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# *Treewidth and Minimum Fill-in: Grouping the Minimal Separators*

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# Treewidth and Minimum Fill-in: Grouping the Minimal Separators

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**Abstract**

We use the notion of potential maximal clique to characterize the maximal cliques appearing in minimal triangulations of a graph. We show that if these objects can be listed in polynomial time for a class of graphs, the treewidth and the minimum fill-in are polynomially tractable for these graphs. We prove that for all classes of graphs for which polynomial algorithms computing the treewidth and the minimum fill-in exist, we can list their potential maximal cliques in polynomial time. Our approach unifies these algorithms. Finally we show how to compute in polynomial time the potential maximal cliques of weakly triangulated graphs, for which the treewidth and the minimum fill-in problems were open.

**Keywords:** graph algorithms, treewidth, minimum fill-in, weakly triangulated graphs

**Résumé**

Nous utilisons la notion de clique maximale potentielle pour caractériser les cliques maximales qui apparaissent dans les triangulations minimales d'un graphe. Nous montrons que si ces objets peuvent être énumérés en temps polynomial pour une classe de graphes, la largeur arborescente et la complétion minimale de ces graphes se calculent aussi en temps polynomial. Nous prouvons que pour toutes les classes de graphes pour lesquels il existe des algorithmes polynomiaux calculant la largeur arborescente et la complétion minimale, nous pouvons générer toutes les cliques maximales potentielles en temps polynomial. Notre approche unifie ainsi ces algorithmes. Enfin, nous montrons comment calculer en temps polynomial les cliques maximales potentielles pour les graphes faiblement triangulés, pour lesquels les problèmes de la largeur arborescente et de la complétion minimale étaient ouverts.

**Mots-clés:** algorithmique des graphes, largeur arborescente, complétion minimale, graphes faiblement triangulés

# 1 Introduction

The notion of *treewidth* was introduced by Robertson and Seymour in [27]. It plays a major role in graph algorithm design. Indeed, it has been shown that many classical NP-hard problems become polynomial and even linear when restricted to graphs with small treewidth. These algorithms often use a tree decomposition or a *triangulation* of the input graph, which is a chordal supergraph, i.e. all the cycles with at least four vertices of the supergraph have a chord. Computing the treewidth consists in finding a triangulation of minimum cliquesize. A related problem is the *minimum fill-in* problem, which consists in finding a triangulation of a graph such that the number of added edges is minimum. This parameter is used in sparse matrix factorization.

Both the treewidth and the minimum fill-in problems are NP-complete. Nevertheless, these parameters can be computed in polynomial time for several classes of graphs such as chordal bipartite graphs [19, 8], circle and circular-arc graphs [15, 31, 23], AT-free graphs with polynomial number of separators [22]. Most of these algorithms use the fact that these classes of graphs have a polynomial number of *minimal separators*. It was conjectured in [17, 18] that the treewidth and the minimum fill-in should be tractable in polynomial time for all the graphs having a polynomial number of minimal separators. The conjecture is still open.

A potential maximal clique of a graph is a vertex set which induces a maximal clique in some minimal triangulation of the graph. Although this seems a purely combinatorial definition, a potential maximal clique corresponds to a local grouping of some minimal separators of the graph. This will lead to a local characterization of a potential maximal clique, and in particular to a polynomial algorithm that, given a graph $G$ and a vertex set $K$, decides if $K$ is a potential maximal clique of $G$. We also show in this paper that if one can list in polynomial time all the potential maximal cliques of some class of graphs, then the treewidth and the minimum fill-in of those graphs can be computed in polynomial time.

We prove that the potential maximal cliques can be enumerated in polynomial time for all the classes of graphs previously mentioned, unifying in this way the cited algorithms.

The class of *weakly triangulated graphs*, introduced in [12], is a class of graphs with polynomial number of separators, probably the only one for which the treewith and minimum fill-in problems were still open. We give an algorithm computing the potential maximal cliques of these graphs. Consequently, the treewidth and the minimum fill-in of weakly triangulated graphs are computable in polynomial time.

# 2 Chordal graphs and minimal separators

Throughout this paper we consider connected, simple, finite, undirected graphs.

A graph $H$ is *chordal* (or *triangulated*) if every cycle of length at least four has a chord, i.e. an edge between two non-consecutive vertices of the cycle. A *triangulation* of a graph $G = (V, E)$ is a chordal graph $H = (V, E')$ such that $E \subseteq E'$. $H$ is a *minimal triangulation* if for any intermediate set $E''$ with $E \subseteq E'' \subset E'$, the graph $(V, E'')$ is not triangulated. For example, the graph of figure 1b is a minimal triangulation of the graph of figure 1a.

Another characterization of minimal triangulations is provided in [28] :

**Lemma 2.1** *Let $H$ be a triangulation of a graph $G$. Then $H$ is a minimal triangulation of $G$ if and only if, for any edge $e$ of $E(H) - E(G)$, the graph $H - e$ is not triangulated.*

Let us define now the treewidth and the minimum fill-in of a graph.

**Definition 2.2** *Let $G$ be a graph. The* treewidth *of $G$, denoted by* $\mathrm{tw}(G)$*, is the minimum, over all triangulations $H$ of $G$, of $\omega(H) - 1$, where $\omega(H)$ is the the maximum cliquesize of $H$.*

**Definition 2.3** *The* minimum fill-in *of a graph $G$, denoted by* $\mathrm{mfi}(G)$*, is the smallest value of* $|E(H) - E(G)|$*, where the minimum is taken over all triangulations $H$ of $G$.*

In other words, computing the treewidth of $G$ means finding a triangulation with smallest cliquesize, while computing the minimum fill-in consists in finding a triangulation with smallest number of edges. In both cases we can restrict our work to minimal triangulations.

Let now $a$ and $b$ be two non adjacent vertices of a graph $G$. A vertex set $S \subseteq V$ is an *$a,b$-separator* if the removal of $S$ from the graph separates $a$ and $b$ in different connected components. $S$ is a *minimal $a,b$-separator* if no proper subset of $S$ separates $a$ and $b$. We say that $S$ is a *minimal separator* of $G$ if there are two vertices $a$ and $b$ such that $S$ is a minimal $a,b$ separator. Notice that a minimal separator can be strictly included in another one. We denote by $\Delta_G$ the set of all minimal separators of $G$.

Let us recall some known results about the minimal separators of a chordal graph. We will use the representation of chordal graphs provided by *clique trees*. For an extensive survey of these notions, see [2, 11]. A *clique* is a complete subgraph of $G$. Consider now the set $\mathcal{K}_G = \{\Omega_1, \ldots, \Omega_p\}$ of maximal cliques of $G$. Let $\mathcal{T}$ be a tree on $\mathcal{K}_G$, i.e. every maximal clique $\Omega \in \mathcal{K}_G$ corresponds to exactly one node of $\mathcal{T}$. We also say that the nodes of $\mathcal{T}$ are *labeled* by the cliques of $\mathcal{K}_G$ and we will simply denote by $\Omega$ the node of the tree labeled by a maximal clique $\Omega$. We say that $\mathcal{T}$ is a *clique tree* of $G$ if it satisfies the *clique-intersection property*: for every pair of distinct cliques $\Omega, \Omega' \in \mathcal{K}$, the set $\Omega \cap \Omega'$ is contained in every clique on the unique path connecting $\Omega$ and $\Omega'$ in the tree $\mathcal{T}$. It is well known (see [2] for a proof) that a graph $G$ is chordal if and only if it has a clique tree.

Figure 1c presents a clique tree of the chordal graph of figure 1b.

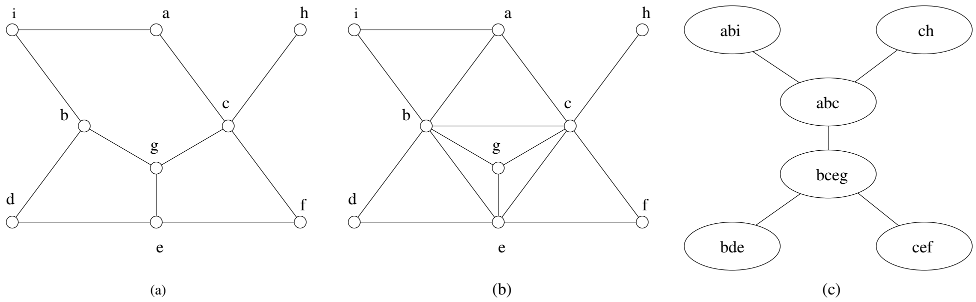


Figure 1: Minimal triangulation and clique tree

The following crucial property is proved in [2] :

**Proposition 2.4** *Let $H$ be a chordal graph and $\mathcal{T}$ be any clique tree of $H$. A vertex set $S$ is a minimal separator of $H$ if and only if $S = \Omega \cap \Omega'$ for some maximal cliques $\Omega$ and $\Omega'$ of $H$ adjacent in the clique tree $\mathcal{T}$.*

In particular, all minimal separators of a chordal graph are cliques. Actually, Dirac [9] has shown that a graph is chordal if and only if all its minimal separators are cliques.

The following proposition (see [2]) gives another relation between a minimal separator and a clique tree of $H$.

**Proposition 2.5** *Let $\mathcal{T}$ be any clique tree of a chordal graph $H$ and let $\Omega$, $\Omega'$ be two maximal cliques of $H$, adjacent in $\mathcal{T}$. Consider the two subtrees of $\mathcal{T}$ obtained by removing the edge between the nodes $\Omega$ and $\Omega'$. Let $\mathcal{T}_\Omega$ be the subtree containing $\Omega$ and $\mathcal{T}_{\Omega'}$ the subtree containing $\Omega'$. We denote by $V_\Omega$ and $V_{\Omega'}$ the union of the labels of $\mathcal{T}_\Omega$, respectively $\mathcal{T}_{\Omega'}$. Then the minimal separator $S = \Omega \cap \Omega'$ separates in $H$ every vertex of $V_\Omega - S$ from every vertex of $V_{\Omega'} - S$.*

Consider for example the chordal graph $H$ of figure 1b and its clique tree of figure 1c. If we take the adjacent cliques $\{a,b,c\}$ and $\{b,c,e,g\}$, then the minimal separator $\{b,c\}$ separates in $H$ every vertex of $\{a,h,i\}$ from every vertex of $\{d,e,f,g\}$.

Let $G$ be a graph and $S$ a minimal separator of $G$. We note $\mathcal{C}_G(S)$ the set of connected components of $G-S$. A component $C \in \mathcal{C}_G(S)$ is a *full* component associated to $S$ if every vertex of $S$ is adjacent to some vertex of $C$. We denote by $\mathcal{C}^*_G(S)$ the set of all full components associated to $S$. For the following lemma, we refer to [11].

**Lemma 2.6** *A set $S$ of vertices of $G$ is a minimal $a,b$-separator if and only if $a$ and $b$ are in different full components associated to $S$.*

If $C \in \mathcal{C}(S)$, we say that $(S,C) = S \cup C$ is a *block* associated to $S$. A block $(S,C)$ is called *full* if $C$ is a full component associated to $S$.

**Definition 2.7** *Two separators $S$ and $T$* cross*, denoted by $S \sharp T$, if $T$ intersects at least two distinct components of $G-S$. If $S$ and $T$ do not cross, they are called* parallel*, denoted by $S \| T$.*

It is easy prove that these relations are symmetric (see [25]). Remark that two minimal separators $S$ and $T$ are parallel if and only if $T$ is contained in some block $(S,C)$ associated to $S$.

Using the fact that a separator cannot separate two adjacent vertices we deduce the following lemma.

**Lemma 2.8** *Let $G$ be a graph, $S$ a minimal separator and $\Omega$ a clique of $G$. Then $\Omega$ is included in some block associated to $S$. In particular, the minimal separators of a chordal graph are pairwise parallel.*

Let $S \in \Delta_G$ be a minimal separator. We denote by $G_S$ the graph obtained from $G$ by completing $S$, i.e. by adding an edge between every pair of non-adjacent vertices of $S$. If $\Gamma \subseteq \Delta_G$ is a set of separators of $G$, $G_\Gamma$ is the graph obtained by completing all the separators of $\Gamma$. The results of [21], concluded in [26], establish a strong relation between the minimal triangulations of a graph and its minimal separators.

**Theorem 2.9** *Let $\Gamma \in \Delta_G$ be a maximal set of pairwise parallel separators of $G$. Then $H = G_\Gamma$ is a minimal triangulation of $G$ and $\Delta_H = \Gamma$.*

*Conversely, let $H$ be a minimal triangulation of a graph $G$. Then $\Delta_H$ is a maximal set of pairwise parallel separators of $G$ and $H = G_{\Delta_H}$.*

In other terms, every minimal triangulation of a graph $G$ is obtained by considering a maximal set $\Gamma$ of pairwise parallel separators of $G$ and completing the separators of $\Gamma$. The minimal separators of the triangulation are exactly the elements of $\Gamma$. For example, if $G$ is the graph of figure 1a and $H$ is the graph of figure 1b, then $\Gamma = \{\{a,b\},\{b,c\},\{c\},\{b,e\},\{c,e\}\}$.

It is important to know that the elements of $\Gamma$, who become the minimal separators of $H$, have strictly the same behavior in $H$ as in $G$. Indeed, the connected components of $H-S$ are exactly the same in $G-S$, for every $S \in \Gamma$. Moreover, the full components are the same in the two graph: $\mathcal{C}^*_H(S) = \mathcal{C}^*_G(S)$.

# 3 Potential maximal cliques

The previous theorem gives a characterization of the minimal triangulations of a graph by means of minimal separators, but it gives no algorithmic information about how we should construct a minimal triangulation in order to minimize its cliquesize or the fill-in. We will prove that the *potential maximal cliques* of a graph suffice to compute its treewidth and its minimum fill-in.

**Definition 3.1** *A vertex set* $\Omega$ *of a graph* $G$ *is called a* potential maximal clique *if there is a minimal triangulation* $H$ *of* $G$ *such that* $\Omega$ *is a maximal clique of* $H$.

In this section, we give several characterizations of the potential maximal cliques of a graph, which will allow us to recognize a potential maximal clique in polynomial time and also to enumerate these objects for several classes of graphs.

### 3.1 Potential maximal cliques and minimal separators

If $K$ is a vertex set of $G$, we denote by $\Delta_G(K)$ the minimal separators of $G$ included in $K$. Our aim is to give a strong relation between a potential maximal clique $\Omega$ and the minimal separators of $\Delta_G(\Omega)$.

**Definition 3.2** *Let* $G$ *be a graph and* $\mathcal{S} \subseteq \Delta_G$ *a set of pairwise parallel separators such that for any* $S \in \mathcal{S}$, *there is a block* $(S, C(S))$ *containing all the separators of* $\mathcal{S}$. *Suppose that* $\mathcal{S}$ *ordered by inclusion has no greatest element. We define the* piece between *the elements of* $\mathcal{S}$ *by*

$$P(\mathcal{S}) = \bigcap_{S \in \mathcal{S}} (S, C(S))$$

Notice that for any $S \in \mathcal{S}$ the block $(S, C(S))$ containing all the separators of $\mathcal{S}$ is unique : if $T \in \mathcal{S}$ is not included in $S$, there is a unique connected component of $S$ containing $T - S$.

**Lemma 3.3** *Let* $H$ *be a chordal graph and let* $\Omega$ *be a maximal clique of* $H$. *Then either* $\Delta_H(\Omega)$ *has a greatest element, or* $P(\Delta_H(\Omega))$ *exists and contains* $\Omega$.

*Proof.* According to lemma 2.8, for any minimal separator $S \in \Delta_H(\Omega)$, the clique $\Omega$ is contained in some bloc $(S, C(S))$ of $S$. It follows that if $\Delta_H(\Omega)$ has no greatest element, then $\Omega$ is contained in $P(\Delta_H(\Omega))$, by definition of the piece between. $\diamond$

**Theorem 3.4** *Let* $\Omega$ *be a maximal clique of a chordal graph* $H$. *If all the separators of* $\Delta_H(\Omega)$ *are contained in some* $S \in \Delta_H(\Omega)$, *then* $\Omega$ *is a block* $(S, C)$ *associated to* $S$. *Otherwise,* $\Omega = P(\Delta_H(\Omega))$.

*Proof.* Suppose that $\bigcup_{T \in \Delta_H(\Omega)} T = S$ with $S \in \Delta_H(\Omega)$. $\Omega$ is included in some block $(S, C)$ of $S$. If $x \in C$ is not in $\Omega$, we take in a clique tree of $H$ a path $\Omega, \Omega', \ldots, \Omega^{(i)}$ of adjacent cliques such that $x \in \Omega^{(i)}$. Then $T = \Omega \cap \Omega'$ is a minimal separator of $H$ included in $\Omega$, so it must be an element of $\Delta_H(\Omega)$. In particular, $T$ is included in $S$. According to proposition 2.5, $T$ separates $x$ and any vertex of $\Omega - T$. Since $T \subseteq S$, clearly $S$ separates $x$ and any vertex of $\Omega - S$, contradicting the fact that $x$ and $\Omega$ are in the same block associated to $S$. It follows that $(S, C) = \Omega$.

Suppose now that no separator $S \in \Delta_H(\Omega)$ contains all the others. According to the lemma 3.3, $P(\Delta_H(\Omega))$ exists. On one hand $\Omega \subseteq P(\Delta_H(\Omega))$ by lemma 3.3. On the other hand, consider $y \in P(\Delta_H(\Omega)) - \Omega$. In a clique tree of $H$, we consider a path $\Omega, \Omega', \ldots, \Omega^{(i)}$ of adjacent cliques such that $y \in \Omega^{(i)}$. Then $S = \Omega \cap \Omega'$ is a minimal separator that belongs to $\Delta_H(\Omega)$ and $S$ separates $y$ from every vertex of $\Omega - S$ by proposition 2.5. Let $T$ be a separator of $\Delta_H(\Omega)$, not included in $S$. Then $S$ separates $y$ and a vertex of $T - S$, so $y$ is not in the bloc $(S, C(S))$ containing $\Omega$, contradicting our choice. It follows that $\Omega = P(\Delta_H(\Omega))$. $\diamond$

Theorem 3.4 gives a relation between a maximal clique of a chordal graph and the minimal separators contained in the clique. We extend this result to the potential maximal cliques of a graph. We also establish the reverse of the theorem 3.4, which will allow us to recognize a potential maximal clique. For this we need some easy lemmas.

**Lemma 3.5** *Let $H$ be a minimal triangulation of a graph $G$ and $T$ be a minimal separator of $G$ such that $H[T]$ is a clique. Then $T$ is also a minimal separator of $H$.*

*Proof.* We know that $H = G_\Gamma$ for a maximal set of pairwise parallel separators $\Gamma \subseteq \Delta_G$ and $\Delta_H = \Gamma$ (theorem 2.9). Let $S \in \Gamma$ be any minimal separator of $H$ and $G$. By lemma 2.8, the clique $T$ will be in some block $(S, C)$ of $H$. Since the blocks associated to $S$ in $G$ are the same as in $H$ (theorem 2.9 and the related remarks) we deduce that $S$ and $T$ are parallel in $G$. Since $\Gamma$ is a *maximal* set of pairwise parallel separators, $T$ must be in $\Gamma = \Delta_H$. $\diamond$

**Corollary 3.6** *In particular, if $T$ is a minimal separator of $G$ contained in $S$ with $S \in \Delta_H$, then $T \in \Delta_H$.*

Consider a graph $G$ and a minimal triangulation $H$ of $G$. An important consequence of lemma 3.5 is that if $\Omega$ is a clique of $H$, then the minimal separators of $H$ contained in $\Omega$ are the same as in $G$.

**Proposition 3.7** *Let $G$ be a graph and $H$ be a minimal triangulation of $G$. Then for any clique $\Omega$ of $H$, we have $\Delta_H(\Omega) = \Delta_G(\Omega)$.*

*Proof.* By theorem 2.9, every minimal separator of $H$ is also a minimal separator of $G$, so $\Delta_H(\Omega) \subseteq \Delta_G(\Omega)$. According to lemma 3.5, since every minimal separator $S \in \Delta_G(\Omega)$ of $S$ induces a clique in $H$, $S$ is also a minimal separator of $H$. $\diamond$

The next step is to prove that if $\Omega$ is a maximal clique of a minimal triangulation $H$ of $G$, then $\Omega$ is also a clique in the graph $G_{\Delta_G(\Omega)}$, obtained by completing in $G$ the minimal separators included in $\Omega$.

**Lemma 3.8** *Let $H$ be a chordal graph, $\Omega$ a maximal clique and $S$ a minimal separator of $H$ intersecting $\Omega$. Then $S \cap \Omega$ is contained in some minimal separator $T$ with $T \subset \Omega$.*

*Proof.* Consider a clique tree $\mathcal{T}$ of $H$. We can write $S = \Omega_1 \cap \Omega_2$ for some maximal cliques $\Omega_1, \Omega_2$ of $H$, adjacent in $\mathcal{T}$ (proposition 2.4). Let us suppose that $\Omega_1$ is closer to $\Omega$ in the clique tree than $\Omega_2$. Let $\Omega'$ be the clique next to $\Omega$ on the path from $\Omega$ to $\Omega_2$ in $\mathcal{T}$, i.e. the path is $\Omega, \Omega', \ldots, \Omega_1, \Omega_2$.

Then $\Omega_1 \cap \Omega \subset \Omega'$ and $\Omega_2 \cap \Omega \subset \Omega'$ by definition of a clique tree. Since $S = \Omega_1 \cap \Omega_2$, we deduce that $S \cap \Omega \subset \Omega'$ and in particular $S \cap \Omega$ is contained in $\Omega \cap \Omega'$, which is a minimal separator by property 2.4. $\diamond$

**Proposition 3.9** *Let $\Omega$ be a potential maximal clique of $G$. Then $\Omega$ is a clique in the graph $G_{\Delta_G(\Omega)}$.*

*Proof.* Let $H$ be a minimal triangulation of $G$ such that $\Omega$ is a maximal clique of $H$. Notice that $H$ exists by definition of a potential maximal clique. By theorem 2.9, $H = G_{\Delta_H}$, so for any edge $\{x, y\} \in E(H) - E(G)$, there is a minimal separator of $H$ containing both $x$ and $y$. Let now $\{x, y\}$ be an edge of $H[\Omega]$, we want to prove that $\{x, y\}$ is also an edge of $G_{\Delta_G(\Omega)}[\Omega]$. If $\{x, y\} \in E(G)$, the assertion is clearly true. Otherwise, let $S$ be a minimal separator of $H$ such that $x, y \in S$. By lemma 3.8, there is a minimal separator $T$ of $H$, contained in $\Omega$ and containing $x$ and $y$. We deduce that $\{x, y\}$ is also an edge of $G_{\Delta_G(\Omega)}$. $\diamond$

We can give now a characterization of the potential maximal cliques $\Omega$ of a graph, using the minimal separators of $\Delta_G(\Omega)$.

**Theorem 3.10** *Let $\Omega$ be a vertex set of $G$ and suppose that $\Delta_G(\Omega)$ has a maximum element $S$, i.e. every $T$ in $\Delta_G(\Omega)$ is included in $S$. Then $\Omega$ is a potential maximal clique if and only if $\Omega$ is some block $(S, C)$ and $G_{\Delta_G(\Omega)}[\Omega]$ is a clique.*

*Proof.*

"$\Rightarrow$" Let $H$ be a minimal triangulation of $G$ such that $\Omega$ is a maximal clique of $H$. By proposition 3.7, $\Delta_H(\Omega) = \Delta_G(\Omega)$ so $S$ is an element of $\Delta_H(\Omega)$, maximum by inclusion. According to theorem 3.4, $\Omega$ is a bloc $(S, C)$ of $S$ in $H$. Since the blocks of $S$ are the same in $H$ and in $G$, $(S, C)$ is also a block in the graph $G$.

By proposition 3.9, $\Omega$ is a clique in the graph $G_{\Delta_G(\Omega)}$. Notice that $G_{\Delta_G(\Omega)}$ is identical to $G_S$.

"$\Leftarrow$" Notice that the separators of $\Delta_G(\Omega)$ are pairwise parallel. Indeed, let $H$ be a minimal triangulation of $G$ such that $S \in \Delta_H$. Then each $T \in \Delta_G(\Omega)$ is a clique in $H$, because $T \subseteq S$. Consequently $\Delta_G(\Omega) \subseteq \Delta_H$, so by theorem 2.9, the elements of $\Delta_G(\Omega)$ are pairwise parallel in $G$.

We prove that $S \cup C$ is a maximal clique of $H$. Let $\Omega'$ be clique of $H$ including $S \cup C$, $\Omega'$ must be in some block associated to $S$ in $H$ (lemma 2.8) and the only choice is $\Omega \subseteq (S, C)$. We conclude that $\Omega$ is a maximal clique of the minimal triangulation $H$ of $G$, and therefore $\Omega$ is a potential maximal clique of $G$. $\diamond$

**Theorem 3.11** *Let $\Omega$ be a vertex set of $G$ and suppose that $\Delta_G(\Omega)$ ordered by inclusion has no greatest element. Then $\Omega$ is a potential maximal clique if and only if $\Omega = P(\Delta_G(\Omega))$ and $G_{\Delta_G(\Omega)}[\Omega]$ is a clique.*

*Proof.* The proof is very similar to the one of the previous theorem.

"$\Rightarrow$" We consider a minimal triangulation $H$ of $G$ such that $\Omega$ is a maximal clique of $H$. According to theorem 3.4, $\Omega = P(\Delta_H(\Omega))$ in $H$. By proposition 3.7, $\Delta_H(\Omega) = \Delta_G(\Omega)$ and the minimal separators of $\Delta_H(\Omega)$ induce the same connected components in $G$ and in $H$. Consequently, $P(\Delta_H(\Omega))$ is the same as $P(\Delta_G(\Omega))$ in $G$, so $\Omega = P(\Delta_G(\Omega))$ in the graph $G$.

By proposition 3.9, $\Omega$ is a clique in $G_{\Delta_G(\Omega)}$.

"$\Leftarrow$" Since $P(\Delta_G(\Omega))$ exists, the minimal separators of $\Delta_G(\Omega)$ are pairwise parallel in $G$. As in the previous theorem, we prove that $\Omega$ is a maximal clique in any minimal triangulation $H = G_\Gamma$ of $G$ with $\Delta_G(\Omega) \subseteq \Gamma$.

Since $\Omega$ is a clique in $G_{\Delta_G(\Omega)}$, $\Omega$ is also a clique in $H$. Consider a maximal clique $\Omega'$ of $H$ containing $\Omega$. Let $x$ be a vertex of $\Omega'$. For any $S \in \Delta_H(\Omega)$, $x$ must be in the block $(S, C_\Omega(S))$ of $H$ containing $\Omega$. So $x \in P(\Delta_H(\Omega))$ in the graph $H$. But $\Delta_H(\Omega) = \Delta_G(\Omega)$ by proposition 3.7, and the piece between these separators is the same in $H$ and in $G$. It follows that $x \in P(\Delta_G(\Omega)) = \Omega$, so $\Omega$ is a maximal clique of $H$. $\diamond$

Notice that theorems 3.10 and 3.11 lead to an algorithm that, given a graph $G$, its minimal separators $\Delta_G$ and a vertex set $K$, verifies in polynomial time if $K$ is a potential maximal clique of $G$.

## 3.2 A simpler characterization of potential maximal cliques

We are going to give here a strong characterization of potential maximal cliques, which does not use minimal separators. We start with some easy observations, following directly from the previous results.

**Corollary 3.12** *Let $\Omega$ be a potential maximal clique of $G$ and let $S \in \Delta_G(\Omega)$. Then $S$ is strictly contained in $\Omega$ and $\Omega - S$ is in a full connected component associated to $S$.*

*Proof.* Consider a minimal triangulation $H$ of $G$ such that $\Omega$ is a maximal clique of $H$. By lemma 3.5, $S$ is a minimal separator of $H$. $S$ is an intersection of two distinct maximal cliques of $H$ by proposition 2.4, so $S$ cannot be a maximal clique of $H$. It follows that $S$ is strictly contained in $\Omega$.

Let us prove now that $\Omega - S$ is in a full connected component associated to $S$ in $G$. Clearly $\Omega - S$ is in a full component associated to $S$ in $H$. Since the full components associated to $S$ in $G$ are the same as in $H$, we conclude that $\Omega - S$ is in a full component associated to $S$ in $G$. $\diamond$

**Corollary 3.13** *Let $\Omega$ be a potential maximal clique of a graph $G$ and let $a$ be any vertex of $V - \Omega$. There is a minimal separator $S \subset \Omega$ that separates $a$ and $\Omega - S$.*

*Proof.* If we are in the case of theorem 3.10, $\Delta_G(\Omega)$ has a greatest element $S$ and $\Omega = (S, C)$. Then clearly $S$ separates any vertex $a \in V - (S, C)$ from any vertex of $C = \Omega - S$. Suppose now that we are under the conditions of theorem 3.11, so $\Delta_G(\Omega)$ has no greatest element and $\Omega = P(\Delta_G(\Omega))$. By definition of $P(\Delta_G(\Omega))$, since $a \notin P(\Delta_G(\Omega))$ we deduce that there is some $S \in \Delta_G(\Omega)$ separating $a$ from $P(\Delta_G(\Omega)) - S$. $\diamond$

Let now $K$ be a set of vertices of a graph $G$. We denote by $C_1(K), \ldots, C_p(K)$ the connected components of $G - K$. We denote by $S_i(K)$ the vertices of $K$ adjacent to at least one vertex of $C_i(K)$. When no confusion is possible we will simply speak of $C_i$ and $S_i$. If $S_i(K) = K$ we say that $C_i(K)$ is a *full component* associated to $K$.

**Lemma 3.14** *Let $\Omega$ be a potential maximal clique of a graph $G$ and let $\Delta_G(\Omega)$ be the set of all minimal separators contained in $\Omega$. Then the elements of $\Delta_G(\Omega)$ are exactly the sets $S_i(\Omega)$.*

*Proof.* We prove that for any $i$, $1 \leq i \leq p$, $S_i$ is a minimal $a, b$-separator for some $a \in C_i$ and $b \in \Omega - S_i$. Corollary 3.13 tells us that there is some minimal separator $S \in \Delta_G(\Omega)$ that separates $a$ from $\Omega - S$; recall that $\Omega - S$ is not empty. Since every vertex in $S_i$ has a neighbor in $C_i$, if $S$ does not contain a vertex $x \in S_i$, $S$ cannot separate $x \in \Omega - S$ from $a$ so we get $S_i \subseteq S$. By corollary 3.12, $\Omega - S$ is in a full component associated to $S$, and therefore of $S_i$. Let $b$ be a vertex of $\Omega - S$. Then $a$ and $b$ are in different full components associated $S_i$, so $S_i$ is a minimal $a, b$-separator by lemma 2.6.

We have to prove now that for any minimal separator $S \subseteq \Omega$, there is some $i, 1 \leq i \leq p$ such that $S = S_i$. We have that $\Omega - S \neq \emptyset$ and $\Omega - S$ is in some full component associated to $S$. Let $C$ be another full component associated to $S$. Then $C$ is a connected component of $G - \Omega$, let us say $C_i$. It follows that $S \subseteq S_i$. Suppose there exists a vertex $x \in S_i - S$. Then $x$ has a neighbor in $C_i$, so $x$ must be in the connected component $C$ of $G - S$, contradicting $C = C_i$. It remains that $S = S_i$. We conclude that the separators of $G$ included in $\Omega$ are exactly the sets $S_i$. $\diamond$

**Remark 3.15** *Actually, each minimal separator $S_i$ separates any vertex of $C_i$ from any vertex of $G - (S_i, C_i)$.*

We also give a "sufficient condition" to characterize the potential maximal cliques, which is somehow the dual of lemma 3.14.

**Theorem 3.16** *Let $K \subseteq V$ be a set of vertices. We denote by $\mathcal{S}$ the set of all $S_i(K)$. $K$ is a potential maximal clique if and only if :*

1. *$G - K$ has no full components associated to $K$.*
2. *$G_{\mathcal{S}}[K]$ is a clique.*

*Proof.* We prove the "only if" part. Suppose that $K$ is a potential maximal clique of $G$. By lemma 3.14, $\mathcal{S} = \Delta_G(K)$. By theorems 3.10 and 3.11, $K$ is a clique in the graph $G_{\mathcal{S}}$. It remains to show that $G - K$ has no full components associated to $K$. Let $C_i$ be any connected component of $G - K$. Then $S_i$ is the neighborhood of $C_i$ in $K$. Since $K$ is a potential maximal clique and $S_i$ is a separator contained in $K$, we have that $S_i$ is strictly contained in $K$, by corollary 3.12. Therefore, $C_i$ is not a full component associated to $K$.

We prove now the "if" part. Let us show at first that for any $i, 1 \leq i \leq p$, $S_i$ is a minimal separator. $S_i$ is clearly a separator and $C_i$ is a full component associated to $S_i$. Let $x$ be a vertex of $K - S_i$. We show that $x$ belongs to a full component associated to $S_i$ and different from $C_i$. We denote by $C_x$ the connected component of $G - S_i$ containing $x$. For any $y \in S_i$, $y$ must have a neighbor in $C_x$. This is true if $x$ and $y$ are adjacent in $G$. If $x$ and $y$ are not adjacent, by the second condition of the theorem, $x$ and $y$ belong to a same $S_j$. $C_j$ being a full component associated to $S_j$, there is a path in $G$ connecting $x$ to $y$ entirely contained in $C_j$ except from $x$ and $y$, we deduce that $C_j \subseteq C_x$. It follows that $y$ has a neighbor in $C_x$ since it has a neighbor in $C_j$. $S_i$ is a minimal separator of $G$ according to lemma 2.6.

Now, given two distinct separators $S_i$ and $S_j$, we have to show that they are parallel. We prove that $K - S_i$ is in a connected component of $G - S_i$. Let $x, y \in K - S_i$. If $x$ and $y$ are adjacent they are clearly in the same component of $G - S_i$. Otherwise, since $G_{\mathcal{S}}[K]$ is a clique, they are in a same $S_k$, so they are connected via $C_k$. So $S_j$ intersects only the component of $G - S_i$ containing $K - S_i$ and consequently $S_i \| S_j$. Therefore $\mathcal{S}$ is a set of pairwise parallel minimal separators.

We have to show that any separator of $G$ included in $K$ is an element of $\mathcal{S}$. Consider a minimal triangulation $H$ of $G$ such that all the elements of $\mathcal{S}$ are minimal separators of $H$. We know that $K$ is a clique in $H$. Now let $U \subseteq K$ be any minimal separator of $G$. Notice that $U$ must be strictly included in $K$, otherwise $G - K$ would have two full components associated to $K$ in $G$, contradicting our choice of $K$. Clearly $U$ is a clique in $H$, so by lemma 3.5, it is a minimal separator of $H$. Since $K$ is a clique in $H$, it must be included in some full block associated to $U$. Let $(U, C)$ be another full block associated to $U$ in $H$, and consequently in $G$. We have that $C$ is a connected component of $G - U$ and $U$ separates $C$ and $K - U$. We deduce that $C$ is also a connected component of $G - K$, let us say $C_i$. By definition of $S_i$, we have $U \subseteq S_i$. Suppose there exists a vertex $x \in S_i - U$, since $x$ has a neighbor in $C$, the connected component $C$ of $G - U$ would contain $x$ contradicting $C = C_i$. So we have $U = S_i$ and $U \in \mathcal{S}$.

We want to prove that $K$ and $\mathcal{S} = \Delta_G(K)$ satisfy the conditions of theorems 3.10 or 3.11. Remark that for any $y \in V - K$, $y$ is in some connected component $C_i$ of $G - K$ and the separator $S_i \in \mathcal{S}$ separates $y$ from $K - S_i$. Suppose now that $\mathcal{S}$ has an element $S$, maximum by inclusion. Let $(S, C)$ be the block associated to $S$ containing $K$. By the previous remark, for any $y \in V - K$, $S$ separates $y$ and $K - S$, so $y \notin (S, C)$. It follows that $(S, C) = K$, so $K$ satisfies all the conditions of theorem 3.10. Now if $\mathcal{S}$ does not have an element maximum by inclusion, clearly $K$ is contained in the piece between the separators of $\mathcal{S}$ in $G$. By the previous remark, $P_G(\mathcal{S})$ does not contain any $y \in V - K$, so $K = P_G(\mathcal{S})$ and therefore we are under the conditions of theorem 3.11. It follows that $K$ is a potential maximal clique of $G$. ◇

**Corollary 3.17** *There is an algorithm that, given a graph $G = (V, E)$ and a vertex set $K \subseteq V$, verifies if $K$ is a potential maximal clique of $G$. The time complexity of the algorithm is $\mathcal{O}(n^3)$, where $n$ is the number of vertices of $G$.*

*Proof.* The algorithm computes the connected components $C_i$ of $G - K$ and their neighborhoods $S_i$. It checks then the two conditions of theorem 3.16. Remark that computing the sets $C_i, S_i$ and verifying that $G - K$ has no full components associated to $K$ can be done in linear time. We can complete each set $S_i$ in $\mathcal{O}(n^2)$ steps, and therefore computing the graph $G_{\mathcal{S}}[K]$ takes $\mathcal{O}(n^3)$ time. ◇

# 4 Triangulating blocks

In this section we prove that the potential maximal cliques of a graph are sufficient to compute its treewidth and its minimum fill-in.

Let $B = (S, C)$ be a block of the graph $G$. The graph $R(S, C) = G_{\mathcal{S}}[S \cup C]$ is called the *realization* of the block $B$. The following lemma, proved in [22], gives a relation between minimal triangulation of a graph and minimal triangulations of some block realizations.

**Lemma 4.1** *Let $S \in \Delta_G$ and let $C_1, C_2, \ldots, C_p$ be the connected components of $G - S$. Suppose that $H_i$ is a minimal triangulation of $R(S, C_i)$ for any $i, 1 \le i \le p$. Then the graph $H = (V, E(H))$ with $E(H) = \bigcup_{i=1}^{p} E(H_i)$ is a minimal triangulation of $G$.*

*Conversely, let $H$ be a minimal triangulation of $G$ with $S \in \Delta_H$. Then $H[S \cup C]$ is a minimal triangulation of the realization $R(S, C)$ for each component $C$ of $G - S$.*

This gives an equation for computing the treewidth and the minimum fill-in of a graph (see [22] for a proof):

**Corollary 4.2** *Let $G$ be a non-complete graph. Then*

$$\operatorname{tw}(G) = \min_{S \in \Delta_G} \max_{C \in \mathcal{C}(S)} \operatorname{tw}(R(S, C))$$

$$\operatorname{mfi}(G) = \min_{S \in \Delta_G} (\operatorname{fill}(S) + \sum_{C \in \mathcal{C}(S)} \operatorname{mfi}(R(S, C)))$$

*where* $\operatorname{fill}(S)$ *is the number of non-edges of $S$.*

We give a version of lemma 4.1 using potential maximal cliques instead of minimal separators. If $\Omega$ is a potential maximal clique of a graph $G$, we denote as usual by $C_i, 1 \le i \le p$ the connected components of $G - \Omega$ and by $S_i, 1 \le i \le p$ the neighborhood of each $C_i$. We say that the blocks $(S_i, C_i)$ are the *blocks associated to* $\Omega$ in $G$.

**Theorem 4.3** *Let $H$ be a minimal triangulation of $G$ and let $\Omega$ be a maximal clique of $H$. Then for each block $(S_i, C_i)$ associated to $\Omega$ in $G$, the graph $H_i = H[S_i \cup C_i]$ is a minimal triangulation of the realization $R(S_i, C_i)$.*

*Conversely, let $\Omega$ be a potential maximal clique of $G$. For each block $(S_i, C_i)$ associated to $\Omega$ in $G$, let $H_i$ be a minimal triangulation of $R(S_i, C_i)$. Then $H = (V, E(H))$ with $E(H) = \bigcup_{i=1}^{p} E(H_i) \cup \{\{x, y\} | x, y \in \Omega\}$ is a minimal triangulation of $G$.*

*Proof.* For proving the first part, notice that by theorem 3.16, $(S_i, C_i)$ are blocks of $G$ and of $H$. Then by lemma 4.1, $H[S_i \cup C_i]$ are minimal triangulations of $R(S_i, C_i)$.

We prove now the second part. Let us prove that $H$ is a triangulated graph. Suppose that $H$ has a chordless cycle of length at least four. Since $\Omega$ is a clique, the cycle is not contained in $\Omega$, so it has a vertex $a$ in some component $C_i$. Since $H[S_i \cup C_i]$ is chordal, the cycle is not contained in $(S_i, C_i)$. Then at least two non consecutive points of the cycle, say $b$ and $c$, are in $S_i$. So the cycle has a chord, namely the edge $\{b, c\}$.

Let us prove now that $H$ is a minimal triangulation of $G$. Let $e = \{x, y\}$ be an edge of $H - G$. Suppose at first that $x$ and $y$ are not both contained in $\Omega$. Then $x$ and $y$ are in some $H_i$. Since $e$ is not in $\Omega$, $e$ is not in $S_i$. It remains that $e$ is an edge of $H_i - R(S_i, C_i)$. By lemma 2.1, $H_i - e$ is not chordal, so $H - e$ is not chordal. Suppose now that $x, y \in \Omega$. Then $x, y$ must be in some $S_i$, since $H[\Omega] = G_{\Delta_G(\Omega)}[\Omega]$ and $e \in H - G$. Consider two full components of $H - S_i$ associated to $S_i$ and a shortest path from $x$ to $y$ in each of them. The two paths form a cycle of length at least four, which is a chordless cycle of $H - e$.

By lemma 2.1, $H$ is a minimal triangulation of $G$. $\diamond$

We want to give a characterization of the minimal triangulations of a realization $R(S, C)$ using the potential maximal cliques $\Omega$ with $S \subset \Omega$ and $\Omega \subseteq (S, C)$ and the minimal triangulations of the realizations of some blocks $(S_i, C_i)$, strictly included in $(S, C)$. This will express the treewidth and the minimum fill-in of a realization from the treewidth, respectively the minimum fill-in of realizations of smaller blocks and we will compute the two parameters by dynamic programming on blocks.

The minimal triangulations of the realizations of non-full blocks are easy reducible to the case of full blocks. For the following lemma, see for example [7] :

**Lemma 4.4** *Let $(S,C)$ be a non-full block of $G$ and let $S^*$ be the set of vertices of $S$ having some neighbor in $C$. Then $(S^*,C)$ is a full block of $G$ and a super graph $H$ of $R(S,C)$ is a minimal triangulation of $R(S,C)$ if and only if $H[S^* \cup C]$ is a minimal triangulation of $R(S^*,C)$.*

We conclude :

**Corollary 4.5** *Let $(S,C)$ be a non-full block of $G$ and let $S^*$ be the vertices of $S$ adjacent in $G$ to at least one vertex of $C$. Then*

$$\mathrm{tw}(R(S,C)) = \max(|S|-1, \mathrm{tw}(R(S^*,C)))$$

$$\mathrm{mfi}(R(S,C)) = \mathrm{mfi}(R(S^*,C))$$

It remains to express the treewidth and the minimum fill-in of realizations of full blocks from realizations of smaller blocks.

**Lemma 4.6** *Let $R(S,C)$ be the realization of some full block $(S,C)$ and let $H(S,C)$ be a minimal triangulation of $R(S,C)$. Then there is a maximal clique $\Omega$ of $H(S,C)$ such that $S \subset \Omega$ and $\Omega$ is a potential maximal clique of $G$.*

*Proof.* Clearly $S$ is a clique in $H(S,C)$. Consider a maximal clique $\Omega$ of $H(S,C)$, containing $S$. Let us show that $S$ is strictly contained in $\Omega$. $C$ is a full component associated to $S$ in $G$, so also in $H(S,C)$. If $S=\Omega$, then $C$ is a full component associated to $\Omega$ in $H(S,C)$. But $\Omega$ is also a potential maximal clique of the chordal graph $H(S,C)$, so by theorem 3.16, $H(S,C)-\Omega$ cannot have full components associated to $\Omega$, which leads to a contradiction. It remains to show that $\Omega$ is a potential maximal clique of $G$. By lemma 4.1, there is a minimal triangulation $H$ of $G$ such that $H[S\cup C] = H(S,C)$. The components associated to $S$ in $H$ are the same as in $G$. Since $S$ separates $C$ from $V-(C\cup S)$, $\Omega$ is a maximal clique of $H$ so by definition it is a potential maximal clique of $G$. $\diamond$

Notice that we have proved that any maximal clique of $H(S,C)$ is a potential maximal clique of $G$ and, moreover, it is a maximal clique of any minimal triangulation $H$ with $H[S\cup C] = H(S,C)$.

By theorem 4.3 and lemma 4.6 we have proved:

**Theorem 4.7** *Let $(S,C)$ be a full block of a graph $G$. Then $H(S,C)$ is a minimal triangulation of $R(S,C)$ if and only if there is a potential maximal clique $\Omega \subseteq (S,C)$ of $G$ such that $S \subset \Omega$ and $H(S,C) = (S\cup C, E(H))$ with $E(H) = \bigcup_{i=1}^{p} E(H_i) \cup \{\{x,y\} | x,y \in \Omega\}$, where $(S_i,C_i)$ are the blocks associated to $\Omega$ in $H(S,C)$ and $H_i$ are minimal triangulations of $R(S_i,C_i)$.*

**Corollary 4.8** *Let $(S,C)$ be a full block of $G$. Then*

$$\mathrm{tw}(R(S,C)) = \min_{S\subset\Omega\subseteq(S,C)} \max(|\Omega|-1, \mathrm{tw}(R(S_i,C_i)))$$

$$\mathrm{mfi}(R(S,C)) = \min_{S\subset\Omega\subseteq(S,C)} \left(\mathrm{fill}(\Omega) - \mathrm{fill}(S) + \sum \mathrm{mfi}(R(S_i,C_i))\right)$$

*where $(S_i,C_i)$ are the blocks associated to $\Omega$ in $R(S,C)$.*

We give in table 1 a sketch of the algorithm that, given a graph and the list of all its potential maximal cliques, computes, by standard dynamic programming techniques, the treewidth and the minimum fill-in of the graph. The first part of the algorithm computes the treewidth and the minimum fill-in of the realization of each block $(S,C)$ of $G$. For computing the treewidth and the minimum fill-in of a realization $R(S,C)$, all we need are the potential maximal cliques

```
Input: G and all its potential maximal cliques
Output: tw(G) and mfi(G)
begin
    compute all the blocks (S,C) and sort them by the number of vertices
    for each block (S,C) taken in increasing order
        if (S,C) is not full then
            compute the neighborhood S* of C
            compute tw(R(S,C)) and mfi(R(S,C)) using tw(R(S*,C))
            and mfi(R(S*,C))
        else {the block (S,C) is full}
            for all p.m.c. Ω with S ⊂ Ω ⊆ (S,C)
                compute the blocks (S_i,C_i) associated to Ω in R(S,C)
            compute tw(R(S,C)) and mfi(R(S,C)) using tw(R(S_i,C_i))
            and mfi(R(S_i,C_i))
        end_if
    end_for
    compute tw(G) and mfi(G) using the treewidth and the minimum
    fill-in of the realizations of its blocks
end
```

Table 1: Algorithm computing the treewidth and the minimum fill-in of a graph

$\Omega$ such that $S \subset \Omega \subseteq (S, C)$ and the treewidth and the minimum fill-in of the realizations of some blocks strictly contained in $(S, C)$ (see corollaries 4.5, 4.8). The treewidth and the minimum fill in of the input graph is computed, like in corollary 4.2, using the treewidth and the minimum fill-in of the realizations of its blocks.

The whole algorithm can be implemented in $\mathcal{O}(bpn+re)$ time, where $n$ and $e$ are the number of vertices, respectively of edges of $G$ and $r$, $b$ and $p$ are the number of minimal separators, blocks, respectively potential maximal cliques of $G$. The algorithm is clearly polynomial in the size of the input, i.e. in $n$ and $p$.

# 5 Application to some classes of graphs

Several classes of graphs have "few" minimal separators, in the sense that the number of minimal separators of such graphs is polynomially bounded in the size of the graph. Moreover, an algorithm given in [20] computes all the minimal separators of these graphs.

For some of these classes of graphs we also have algorithms computing the treewidth and the minimum fill-in in polynomial time, using the minimal separators (cf. [19, 8, 3, 31, 23, 14, 25]). Different proofs have been given for each of these algorithms. We have remarked that, for computing the treewith or the minimum fill-in of a graph, all these algorithms compute, in an implicit manner, all the potential maximal cliques of the input graph. Therefore, our approach unifies the cited algorithms (see also [4]).

We will also show how to compute in polynomial time the potential maximal cliques for a new class of graphs, namely the weakly triangulated graphs (see also [5]).

## 5.1 AT-free graphs

We say that three vertices $(x, y, z)$ of a graph form an *asteroidal triple* of a graph $G$ if between every two of them there exists a path avoiding the neighborhood of the third. A graph is *AT-free* if it has no asteroidal triple. The notion of asteroidal triple was introduced by Lekkerkerker and Boland [24], in relation to interval graphs.

Notice that the treewidth and the minimum fill-in problems are NP-complete even restricted to the class of co-bipartite graphs [1, 33], which is contained in the class of AT-free graphs. It was shown in [22, 25] that the treewidth and the minimum fill-in are tractable in polynomial time for AT-free graphs with polynomial number of minimal separators. Among the AT-free graphs with few minimal separators, we find the *cocomparability graphs* of bounded dimension and in particular the *permutation graphs* or the *d-trapezoid graphs*, which have a polynomial number of separators for any fixed $d$.

The following proposition gives an easy way to compute the potential maximal cliques of a AT-free graph $G$, in a time which is polynomial in the size of $G$ and in the number of its minimal separators.

**Proposition 5.1** *Let $\Omega$ be a potential maximal clique of an AT-free graph $G$. Then $\Delta_G(\Omega)$ has at most two elements maximal by inclusion.*

*Proof.* Recall that we denoted by $C_1, C_2, \ldots C_p$ the connected components of $G - \Omega$ and by $S_i$ the neighborhood of $C_i$ in $\Omega$. By theorem 3.16, the elements of $\Delta_G(\Omega)$ are exactly the vertex sets $S_1, \ldots, S_p$. Suppose that $\Delta_G(\Omega)$ has three elements maximal by inclusion, say $S_1, S_2, S_3$. Let $x_1, x_2$ and $x_3$ be three vertices of $C_1$, $C_2$ and respectively $C_3$. We show that $(x_1, x_2, x_3)$ is an asteroidal triple. Let $y$ be a vertex of $S_2 - S_1$. There is a path from $y$ to $x_2$ that avoids $S_1$, so $y$ and $x_2$ are in the same connected component of $G - S_1$. Since $\Omega$ is contained in a block $(S_1, C(S_1))$ associated to $S_1$, we deduce that $y$ and $x_2$ are in the connected component $C(S_1)$ of $G - S_1$. For the same reasons, $x_3$ is in the same connected component $C(S_1)$ of $G - S_1$. Therefore, there is a path from $x_2$ to $x_3$ in $C(S_1)$, which clearly avoids the neighborhood of $x_1$. By symmetry, $x_1, x_2$ and $x_3$ form an asteroidal triple. $\diamond$

**Corollary 5.2** *An AT-free graph $G$ has $\mathcal{O}(|\Delta_G|^2 + n|\Delta_G|)$ potential maximal cliques, computable in a time polynomial in the number $n$ of vertices of $G$ and the number $|\Delta_G|$ of its minimal separators.*

*Proof.* In an AT-free graph, we have two types of potential maximal cliques: the potential maximal cliques $\Omega$ such that $\Delta_G(\Omega)$ has two distinct elements maximal by inclusion, and the potential maximal cliques such that $\Delta_G(\Omega)$ has one element maximum by inclusion.

Let us prove that if $\Omega$ is a potential maximal clique of $G$ and $S_1, S_2$ are two distinct elements maximal by inclusion in $\Delta_G(\Omega)$, then $\Omega = P(S_1, S_2)$. For any $T \in \Delta_G(\Omega)$, let $(T, C(T))$ be the unique block of $T$ containing $\Omega$. If $T \subseteq S_1$, then $(T, C(T))$ contains the block $(S_1, C(S_1))$. It follows that $\cap_{T \in \Delta_G(\Omega)}(T, C(T) = (S_1, C(S_1)) \cap (S_2, C(S_2))$, so $P(\Delta_G(\Omega)) = P(S_1, S_2)$. Thus, we have $\mathcal{O}(|\Delta_G|^2)$ potential maximal cliques $\Omega$ with two maximal elements in $\Delta_G(\Omega)$. Clearly, all these potential maximal cliques can be computed in polynomial time.

If $\Omega$ is a potential maximal clique and $S$ is the unique element maximum by inclusion in $\Delta_G(\Omega)$, then by theorem 3.10 we have $\Omega = (S, C)$, where $(S, C)$ is a full block associated to $S$. Consequently, we have at most $n|\Delta_G|$ potential maximal cliques of this type, computable in polynomial time. $\diamond$

**Remark 5.3** *One can prove that an AT-free graph has at most $\mathcal{O}(n^2|\Delta_G|)$ potential maximal cliques. This result follows directly from [25].*

The notion of asteroidal triple can be extended to more than three vertices. We say that vertex set $A$ is an *asteroidal set* if for any $a \in A$, $A - \{a\}$ is contained in a same connected component of $G - N(a)$, where $N(a)$ is the neighborhood of $a$ in $G$. In particular, an asteroidal triple is an asteroidal set of cardinality 3. The *asteroidal number* of a graph is $\mathrm{na}(G) = \max\{|A|,\ \text{A is an asteroidal set of } G\}$. In [6] it is proved that the treewidth and the minimum fill-in are polynomially tractable for graphs with bounded asteroidal number and with few minimal separators. Like for the AT-free graphs, a potential maximal clique of $G$ has at most na$(G)$ maximal elements, so a graph has at most $\mathcal{O}(|\Delta_G|^{\mathrm{na}(G)} + n|\Delta_G|)$ potential maximal cliques. We deduce like in corollary 5.2 that if na$(G)$ is bounded by a constant $c$ and if $G$ has few minimal separators, then all the potential maximal cliques of $G$ can be listed in polynomial time.

## 5.2 Circle and circular arc graphs

*Circle* and *circular arc graphs* are obtained from an intersection model. A graph $G = (V, E)$ is a *circle graph* if and only if we can associate each vertex of the graph to a chord of a circle such that two vertices are adjacent in the graph if and only if the corresponding chords cross. The circle and its chords are the *circle model* $D(G)$ of the graph. In the same manner, a graph $G$ is a *circular arc graph* if each vertex can be associated to an arc of a circle and two vertices are adjacent if and only if the corresponding arcs overlap. The circle and its circular arcs are said to be the *circular arc model* $D(G)$ of the graph.

Without loss of generality, we can assume that no two chords of the circle model, respectively no two arcs of the circular arc model share an endpoint. Both for circle and circular arc graphs, there are recognition algorithms working in $\mathcal{O}(n^2)$ time, which also compute the circle, respectively the circular arc model of a graph (see [29, 10]). Therefore, we will assume that an intersection model (i.e. a circle, respectively a circular-arc model) of the input graph is always given.

The treewidth and the minimum fill-in problems have been solved for the circle and the circular arc graphs in $\mathcal{O}(n^3)$ time [15, 31, 23]. We show here that all the potential maximal cliques of a circle or a circular arc graph can be listed in polynomial time. We will use the results of [23] in order to prove this assertion. Actually, the cited algorithms compute, in an implicit manner, all the potential maximal cliques of the input graph.

We will present here in details only the circle graphs. The results can be extended to circular arc graphs using the same techniques.

For these "geometrical" classes of graphs, the minimal separators can be modeled by *scanlines*. In the circle model of a circle graph, we add a *scanpoint* between every two consecutive endpoints of the chords. The scanlines are the straight line segments between two scanpoints.

Let $G$ be a circle graph and $D(G)$ be a circle model of $G$. For each scanline $s$, we denote by $S(s)$ the set of all vertices of $G$, corresponding to the chords of $D(G)$ that intersect $s$.

Kloks [15] has proved the following proposition:

**Proposition 5.4** *Let $G$ be a circle graph and let $D(G)$ be a circle model of $G$. For any minimal separator $S$ of $G$, there is a scanline $s$ in $D(G)$ such that $S = S(s)$.*

Since the circle model has $n$ chords, we have $2n$ scanpoints and consequently $n(2n-1)$ scanlines. Therefore, a circle graph has $\mathcal{O}(n^2)$ minimal separators.

Kloks, Kratsch and Wong [23] give a characterization of minimal triangulations of a circle graph $G$ using scanlines of the circle model. The scanpoints of $D(G)$ form a convex polygon $P$. A *planar triangulation* of $P$ is a set $T$ of non-crossing diagonals of $P$, dividing the interior of $P$ in triangles. Notice that if $P$ has $n$ vertices, then $T$ has $n-3$ diagonals dividing the interior of $P$ in $n-2$ triangles.

If $T$ is a planar triangulation of $P$, the graph $H(T)$ is defined as the graph with the same vertex set as $G$, and two vertices $x$ and $y$ are adjacent in $H(T)$ if and only if the chords corresponding to $x$ and $y$ in $D(G)$ intersect a same triangle of $T$. Clearly, $H(T)$ is a supergraph of $G$. Moreover, it is proved in [23] that $H(T)$ is a triangulation of $G$. But we will only use the following theorem of [23]:

**Theorem 5.5** *Let $G$ be a circle graph, $D(G)$ a circle model of $G$ and $P$ its polygon of scanpoints. Then for any minimal triangulation $H$ of $G$ there is a planar triangulation $T$ of $P$ such that $H = H(T)$.*

All we have to do is to notice how maximal cliques are formed in $H(T)$. Let $Q$ be a triangle of a planar triangulation $T$ of $P$. We denote by $S(Q)$ the set of all vertices of $G$, for which the corresponding chords intersect $Q$.

**Proposition 5.6** *Let $G$ be a circle graph, $D(G)$ a circle model of $G$ and $T$ a planar triangulation of its polygon $P$ of scanpoints. Then for any maximal clique $\Omega$ of $H(T)$, there is a triangle $Q$ of $T$ such that $\Omega = S(Q)$.*

*Proof.* Clearly, for any triangle $Q$ of $T$, the vertex set $S(Q)$ induces a clique in $H(T)$.

We want to prove that all the chords corresponding to vertices of $\Omega$ intersect a same triangle $Q$. Let $s$ be any diagonal of $T$ and let $x, y$ be two adjacent vertices of $H(T)$. The scanline $s$ induces two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ in the polygon $P$. Since $x$ and $y$ are adjacent in $H(T)$, the chords of $D(G)$ corresponding to $x$, respectively $y$ cannot be in different regions induced by $s$. Consequently, all the chords corresponding to vertices of $\Omega$ intersect a same region induced by $s$, say $\mathcal{R}_1$. If $\mathcal{R}_1$ is a triangle, we have found our triangle $Q$. Otherwise, let $s'$ be a diagonal of $T$ contained in $\mathcal{R}_1$. Then $s'$ divides $\mathcal{R}_1$ in two smaller regions $\mathcal{R}'_1$ and $\mathcal{R}'_2$. For the same reasons as previously, the chords corresponding to vertices of $\Omega$ intersect a same sub-region, say $\mathcal{R}'_1$. We can iterate the process until finding a triangle $Q$ such that $\Omega \subseteq S(Q)$.

We have that $\Omega \subseteq S(Q)$, $S(Q)$ is a clique in $H(T)$ and $Q$ is a maximal clique of $H(T)$. It follows that $\Omega = S(Q)$. $\diamond$

**Corollary 5.7** *A circle graph has $\mathcal{O}(n^3)$ potential maximal cliques, computable in polynomial time.*

*Proof.* Let $\Omega$ be a potential maximal clique of $G$ and $H$ a minimal triangulation of $G$ such that $\Omega$ is a maximal clique of $H$. By theorem 5.5, there is a planar triangulation $T$ of the polygon of scanpoints such that $H = H(T)$. By proposition 5.6, there is a triangle $Q$ of scanlines such that $\Omega = S(Q)$.

Consequently, for any potential maximal clique $\Omega$ there is a triangle of scanpoints $Q$ such that $\Omega = S(Q)$. Since we have at most $\mathcal{O}(n^3)$ triangle of scanpoints, $G$ has at most $\mathcal{O}(n^3)$ potential maximal cliques.

For listing all the potential maximal cliques of $G$, it is sufficient to list all the triangles of scanpoints $Q$, to compute $S(Q)$ and to check if $S(Q)$ is a potential maximal clique of $G$. Clearly, this enumeration can be done in polynomial time. $\diamond$

So for circle graphs, all the potential maximal cliques can be listed in polynomial time. Notice that the algorithms of [15, 23] that compute the treewidth and the minimum fill-in of circle graphs, are looking for triangulations of type $H(T)$ of $G$. In particular, all the minimal triangulations of $G$ are of this type. Moreover, these algorithms compute all the sets $S(Q)$, for all the triangles of scanlines. So they implicitly use all the potential maximal cliques of the input graph. The fact that they are using triangulations of type $H(T)$ of $G$ instead of using only minimal triangulations leads to an efficient algorithm, in time $\mathcal{O}(n^3)$.

For characterizing the potential maximal cliques of a circle graph, we have used the global characterization of its minimal triangulations given by theorem 5.5. We could have proved directly that, under certain conditions, two minimal separators of a circle graph are parallel if and only if the corresponding scanlines do not cross (they may have a common endpoint). Using theorem 3.16 and the same arguments as for the proof of proposition 5.6, we can prove that, if $\Omega$ is potential maximal clique of $G$, then the scanlines corresponding to the minimal separators of $\Delta_G(\Omega)$ determine a region $\mathcal{R}$ such that $\Omega = S(\mathcal{R})$ (the chords corresponding to vertices of $\Omega$ are exactly the chords intersecting $\mathcal{R}$). One can deduce directly the proposition 5.6 (see [32]).

For the class of circular arc graphs, the results are very similar. Let $G$ be a circular arc graph and $D(G)$ be a circular arc model of $G$. For a point $p$ of the circle, we denote by $S(p)$

the vertices of $G$ for which the corresponding arcs contain $p$. We place a point $p$ between every two consecutive endpoints $u$ and $v$ of the arcs of $D(G)$. We say that $p$ is a scanpoint if $|S(p)| < \min(|S(u)|, |S(v)|)$. Once again, a straight line $s$ between two scanpoints is called a scanline. If $p_1$ and $p_2$ are the scanpoints determining the scanline $s$, we define $S(s) = S(p_1) \cup S(p_2)$, i.e. the vertices of $G$ such that the corresponding arcs contain a point of the scanline. If $Q$ is a triangle of scanpoints of vertices $p_1, p_2, p_3$ then we put $S(Q) = S(p_1) \cup S(p_2) \cup S(p_3)$. We can also define a triangulation $T$ of the polygon $P$ of scanpoints and the supergraph graph $H(T)$ of $G$ in which two vertices $x$ and $y$ are adjacent if they are adjacent in $G$ or if there is a scanline $s \in T$ with $x, y \in S(s)$.

Kloks, Kratsch and Wong [23] have proved the following results:

**Theorem 5.8** *Let $G$ be a circular arc graph, $D(G)$ a circular arc model of $G$ and $P$ its polygon of scanpoints. Then for any minimal triangulation $H$ of $G$ there is a planar triangulation $T$ of $P$ such that $H = H(T)$.*

We can give the same characterization of the potential maximal cliques as in the case of circle graph. The proof is almost the same as for proposition 5.6, so we omit it.

**Proposition 5.9** *Let $G$ be a circular arc graph, $D(G)$ a circular arc model of $G$ and $T$ a planar triangulation of its polygon of scanpoints $P$. Then for any maximal clique $\Omega$ of $H(T)$, there is a triangle of scanpoints $Q$ of $T$ such that $\Omega = S(Q)$.*

Clearly, circular arc graphs have a polynomial number of potential maximal cliques and we can list these potential maximal cliques in polynomial time.

**Corollary 5.10** *A circular arc graph has $\mathcal{O}(n^3)$ potential maximal cliques, computable in polynomial time.*

Once again, the algorithms of [31, 23] compute all the sets of type $S(Q)$, so in particular they use the potential maximal cliques of the input graph.

## 5.3 Weakly triangulated graphs

We consider now two non-adjacent vertices $x, y$ of an arbitrary graph $G$. Let $G'$ be the graph obtained from $G$ by adding the edge $\{x, y\}$. We will show in this section that the potential maximal cliques of $G$ can be computed from the minimal separators of $G$ and the potential maximal cliques of $G'$. We will use this technique to compute all the potential maximal cliques of any weakly triangulated graph.

Let once again $\Omega$ be a potential maximal clique clique of $G$. Let $C_1, \ldots, C_p$ be the connected components of $G - \Omega$ and let $S_i$ be the set of vertices of $\Omega$ having at least a neighbor in $C_i$. We want to describe the behavior of $\Omega$ and $\mathcal{S}$ in the graph $G'$. We deduce directly from theorem 3.16 :

**Lemma 5.11**

1. *If $x, y \in \Omega$ or there is an $i, 1 \le i \le p$ such that $x, y \in C_i$ or $x \in S_i$ and $y \in C_i$ then $\Omega$ is a potential maximal clique of $G'$ and the elements of $\Delta_{G'}(\Omega)$ are $S_1, S_2, \ldots, S_p$.*

2. *If $y \in C_1$, $x \in \Omega - S_1$ and $\Omega \neq S_1 \cup \{x\}$, then $\Omega$ is a potential maximal clique of $G'$ and the elements of $\Delta_{G'}(\Omega)$ are $S_1 \cup \{x\}, S_2, \ldots, S_p$.*

3. *If $x \in C_1$, $y \in C_2$ and $\Omega \neq S_1 \cup S_2$ then $\Omega$ is a potential maximal clique of $G'$ and the elements of $\Delta_{G'}(\Omega)$ are $S_1 \cup S_2, S_3, \ldots, S_p$.*

*Proof.* By theorem 3.16, $G-\Omega$ has no full components associated to $\Omega$ and $G_{\Delta_G(\Omega)}[\Omega]$ is a clique. Notice that, by lemma 3.14, the elements of $\Delta_G(\Omega) = \mathcal{S}(\Omega)$ are exactly the sets $S_1, \dots, S_p$.

If we are in the first case of the lemma, then the connected components of $G'-\Omega$ are exactly the same in $G-\Omega$, and their neighborhoods in $G'$ are also the same. If we denote by $\mathcal{S}'(\Omega)$ the neighborhoods of the connected components of $G'-\Omega$, then $\mathcal{S}'(\Omega)$ consists in $S_1, S_2, \dots, S_p$. Clearly, $G'-\Omega$ has no full components associated to $\Omega$ and $G_{\mathcal{S}'(\Omega)}[\Omega]$ is a clique. By theorem 3.16, $\Omega$ is a potential maximal clique of $G'$.

If we are in the second case, the connected components of $G'-\Omega$ are $C_1, C_2 \dots, C_p$ and their neighborhoods in $G'$ are $S_1 \cup \{x\}, S_2, \dots, S_p$. If $\Omega \neq S_1 \cup \{x\}$, then $G'-\Omega$ has no full components associated to $\Omega$, and once again $\Omega$ is a potential maximal clique of $G'$, by theorem 3.16.

In the third case, the connected components of $G'-\Omega$ are $C_1 \cup C_2, C_3, \dots, C_p$ and their neighborhoods are respectively $S_1 \cup S_2, S_3, \dots, S_p$. The fact that $\Omega \neq S_1 \cup S_2$ insures that $G'-\Omega$ has no full components associated to $\Omega$. Clearly $G_{\mathcal{S}'(\Omega)}[\Omega]$ is a clique, so by theorem 3.16, $\Omega$ is a potential maximal clique of $G'$. ◇

It follows directly :

**Theorem 5.12** *Let $\Omega$ be a potential maximal clique of $G$. Let $x, y$ be two non-adjacent vertices of $G$ and let $G' = G \cup \{x, y\}$. Two cases are possible :*

1. *$\Omega$ can be written as $S_1 \cup \{x\}$, $S_1 \cup \{y\}$ or $S_1 \cup S_2$, where $S_1, S_2$ are minimal $x, y$-separators of $G$.*
2. *$\Omega$ is a potential maximal clique of $G'$.*

The weakly triangulated graphs were introduced in [12]. A graph $G$ is called *weakly triangulated* if neither $G$ nor its complement $\overline{G}$ have an induced cycle with strictly more than four vertices. This class contains the chordal graphs, the chordal bipartite graphs and the distance hereditary graphs.

We denote by $N(x)$ the neighbors of the vertex $x$. We say that two vertices $x, y$ of a graph $G$ form a *two-pair* if their common neighbors $N(x) \cap N(y)$ form an $x, y$-separator. The following theorem was proved in [13]:

**Theorem 5.13** *If $G$ is a weakly triangulated graph, then every induced subgraph of $G$ that is not a clique contains a two-pair.*

Spinrad and Sritharan give in [30] an algorithm recognizing the weakly triangulated graphs, based on the following theorem:

**Theorem 5.14** *Let $G = (V, E)$ be a graph and let $\{x, y\}$ be a two-pair of $G$. Let $G' = (V, E')$ be the graph obtained from $G$ by adding the edge $\{x, y\}$. Then $G$ is weakly triangulated if and only if $G'$ is weakly triangulated.*

Notice that a clique is a weakly triangulated graph. The recognition algorithm considers an input graph $G$ and, while $G$ has a two-pair $\{x, y\}$, it adds the edge between $x$ and $y$ to $G$. At the end of the loop, either $G$ became a clique, in which case the initial graph was weakly triangulated by theorem 5.14, or $G$ is not a clique and it has no two-pair, in which case the input graph could not be weakly triangulated by theorems 5.13 and 5.14.

We denote by $\overline{e}$ the number of edges of $\overline{G}$. Let now $G = (V, E)$ be a weakly triangulated graph and let $f_1 = \{x_1, y_1\}, \dots, f_{\overline{e}} = \{x_{\overline{e}}, y_{\overline{e}}\}$ be the edges added to $G$ by the recognition algorithm in this order. In particular, $(V, E \cup \{f_1, \dots, f_{\overline{e}}\})$ is a clique. We denote by $G_i$ the graph $(V, E \cup \{f_1, f_2, \dots, f_i\})$, with $0 \leq i \leq \overline{e}$ (so $G_0 = G$ and $G_{\overline{e}}$ is a clique). We will describe the minimal separators, respectively the potential maximal cliques of $G_i$ using the minimal separators, respectively the potential maximal cliques of $G_{i+1}$, for any $i < \overline{e}$.

It is known that a weakly triangulated graph has at most $\overline{e}$ minimal separators (Kloks [16]). We give here the proof of this result, because we will reuse the same technique for counting and listing the potential maximal cliques of a weakly triangulated graph.

**Theorem 5.15** *Let $G$ be a non complete weakly triangulated graph and let $\{x, y\}$ be a two-pair of $G$. Let $S_{xy}$ be the set $N(x) \cap N(y)$. Consider the graph $G'$ obtained from $G$ by adding the edge $\{x, y\}$. Then $\Delta_G \subseteq \Delta_{G'} \cup \{S_{xy}\}$.*

*Proof.* Notice that $S_{xy}$ is a minimal $x, y$-separator of $G$, by definition of a two-pair.

Let $S$ be any minimal separator of $G$.

Suppose at first that $S$ separates $x$ and $y$, and let $C_x, C_y$ be the connected components of $G - S$ containing $x$, respectively $y$. If both $C_x$ and $C_y$ are full components associated to $S$, then $S$ is a minimal $x, y$-separator by lemma 2.6. Notice that $N(x) \subset C_x \cup S$ and $N(y) \subset C_y \cup S$. We deduce that $N(x) \cap N(y) \subseteq S$, and since $N(x) \cap N(y) = S_{xy}$ is a $x, y$-separator in $G$ by definition of a two-pair, it follows that $S = S_{xy}$. Suppose now that $C_x$ and $C_y$ are not both full components of $S$ in $G$. The connected components of $G' - S$ are the same as in $G - S$, except for $C_x$ and $C_y$ which form a unique component $C_x \cup C_y$ in $G' - S$. If $G - S$ has two full components $D$ and $E$ associated to $S$, both different from $C_x$ and $C_y$, then $D$ and $E$ are full components associated to $S$ in $G'$, so $S$ is a minimal separator of $G'$ by lemma 2.6. Otherwise, $G - S$ has a unique full component $D$ associated to $S$ and different from $C_x$ and $C_y$, so at least one of $C_x, C_y$ must be a full component associated to $S$ in $G$. Therefore, $D$ and $C_x \cup C_y$ are full components associated to $S$ in $G'$, so $S$ is a minimal separator of $G'$ by lemma 2.6.

If $S$ does not separate $x$ and $y$, the connected components of $G' - S$ are the same as in $G - S$, so $G' - S$ has two full components associated to $S$. Consequently, $S$ is a minimal separator of $G'$. $\diamond$

So for any $i < \overline{e}$, the graph $G_i$ has at most one more minimal separator than $G_{i+1}$. We deduce:

**Corollary 5.16** *A weakly triangulated graph $G$ has at most $\overline{e}$ minimal separators, where $\overline{e}$ is the number of edges of $\overline{G}$.*

We can conclude directly from theorem 5.12 and lemma 5.16 that all the potential maximal cliques of a weakly triangulated graph can be computed in polynomial time. But we can refine the results of the third case of lemma 5.11.

**Lemma 5.17** *Let $G$ be a graph, let $x, y$ be a two-pair of $G$ and let $\Omega$ be a potential maximal clique of $G$ such that $x$ and $y$ are in different connected components of $G - \Omega$. Then $\Omega$ is a potential maximal clique of $G' = G \cup \{x, y\}$.*

*Proof.* We use the fact that if $x, y$ is a two-pair of a graph $G$, then $S_{xy} = N(x) \cap N(y)$ is the only $x, y$-minimal separator of $G$. Let $C_1$ and $C_2$ the connected components of $G - \Omega$ containing $x$, respectively $y$. Like previously, let $S_1$ and $S_2$ the sets of vertices of $\Omega$ having a neighbor in $C_1$, respectively $C_2$. We want to prove that both $S_1$ and $S_2$ contain $S_{xy}$ and at least one of them is equal to $S_{xy}$. By remark 3.15, $S_1$ separates $x$ and $y$, so it must contain $N(x) \cap N(y) = S_{xy}$. The same holds for $S_2$. Suppose that there is some vertex $a \in S_1 - S_{xy}$ and some vertex $b \in S_2 - S_{xy}$. Since $a$ has a neighbor in $C_1$, $a$ is in the connected component $G - S_{xy}$ containing $x$. Also $b$ is in the connected component of $G - S_{xy}$ containing $y$. Then $\Omega$ intersects different connected components of $G - S_{xy}$, which is, by corollary 3.12, a contradiction to the fact that $S_{xy} \subset \Omega$. It remains that $S_1 = S_{xy}$ or $S_2 = S_{xy}$. Therefore, $S_1 \cup S_2$ is equal to $S_1$ or $S_2$, and consequently $\Omega \neq S_1 \cup S_2$ by corollary 3.12. By the third part of lemma 5.11, $\Omega$ will be a potential maximal clique of $G'$. $\diamond$

We deduce from theorem 5.12 and lemma 5.17 :

**Corollary 5.18** *Let $\Omega$ be a potential maximal clique of $G$. Let $x, y$ be a two-pair of $G$ and let $G' = G \cup \{xy\}$. Let $S_{xy} = N(x) \cap N(y)$. Two cases are possible :*

1. *$\Omega$ can be written as $S_{xy} \cup \{x\}$ or $S_{xy} \cup \{y\}$.*
2. *$\Omega$ is a potential maximal clique of $G'$.*

**Corollary 5.19** *A weakly triangulated graph $G$ has at most $2\overline{e} + 1$ potential maximal cliques.*

*Proof.* We consider the sequence of graphs $G_0 = G, G_1, \ldots, G_{\overline{e}}$ previously defined. Since $G_{i+1}$ is obtained from $G_i$ by adding an edge between a two-pair, by corollary 5.18 the graph $G_i$ has at most two more potential maximal cliques than $G_{i+1}$. Clearly $G_{\overline{e}}$, which is a clique, has a unique potential maximal clique. $\diamond$

**Corollary 5.20** *The treewidth and the minimum fill-in of weakly triangulated graphs can be computed in polynomial time.*

The complexity of computing all the potential maximal cliques is $\mathcal{O}(n^3\overline{e})$. It is sufficient to use the weakly triangulated graphs recognition algorithm of [30] to generate the two-pairs $\{x_i, y_i\}$ and the sets $S_{x_i y_i}$. This takes $\mathcal{O}(n^2\overline{e})$ time. One can check in $\mathcal{O}(n^3)$ time if a set is a potential maximal clique, by corollary 3.17. So the list of all potential maximal cliques is computable in $\mathcal{O}(n^3\overline{e})$.

For weakly triangulated graphs, the number of minimal separators is $r = \mathcal{O}(\overline{e})$, the number of blocks is $b = \mathcal{O}(n\overline{e})$ and the number of potential maximal cliques is $p = \mathcal{O}(\overline{e})$. According to the complexity of the algorithm of corollary 3.17, the treewidth and the minimum fill-in are computable in $\mathcal{O}(n^2\overline{e}^2 + n^3\overline{e})$ time.

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