, we have that $\text{pwd}(G) \leq \min w(LT) - 1$. By Theorem 4, there is a 2-monotone folding (G, Q) of G such that $\text{pwd}(\texttt{FillFolding}(G, Q)) = \text{pwd}(G)$. The conclusion follows by Lemma 6. □

The algorithm for computing a linear triangulation of $\mathcal{P}_G$ of minimum width is very similar to the one of [9]. Due to space restrictions, its description is given in Appendix A.

Theorem 6. *The pathwidth of circular-arc graphs can be computed in $\mathcal{O}(n^2)$ time.*

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A Proof of Theorem: 6

It remains to give an algorithm computing a linear planar triangulation LT of $\mathcal{P}_G$, of minimum width. The algorithm is practically the same as in [9], except that in their case the planar triangulation is not necessarily linear and that the width function is slightly different. Therefore we only give a short description of the algorithm.

Observe that, for computing linear planar triangulations, we only need $\mathcal{O}(n^2)$ triangles. Indeed, for each triangle corresponding to a face, two endpoints are consecutive on the polygon. Assume first that we are given the widths of all the triangles of this type. Let $s_0, s_1, \cdots, s_{p-1}$ be the scanpoints of $\mathcal{P}_G$, ordered by the counter-clockwise orientation of the polygon. Let $w(i,j)$ the optimum width, over all linear triangulations $LT(i,j)$ of the polygon formed by the points $s_i, s_{i+1} \ldots, s_j$ (indices are considered modulo p). We point out that, if (s_i, s_j) is a diagonal of $\mathcal{P}_G$, then it is also considered as a diagonal of the restricted polygon, meaning that $LT(i,j)$ is not allowed to have a face with (s_i, s_j) and two other diagonals as edges.

If $j = i + 2$, then $w(i,j)$ is the width of the triangle (s_i, s_{i+1}, s_j). If $j > i + 2$, for using the diagonal (s_i, s_j) we must also use one of the diagonals (s_{i+1}, s_j) or (s_i, s_{j-1}). Therefore:

$$w(i,j) = \min \left(w(s_i, s_{i+1}, s_j) + w(i+1, j), w(s_i, s_{j-1}, s_j) + w(i, j-1) \right)$$

The optimum width of a linear triangulation is

$$w(\mathcal{P}_G) = \min_{0 \le i,j < p, j \ge i+2} \max(w(i,j), w(j,i))$$

All these equations are easily transformed into an $\mathcal{O}(n^2)$ dynamic programming algorithm for computing the linear planar triangulation of optimum width. Within the same time bounds, we can equally obtain, like in Lemma 5, an optimum path decomposition of the input graph.

The widths of the triangles can be computed in $\mathcal{O}(n^3)$, so the pathwidth of G can be computed in $\mathcal{O}(n^3)$. Following the principles of [9], we can improve this running time by computing the widths of the needed triangles in $\mathcal{O}(n^2)$. Each triangle T of a linear planar triangulation is of the type (s_i, s_{i+1}, s_j), with two consecutive scanpoints. One of the two, say s_i, is a clique scanpoint. Hence the width of T is $|V(s_j) \cup V(s_i)|$. Thus we only need to compute, for every couple $0 \le j < i < p$ the cardinality of $V(s_j) \cup V(s_i)$. During a preprocessing step we explore the clique-semiseparator cycle (in counter-clockwise order). For each scanpoint s_i, we distinguish two situations, depending whether s_i corresponds to a semi-separator or a clique. In the first case, s_i is a clique bag. We compute the arcs of $V(s_i) \setminus V(s_{i-1})$. Moreover, for each $j, 1 \le j < p$ let $Add(i,j)$ be the number of circular-arcs x of $V(s_i) \setminus V(s_{i-1})$ such that the right-end point $r(x)$ is between i and $j-1$ according to the cyclic order. Using a bucket sort, these quantities can be computed in $O(n)$ for each i. In a similar way, if i is a semi-separator scanpoint we take into account the vertices of $V(s_{i-1}) \setminus V(s_i)$. Let $Sub(i,j)$ the number of arcs $x \in V(s_{i-1}) \setminus V(s_i)$ such that the left-end point $l(x)$ is between $j+1$ and i in the cyclic order. Fix a value j, $0 \le j < p$. Consider each i, from $j+1$ to $j-1$ in cyclic order. The value $|V(s_j) \cup V(s_{j+1})|$ is computed directly. Now observe that if s_i is a clique scanpoint, we have $|V(s_j) \cup V(s_i)| = |V(s_j) \cup V(s_{i-1})| + Add(i,j)$. Indeed, the only arcs of $V(s_j) \cup (V(s_i) \setminus V(s_{i-1}))$ ar the arcs of $V(s_i) \setminus V(s_{i-1})$ whose right endpoint is strictly smaller that j in the cyclic order. Similarly, if s_i is a semi-separator scanpoint then $|V(s_j) \cup V(s_i)| = |V(s_j) \cup V(s_{i-1})| - Sub(i,j)$. Thus we need a constant time to compute $|V(s_j) \cup V(s_i)|$, and $\mathcal{O}(n^2)$ to compute the weights of the useful triangles.