Approximating the treewidth of AT-free graphs

Vincent Bouchitté and Ioan Todinca

LIP-École Normale Supérieure de Lyon
46 Allée d'Italie, 69364 Lyon Cedex 07, France
email: {Vincent.Bouchitte, Ioan.Todinca}@ens-lyon.fr

Abstract. Using the specific structure of the minimal separators of AT-free graphs, we give a polynomial time algorithm that computes a triangulation whose width is no more than twice the treewidth of the input graph.

1 Introduction

The *treewidth* of graphs, introduced by Robertson and Seymour [12], has been intensively studied in the last years, mainly because many NP-hard problems become solvable in polynomial and even in linear time when restricted to graphs with small treewidth. These algorithms use a tree-decomposition of small width of the graph. A tree-decomposition or a *triangulation* of a graph is a chordal supergraph, i.e. all the cycles of the supergraph of length strictly more than three have a chord. Computing the treewidth of a graph corresponds to finding a triangulation with the smallest cliquesize. In particular, we can restrict ourselves to triangulations minimal by inclusion, that we call *minimal triangulations.*

Computing the treewidth of arbitrary graphs is NP-hard. Nevertheless, the treewidth can be computed in polynomial time for several well-known classes of graphs, for example the chordal bipartite graphs, the circle and circular-arc graphs, and permutation graphs. All these algorithms use the *minimal separators* of the graph and the fact that these classes of graphs have "few" minimal separators, in the sense that the number of the separators is polynomially bounded in the size of the graph.

Characterizations of the minimal triangulations of a graph by the minimal separators have already been given in [3, 11]. Their approach is a global vision of all the minimal separators of a graph that does not yield an algorithmic construction of minimal triangulations with small cliquesize. We gave a local characterization of minimal triangulations [4] that we used to compute the treewidth for particular classes of graphs, including those previously mentioned.

By studying the potential maximal cliques of a graph, which are the maximal cliques appearing in at least one minimal triangulation of the graph, we proved in [5, 6] that the minimal separators are sufficient to compute the treewidth of a graph.

It has been proved by Bodlaender et al [2] that the treewidth can be approximated within a $\mathcal{O}(\log n)$ multiplicative factor. In addition, they showed that the treewidth is not approximable within an additive constant unless $P = NP$.

The existence of a polynomial time approximation algorithm which is no more than a constant times the optimal value is a question that still remains open. In this paper, we consider this problem for the class of AT-free graphs. Computing the treewidth is still NP-hard for these graphs [1]. Using the particular structure of their minimal separators, we give a 2-approximation algorithm for the treewidth of AT-free graphs.

2 Preliminaries

Throughout this paper we consider connected, simple, finite, undirected graphs. Let $G = (V, E)$ be a graph. For a vertex set $V' \subseteq V$ of G, we denote by $N_G(V')$ the neighborhood of V' in $G \backslash V'$.

A graph H is *chordal* (or *triangulated*) if every cycle of length at least four has a chord. A *triangulation* of a graph $G = (V, E)$ is a chordal graph $H = (V, E')$ such that $E \subseteq E'$. H is a *minimal triangulation* if for any intermediate set E'' with $E \subseteq E'' \subset E'$, the graph (V, E'') is not triangulated. A *clique* of G is a complete subgraph of G.

Definition 1. *Let G be a graph. The* treewidth *of G, denoted by* $\mathrm{tw}(G)$*, is the minimum, over all triangulations H of G, of $\omega(H) - 1$, where $\omega(H)$ is the the maximum cliquesize of H.*

In other words, computing the treewidth of G means finding a triangulation with smallest cliquesize. In particular, the treewidth is always achieved by some minimal triangulation of the graph.

2.1 Minimal separators and chordal graphs

The *minimal separators* play a crucial role in the characterization of the minimal triangulations of a graph.

A subset $S \subseteq V$ is an *a, b-separator* for two nonadjacent vertices $a, b \in V$ if the removal of S from the graph separates a and b in different connected components. S is a *minimal a, b-separator* if no proper subset of S separates a and b. We say that S is a *minimal separator* of G if there are two vertices a and b such that S is a minimal a, b separator. Notice that a minimal separator can be strictly included into another. We denote by Δ_G the set of all minimal separators of G.

Let G be a graph and S a vertex set of G. We note $\mathcal{C}_G(S)$ the set of connected components of $G \backslash S$. A component $C \in \mathcal{C}_G(S)$ of $G \backslash S$ is a *full component associated to S* if every vertex of S is adjacent to some vertex in C. For the following lemma, we refer to [8].

Lemma 1. *A set S of vertices of G is a minimal a, b-separator if and only if a and b are in different full components associated to S.*

Let S be a minimal separator of G. If $C \in \mathcal{C}_G(S)$, we say that $(S, C) = S \cup C$ is a *one-block* associated to S. A one-block (S, C) is called *full* if C is a full component associated to S. If (S, C) is a full one-block, then $S = N_G(C)$. If (S, C) is not full, then $S^* = N_G(C)$ is a minimal separator of G, strictly contained in S.

Let S be a minimal separator of G. We say that S *crosses* a vertex set A if S separates two vertices $x, y \in A$ (i.e. S is an x, y-separator). We say that S *separates* two vertex sets A and B if S separates each vertex of $A \backslash S$ from each vertex of $B \backslash S$.

Let S and T be two minimal separators. If S crosses T, we write $S \# T$. Otherwise, S and T are called *parallel*, denoted by $S \| T$. It is easy to prove that these relations are symmetric. Remark that S and T cross if and only if then T intersects each full component associated to S. Conversely, S and T are parallel if and only if T is contained in some one-block (S, C_T) associated to S. In particular, if $T \subseteq S$, then S and T are parallel.

Let $S \in \Delta_G$ be a minimal separator. We denote by G_S the graph obtained from G by *completing* S, i.e. by adding an edge between every pair of non-adjacent vertices of S. If $\Gamma \subseteq \Delta_G$ is a set of separators of G, G_Γ is the graph obtained by completing all the separators of Γ. The results of [9], concluded in [11], establish a strong relation between the minimal triangulations of a graph and its minimal separators.

Theorem 1. *Let $\Gamma \in \Delta_G$ be a maximal set of pairwise parallel separators of G. Then $H = G_\Gamma$ is a minimal triangulation of G and $\Delta_H = \Gamma$.*

Let H be a minimal triangulation of a graph G. Then Δ_H is a maximal set of pairwise parallel separators of G and $H = G_{\Delta_H}$.

In other terms, every minimal triangulation of a graph G is obtained by considering a maximal set Γ of pairwise parallel separators of G and completing the separators of Γ. The minimal separators of the triangulation are exactly the elements of Γ.

It is important to know that the elements of Γ, who become the separators of H, have strictly the same behavior in H as in G. Indeed, the connected components of $H \backslash S$ are exactly the same as in $G \backslash S$, for every $S \in \Gamma$. Moreover, the full components associated to S are the same in the two graphs.

2.2 Blocks

The following definitions are strongly related with the *blocking sets* and the *blocks* introduced in [7].

Definition 2. *Let G be a graph and $\mathcal{S} \subseteq \Delta_G$ a set of pairwise parallel separators such that for any $S \in \mathcal{S}$, there is a one-block $(S, C(S))$ containing all the elements of $\mathcal{S}$. Suppose that $\mathcal{S}$, ordered by inclusion, has no greatest element. We define the* piece between *the elements of $\mathcal{S}$ by*

$$P(\mathcal{S}) = \bigcap_{S \in \mathcal{S}} (S, C(S))$$

Notice that for any $S \in \mathcal{S}$ the one-block of S containing all the separators of $\mathcal{S}$ is unique: if $T \in \mathcal{S}$ is not included in S, there is a unique connected component of $G \backslash S$ containing $T \backslash S$.

Definition 3. *Let B be a vertex set of a graph G. We denote by $C_1, \ldots, C_p$ the connected components of $G \backslash B$ and by S_i the neighborhood of C_i. We will say that B is a* block *of G if the sets S_i are minimal separators of G and one of the following conditions holds:*

- *$B = G$.*
- *There is an $i \in [1, p]$ such that B is a one-block (S_i, C).*
- *$B = P(S_1, \ldots, S_p)$.*

If B is a block, we say that the minimal separators $S_1, \ldots, S_p$ border *B. The block B is* full *if $G = B$ or, for each S_i, B is contained in a full one-block associated to S_i.*

Remark 1. Observe that if $\{T_1, T_2, \ldots T_q\}$ is a set of minimal separators such that the piece between them exists, then $B = P(T_1, \ldots, T_p)$ is a block. Moreover, any of the minimal separators bordering B is contained in one of $T_1, \ldots, T_q$. Also if B is a one-block (S, C), then any of the minimal separators bordering B is contained in S.

A block is also characterized by the following property:

Proposition 1. *Let B be a vertex set of G. Consider the connected components $C_1, \ldots, C_p$ of $G \backslash B$ and their neighborhoods $S_1, \ldots, S_p$. If for all i, $1 \leq i \leq p$ we have that S_i is a minimal separator, B is contained in some one-block (S_i, D_i) and $B \neq S_i$, then B is a block of G.*

Proof. Suppose that the set $\{S_1, S_2, \ldots, S_p\}$ has no greatest element. We show that $P(S_1, \ldots, S_p)$ exists and is equal to B. For each S_i, all the minimal separators $S_1, S_2, \ldots, S_p$ are contained in a one-block (S_i, D_i), so $P(S_1, \ldots, S_p) = \bigcap_i (S_i, D_i)$. In particular, $B \subseteq P(S_1, \ldots, S_p)$. Conversely, we prove that $P(S_i, \ldots, S_p)$ is contained in B. Let x be a vertex of $G \backslash B$, so x is contained in some C_i. Hence, $x \notin (S_i, D_i)$, so $x \notin P(S_1, \ldots, S_p)$.

Suppose now that $\{S_1, S_2, \ldots, S_p\}$ has a greatest element, say S_1. We show that $B = (S_1, D_1)$. Clearly, $B \subseteq (S_1, D_1)$. Suppose there is a vertex x in $(S_1, D_1) \backslash B$ and let C_i the component of $G \backslash B$ containing x. We have that $S_i \subseteq S_1$, so C_i is also a connected component of $G \backslash S_1$. Since $x \in D_1$, it follows that $D_1 = C_i$, so D_1 is a connected componend of $G \backslash B$. Therefore, $D_1 \cap B = \emptyset$, contradicting $B \neq S_1$. □

Proposition 2. *Let G be a graph and S, T two minimal separators of G such that $S \# T$. Then $S \cup T$ is a full block of G.*

Proof. Let $B = S \cup T$. The assertion is true if $B = G$. Suppose that $B \neq G$. Consider the connected components $C_1, \ldots, C_p$ of $G \backslash B$ and their neighborhoods $S_1, \ldots, S_p$.

We prove first that each S_i is a minimal separator of G and B is contained in a full one-block (S_i, D_i). The vertex set C_i is contained in a connected component D' associated to S and in a connected component E' associated to T. Let D, E be two full components associated to S, respectively T, such that $D \neq D'$ and $E \neq E'$. Let us show that $D \cup E$ is contained in a same full component associated to S_i. Since $C_i \subseteq D'$ and D' is a connected component of $G \backslash S$, $N(C_i)$

is contained in $D' \cup N(D') \subseteq (S, D')$. It follows that $S_i = N(C_i)$ does not intersect D. Thus, D is contained in a connected component C_D of $G \backslash S_i$. For similar reasons, E is contained in a connected component C_E of $G \backslash S_i$. It remains to prove that $C_D = C_E$. Since $S \# T$ and D is a full component associated to S, there is an $x \in T \cap D$. Then $x \in T$ has at least one neighbor y in the full component E associated to T. We have two adjacent vertices $x \in C_D$ and $y \in C_E$, so $C_D = C_E$. We denote $D_i = C_D$. Since $S_i \subseteq S \cup T$ and D, E are full components associated to S, respectively T, we deduce that each vertex of S_i has a neighbor in $D_i \supseteq D \cup E$. So D_i is a full component associated to S_i, different from C_i. By lemma 1, S_i is a minimal separator of G. We show now that $B \subseteq (S_i, D_i)$. Indeed, $D \subseteq D_i$, so $S = N(D)$ is contained in $N(D_i) \cup D_i = (S_i, D_i)$. In the same way, $T \subseteq (S_i, D_i)$. We conclude that B is contained in the full one-block (S_i, D_i). Also notice that $D_i \cap B \neq \emptyset$, because D_i contains D and $D \cap T \neq \emptyset$. Thus, $B \neq S_i$.

By proposition 1, B is a full block of G.

□

Proposition 3. *Let B be a block of G and $S \subseteq B$ be a minimal separator of G. For any connected component C associated to S which intersects B we have that $(S, C) \cap B$ is a block. In other words, if $S \subseteq B$ crosses B, then S splits the block B into smaller blocks.*

We omit here the proof of proposition 3. If B is a block $S \cup T$ with $S \# T$, then any full component associated to S intersects T. We obtain:

Corollary 1. *Let $S \# T$ be two minimal separators of G. Let C be a full component associated to S. Then $S \cup (T \cap C)$ is a block of G.*

3 AT-free graphs and minimal separators

We say that three vertices (x, y, z) of a graph form an *asteroidal triple* of a graph G if between every two of them there exists a path avoiding the neighborhood of the third. A graph is *AT-free* if it has no asteroidal triple.

For characterizing AT-free graphs we rather use the notion of *asteroidal triple of separators.*

Definition 4. *Let S, T and U be three minimal separators of G, none of them being contained into another. We say that (S, T, U) form an* asteroidal triple of separators *if $P(S, T, U)$ exists.*

Proposition 4 ([7]). *A graph G is AT-free if and only if it has no asteroidal triple of separators.*

In particular, an AT-free graph cannot have three-blocks, that is blocks of type $P(S, T, U)$ with none of S, T, U contained into another. So any block B of an AT-free graph is a one-block (S, C), a two-block $P(S, T)$ or the graph G itself.

Consider three minimal separators $U \subseteq P(S, T)$. The following proposition will allow us to check if S, T, U form an asteroidal triple of separators.

Proposition 5. *Let S, T and U be three minimal separators of G such that $U \subseteq P(S, T)$. Then P, S and T do not form an asteroidal triple of separators if and only if they satisfy one of the following conditions:*

1. *U contains S or T.*
2. *U is contained in S or T.*
3. *U separates S and T.*

Let us reconsider corollary 1 for the case of AT-free graphs.

Proposition 6. *Consider an AT-free graph G, a minimal separator S of G and a full component C associated to S. Let T be a minimal separator crossing S. Suppose that $C \backslash T \neq \emptyset$ and let $U = N(C \backslash T)$. Then $S \cup (T \cap C) = P(S, U)$.*

Proof. By corollary 1, $B = S \cup (T \cap C)$ is a block of G.

Let $C_1, C_2, \ldots, C_p$ be the connected components of $G \backslash B$. The C_i's are either connected components of $G \backslash S$ different from C or connected components of $G[C \backslash T]$. Let $S_1, \ldots, S_p$ be the neighborhoods of $C_1, \ldots, C_p$. Notice that, if C_i is a connected component of $G \backslash S$ different from C, then $S_i \subseteq S$. In particular, if C_i is a full component associated to S with $C_i \neq C$, we have $S_i = S$.

Let us show that S is maximal by inclusion in $\{S_1, \ldots, S_p\}$. Suppose that S is strictly contained in some S_i, so $S \subset N(C_i)$. Then C_i is a connected component of $G[C \backslash T]$. Let D be the connected component of $G \backslash T$ containing C_i. We have $N(C_i) \subseteq D \cup N(D) \subseteq (T, D)$, so $S \subseteq (T, D)$. This contradicts the fact that S and T cross.

Let C_i be any connected component of $G[C \backslash T]$ (such a C_i exists because $C \backslash T$ is non-empty). Let us show that $S_i \not\subseteq S$. If $S_i \subseteq S$, then C_i is also a connected component of $G \backslash S$, so $C_i = C$. This implies $B \cap C = \emptyset$, contradicting the fact that T intersects the full component C associated to S. It follows that $S_i \not\subseteq S$.

Thus, the set of minimal separators bording B has at least two elements maximal by inclusion, S and S'. Since an AT-free graph has only one-blocks and two-blocks, we deduce $B = P(S, S')$. It remains to show that $S' = U$. For any connected component C_i of $G[C \backslash T]$, we have $S_i \not\subseteq S$, so $S_i \subseteq S'$ by remark 1. It follows that S' contains $U = N(C \backslash T)$. It is easy to see that any minimal separator bordering B is contained in S or in $N(C \backslash T)$, so $S' \subseteq U$. We conclude that U is a minimal separator of G and $S \cup (T \cap C) = P(S, U)$. □

Notice that the minimal separator U of proposition 6 is also one of the minimal separators bordering the block $S \cup T$. In particular, U is parallel to both S and T. Since U intersects at most one component associated to T, there is at least one full component E associated to T such that $U \cap E = \emptyset$ (cf. figure 1).

Corollary 2. *Let S and T be two crossing separators of an AT-free graph G. Let C be a full component associated to S such that $C \backslash T \neq \emptyset$. Consider the block $S \cup (T \cap C) = P(S, U)$. Let E be a full component associated to T such that $E \cap U = \emptyset$. Then $T \cap C$ is contained in the neighborhood of $S \cap E$.*

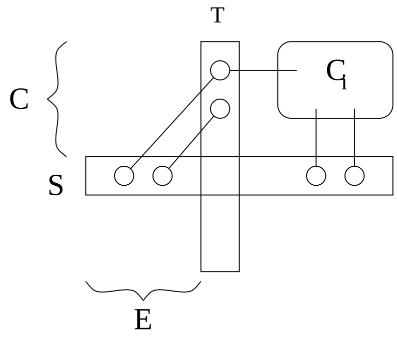

Fig. 1. Crossing separators in an AT-free graph

Proof. We denote $E_C = E \cap C$. Let us prove that $E_C = \emptyset$. Suppose that E_C is non-empty. The neighborhood of E_C is contained in $S \cup T$. We show first that $N(E_C)$ is not contained in T. If $N(E_C) \subseteq T$, then E_C is a connected component of $G \backslash T$, so we must have $E_C = E$. But $E_C \subseteq C$, so the neighborhood of E_C is included in $C \cup N(C) = (S, C)$. Let now D be a full component associated to S, different from C and let $y \in T \cap D$. We have $y \notin N(E_C)$, contradicting the fact that $E_C = E$ is a full component associated to T. Thus, $N(E_C)$ is not contained in T, so there is a vertex y in $N(E_C)\backslash T$. Clearly, $y \in E$. By proposition 6, $U = N(C\backslash T)$, so $y \in U$. This contradicts the fact that E does not intersect U.

We have proved that $E \cap C = \emptyset$. Let now x be any vertex of $T_C = T \cap C$, we show that x has a neighbor in $S_E = S \cap E$. E is a full component associated to T, so x has a neighbor x' in E. Since $x \in C$, we have that x' is in C or in $N(C) = S$. But $E \cap C = \emptyset$, so the only choice remaining is $x' \in S \cap E$. □

Proposition 7. *Let S be a minimal separator of the AT-free graph G and let C be a full component associated to S. Consider any minimal separator T, crossing S. Let $x, y \in C\backslash T$ such that x and y have neighbors in $S\backslash T$. Then x and y are in a same connected component of $G\backslash T$.*

Proof. By proposition 6, we know that $U = N(C\backslash T)$ is a minimal separator and the block $B = S \cup (T \cap C)$ can be written $P(S, U)$. Recall that T and U are parallel. Suppose that T separates x and y and let $x', y' \in S\backslash T$ be two neighbors of x, respectively y. Clearly, T separates x' and y'. But $x', y' \in N(C\backslash T) = U$, so T crosses U – contradiction. □

4 The algorithm

We begin by an informal description of our algorithm. We are given an AT-free graph G and an integer k and we want to decide if $\mathrm{tw}(G) > k$ or to find a triangulation of width at most $2k$. The time complexity of the algorithm we obtain is polynomial and does not depend on the value of k, so it is easy to derive a polynomial algorithm for a 2-approximation of the treewidth of AT-free graphs.

The algorithm relies on this property of AT-free graphs which will be proved in the next section. Given an AT-free graph G of treewidth at most k and a one-block (S, C) with $|S| \leq k$ and $|C| > k + 1$, there is a minimal separator $U \subseteq (S, C)$ such that $|P(S, U)| \leq 2k + 1$ and $|U| \leq k$. Moreover, the separator U can be computed in polynomial time.

Consider any minimal separator S of G and any connected component C of $G\backslash S$. We denote by $R(S, C)$ the realization of (S, C), which is the graph $G_S[S \cup C]$ (see [10]). We have to show that if $|S| \leq k$ then either we can compute a triangulation $H(S, C)$ of $R(S, C)$ of width at most $2k$ or we can decide that $\mathrm{tw}(G) > k$. Indeed, a triangulation of G is obtained by taking $\bigcup_C H(S, C)$ [10].

The algorithm works in this way. If $|C| \leq k + 1$, we simply complete (S, C) into a clique. If $|C| > k + 1$, we are looking for a minimal separator $U \subseteq (S, C)$ such that $|U| \leq k$ and $|P(S, U)| \leq 2k + 1$. If such a separator does not exist we assert that $\mathrm{tw}(G) > k$. If U exists, observe that U splits the block (S, C) into smaller blocks, namely the block $P(S, U)$ and blocks of the form (U, C') with $C' \subset C$. We then apply this process recursively on each sub-block (U, C') to find a triangulation $H(U, C')$ of $R(U, C')$ of width at most $2k$ or output that $\mathrm{tw}(G) > k$. The block $P(S, U)$ will be simply completed into a clique $K(S, U)$. Then $H(S, C)$ is $\bigcup_{C'} H(U, C') \cup K(S, U)$.

Finding the first separator S with $|S| \leq k$ is easy, it is sufficient to look for a separator of minimum size. It is well-known that this computation can be done in polynomial time.

More delicate is the search for the minimal separator U in the general framework and we use a quite indirect method. We will consider the co-bipartite graph $G' = G_{\{S,C\}}[S \cup C]$ obtained from $G[S \cup C]$ by completing S and C. We show that if $\mathrm{tw}(G) < k$, there is a minimal separator X of G' of size at most k. Conversely, we prove that if G' has a minimal separator X with $|X| \leq k$, the vertex set $S \cup X$ is actually a block $P(S, U)$ of G, with $U = N_G(C\backslash X)$. The minimal separator U will satisfy $|P(S, U)| \leq 2k + 1$ and $|U| \leq k$.

We give now the pseudo-code of the algorithm. Suppose that we have a procedure **triangulate_realization** which, given G, k and a one-block (S, C) computes a triangulation of $R(S, C)$ of width at most $2k$ or outputs that $\mathrm{tw}(G) > k$. Here is the main program:

```
approximate_triangulation
Input: an AT-free graph G and an integer k
Output: tw(G) > k or a triangulation H of width at most 2k
begin
    compute a minimal separator S of G of minimum size
    if |S| > k then
        return "tw(G) > k"
    for all one-blocks (S,C) associated to S do
        H(S,C) ← triangulate_realization(G,S,C,k)
        if the call outputs "tw(G) > k" then
            return "tw(G) > k"
    return ⋃_C H(S,C)
end
```

Our goal is to implement the procedure **triangulate_realization** in polynomial time. Before writing down its code, it remains to discuss a point we have hidden until now. It is possible that the algorithm fails to find the separator U even if $\text{tw}(G) \leq k$. This case occurs only when the minimal separator S belongs to all the minimal triangulations of width k. So to handle this problem, we have just to decide if $\text{tw}(R(S,C)) > k$ or to find a triangulation of $R(S,C)$ of width at most $2k$, that is we have to apply the main algorithm on the realization of the block.

```
triangulate_realization
Input: G, S, C and k
Output: tw(G) > k or a triangulation H(S,C) of R(S,C) of width at most 2k
begin
    if |S ∪ C| ≤ 2k + 1 then
        return the clique on vertex set S ∪ C
    /* we have to find the separator U*/
    G' ← G_{S,C}[S ∪ C]
    compute a minimum size separator X of G'
    if |X| > k then
        /* we still have the possibility that tw(R(S,C)) ≤ k */
        return approximate_triangulation(R(S,C), k)
    U ← N_G(C\X)
    for all the one-blocks (U,C') contained in (S,C) do
        H'(U,C') ← triangulate_realization(G,U,C',k)
        if the call outputs "tw(G) > k" then
            return "tw(G) > k"
    return ⋃_{C'} H'(U,C') ∪ G_{P(S,U)}[S ∪ U]
end
```

It still remains to clarify a detail. If (S,C) is not a full block, we first compute $S^* = N(C)$. Note that $S^* \subseteq S$ is a minimal separator and (S^*,C) is a full one-block. Instead of computing a triangulation for $R(S,C)$ we have to compute a triangulation of $R(S^*,C)$.

5 Main theorem

In this section we consider an AT-free graph G of treewidth at most k. We take any minimal triangulation H of G, of width at most k (so $\omega(H) \leq k+1$). Consider now any minimal separator S of G and a full component C associated to S. Suppose that S is not a minimal separator of H and that $|C| > k+1$.

Under these conditions, our main result is the following:

Theorem 2. *There is a minimal separator $U \subset (S,C)$ of G such that $|U| \leq k$ and $|P(S,U)| \leq |S| + k$. Moreover, the minimal separator U can be computed in polynomial time.*

The proof of this theorem will be made in several steps. We will first concentrate on the existence of the minimal separator U. Our proofs will characterize U and will give some further information about the block $P(S, U)$. Then we obtain an algorithm computing U.

Since S is not a minimal separator of H and Δ_H is a maximal set of pairwise parallel separators of G by theorem 1, we have at least one minimal separator $T \in \Delta_H$ that crosses S. Take T among all the minimal separators crossing S such that $T \cap C$ is maximal by inclusion. Observe that $C \backslash T \neq \emptyset$, because $|C| > k + 1$ and $|T| \leq k$. By proposition 6, B is a block of G and there is some $U \subseteq B$ such that $B = P(S, U)$. We denote $X = U \cup (T \cap C)$. We distinguish the "easy" case when $|X| \leq k$, and the "difficult" case, when $|X| > k$.

5.1 Easy case: $|X| \leq k$

Proposition 8. *Consider the co-bipartite graph $G' = G_{\{S,C\}}[S \cup C]$ obtained from $G[S \cup C]$ by completing the sets S and C into cliques. Then X is a minimal separator of G'.*

Proof. We show that $C \backslash X$ and $S \backslash X$ are full components associated to X in G'.

Observe that $C \backslash X \neq \emptyset$. Indeed, $|X| \leq k$, and $|C| > k + 1$, so $C \backslash X \neq \emptyset$. We prove that $N_{G'}(C \backslash X) = X$. By proposition 6, the neighborhood of $C \backslash T$ in G is U. Clearly, the neighborhood of $C \backslash T$ in G' is $(T \cap C) \cup U = X$, so $C \backslash T = C \backslash X$ is a full component associated to X in G'.

Since S is a clique of G', $S \backslash X$ is contained in a connected component D of $G' \backslash X$. This component is different from $C \backslash X$, and since $V(G') = (S \backslash X) \cup X \cup (C \backslash X)$ the only choice is $D = S \backslash X$. Since T and U are parallel in G, there is a full component E associated to T in G such that $E \cap U = \emptyset$. Notice that $S \backslash X$ is not empty, in particular it contains $S_E = S \cap E$. Let us show that X is in the neighborhood of $S \backslash X$ in G'. According to corollary 2, $T \cap C$ is contained in the neighborhood of S_E in the graph G, so $X \backslash S = T \cap C$ is contained in $N_{G'}(S \backslash X)$. Also $X \cap S$ is contained in $N_{G'}(S \backslash X)$, because S is a clique in G'. Hence, $N_{G'}(S \backslash X) = X$. We conclude that $S \backslash X$ is a full component associated to X in G'.

We have proved that $G' \backslash X$ has two full components associated to X. By lemma 1, X is a minimal separator of G'. □

So there is a minimal separator X of G' of size at most k. Conversely, we show that if G' has a minimal separator Y of size at most k, we will be able to compute a minimal separator U like in theorem 2.

Proposition 9. *Let Y be a minimal separator of the graph $G' = G_{\{S,C\}}[S, C]$. Then $S \cup Y$ is a two-block $P(S, U)$ of G. Moreover, $U = N_G(C \backslash Y)$ and U is contained in Y.*

Proof. We denote $B = S \cup Y$. We show that B is not contained in S or in Y. Since Y is a minimal separator of the cobipartite graph G', the connected components of $G' \backslash Y$ are exactly $S \backslash Y$ and $C \backslash Y$. In particular, both of them are non empty, so $S \not\subseteq Y$. Also, $Y \not\subset S$. Indeed, if $Y \subset S$, let x be a vertex of $S \backslash Y$ and let $y \in C$ be a neighbor of x in the graph G. Clearly, x and y are also adjacent in G', contradicting the fact that $S \backslash Y$ and $C \backslash Y$ are in different connected components of $G' \backslash Y$. Thus, $B \not\subseteq S$ and $B \not\subseteq Y$.

Let $C_1, \ldots, C_p$ be the connected components of $G \backslash B$ and let $S_1, \ldots, S_p$ be their neighborhoods in G. The C_i's are either connected components of $G \backslash S$ different from C or connected components of $G[C \backslash Y]$. In particular, if C_i is a full component associated to S, different from C, then $S_i = S$.

We will see that each S_i is a minimal separator of G and that B is contained in some full block (S_i, D_i). Consider first the case when C_i is a connected component of $G \backslash S$, different from C. Then $S_i \subseteq S$. Clearly, $C \cap S_i = \emptyset$, so C is contained in some connected component D_i of $G \backslash S_i$, with $D_i \neq C_i$. Since each vertex of S has a neighbor in C, it follows that every vertex of S_i has a neighbor in D_i. Thus, D_i is a full component associated to S_i. Both C_i and D_i are full components associated to S_i, so S_i is a minimal separator by lemma 1. Since $C \subseteq D_i$, we have that $S = N_G(C)$ is contained in $D_i \cup N_G(D_i) = (S_i, D_i)$. We conclude that $B \subseteq (S, C) \subseteq (S_i, D_i)$. Consider now the case when C_i is a connected component of $G[C \backslash Y]$. Notice that S_i is contained in Y. Let D be a full component associated to S, different from C. Since $S_i \subseteq (S, C)$, S_i does not intersect D, so D is contained in a connected component D_i associated to S_i. Let us prove first that $S \subseteq S_i \cup D_i$. Indeed, $S = N_G(D)$, so S is contained in $D_i \cup N_G(D_i) \subseteq S_i \cup D_i$. We show now that $Y \backslash S$ is also contained in $S_i \cup D_i$. In the graph G', the vertex set $S \backslash Y$ is a full component associated to Y, so any vertex $x \in Y$ has a neighbor $y \in S \backslash Y$. In particular, if $x \in Y \backslash S$, then y is also a neighbor of x in the graph G (the edges between $S \backslash Y$ and $Y \backslash S$ are the same in G and in G'). Thus, $Y \backslash S$ is contained in $N_G(S \backslash Y)$. Since $S_i \subseteq Y$ we have $S \backslash Y \subseteq S \backslash S_i \subseteq D_i$, so $Y \backslash S \subseteq D_i \cup N_G(D_i) \subseteq S_i \cup D_i$. Therefore, $B \subseteq S_i \cup D_i$. Observe now that D_i is a full component associated to S_i. Indeed, each vertex of $S_i \cap S$ has a neighbor in D, so in D_i, and each vertex of $S_i \backslash S \subseteq Y \backslash S$ has a neighbor in $S \backslash Y$, so in D_i. Thus, D_i is a full component associated to S_i. Clearly $D_i \neq S_i$, so by lemma 1, S_i is a minimal separator.

We have proved that each S_i is a minimal separator of G and that B is contained in some full one-block (S_i, D_i). Moreover B is not contained in any S_i: recall that each S_i is contained in S or in Y, but $B \not\subseteq Y$ and $B \not\subseteq S$, so $B \not\subseteq S_i$. By proposition 1, B is a block of G.

S is one of the minimal separators bordering B. We show now that S is maximal by inclusion among $\{S_1, \ldots, S_p\}$. Any minimal separator S_i is contained in S or in Y. Since $S \not\subseteq Y$, S is inclusion maximal among the minimal separators bordering B. We want to prove that S is not the only inclusion-maximal element of $\{S_1, \ldots, S_p\}$. Let C_i be any connected component of $G[C \backslash Y]$. We show that $S_i \not\subseteq S$. Indeed, if $S_i \subseteq S$, then C_i is also a connected component of $G \backslash S$, so the only choice is $C_i = C$. This implies $Y \cap C = \emptyset$, contradicting $Y \not\subseteq S$. The set of minimal separators bordering B has at least two inclusion maximal elements, S and S'. The AT-free graph G has only one-blocks and two-blocks, so B is a two-block $P(S, S')$. It remains to prove that $S' = N_G(C \backslash Y)$. Since $S' \not\subseteq S$, S' is the neighborhood of a connected component C_j of $G[C \backslash Y]$. Thus, $S' \subseteq N_G(C \backslash Y)$. For any connected component $C_i \subseteq C \backslash Y$, we have $S_i \not\subseteq S$, so $S_i \subseteq S'$ by remark 1. Consequently, S' contains $N_G(C \backslash Y)$. We conclude that $S' = N_G(C \backslash Y) = U$. Thus, $B = P(S, U)$. Moreover, $U = N_G(C \backslash Y) \subseteq N'_G(C \backslash Y)$, so $U \subseteq Y$. □

We deduce:

Corollary 3. *Let G be an AT-free graph and let (S, C) be a full block of G. We denote by G' the cobipartite graph $G_{\{S,C\}}[S \cup C]$. If G' has a minimal separator of size at most k, then we can compute in polynomial time a minimal separator U of G such that $P(S, U) \leq |S| + k$ and $|U| \leq k$.*

Proof. We compute a minimum size separator Y of G'. This can be done in polynomial time. Clearly, Y is a minimal separator of G' and $|Y| \leq k$. According to proposition 9, the vertex set $S \cup Y$ is a two-block $P(S, U)$ of G. Therefore, $|P(S, U)| \leq |S| + |Y| \leq |S| + k$. By proposition 9, $U \subseteq Y$ so $|U| \leq k$. Moreover, $U = N_G(C \backslash Y)$, so U can be computed in polynomial time. □

5.2 Difficult case: $|X| > k$

Let us consider now the case when $X = (T \cap C) \cup U$ has strictly more that k vertices. Our goal is to show that also in this case there is a minimal separator Y of $G' = G_{\{S,C\}}[S \cup C]$ such that $|Y| \leq k$. Thus, the corollary 3 will allow us to compute the minimal separator U like in theorem 2.

Due to space restriction, we only give here an outline of the proof. A full description is given in appendix A.

The construction of the vertex set Y is rather technical and will be made in several steps. We will show that there is a minimal separator T' of H such that T' crosses X. We prove then that $P(T, T')$ exists. We will consider the block $B = P(T, T')$ and we put $Y = (B \cap C) \cup N_G(C \backslash B)$.

The first crucial observation is that Y is strictly contained in the block B. Moreover, Y is a minimal separator of the co-bipartite graph $G' = G_{\{S,C\}}[S \cup C]$.

The second step is to prove that $|Y| \leq k$. This is true if our block B is a clique of H, because the cliques of H have at most $k+1$ vertices. If B is not a clique, we can split it into smaller blocks, and we show that Y is strictly contained in one of these smaller blocks. We reiterate this process until we obtain a block B' that can not be splitted. This implies that B' is a clique, and since $Y \subset B$ we conclude that $|Y| \leq k$.

Using corollary 3, we deduce that in this case we can also compute the minimal separator U such that $|U| \leq k$ and $|P(S,U)| \leq |S| + k$, which proves the theorem 2.

References

1. S. Arnborg, D.G. Corneil, and A. Proskurowski. Complexity of finding embeddings in a k-tree. *SIAM J. on Algebraic and Discrete Methods*, 8:277–284, 1987.
2. H. Bodlaender, J.R. Gilbert, H. Hafsteinsson, and T. Kloks. Approximating treewidth, pathwidth, and minimum elimination tree height. *J. of Algorithms*, 18:238–255, 1995.
3. H.L. Bodlaender, T. Kloks, and D. Kratsch. Treewidth and pathwidth of permutation graphs. *SIAM J. on Discrete Math.*, 8:606–616, 1995.
4. V. Bouchitté and I. Todinca. Minimal triangulations for graphs with "few" minimal separators. In *Proceedings 6th Annual European Symposium on Algorithms (ESA'98)*, volume 1461 of *Lecture Notes in Computer Science*, pages 344–355. Springer-Verlag, 1998.
5. V. Bouchitté and I. Todinca. Treewidth and minimum fill-in of weakly triangulated graphs. In *Proceedings 16th Symposium on Theoretical Aspects of Computer Science (STACS'99)*, volume 1563 of *Lecture Notes in Computer Science*, pages 197–206. Springer-Verlag, 1999.
6. V. Bouchitté and I. Todinca. Listing all potential maximal cliques of a graph. In *Proceedings 17th Annual Symposium on Theoretical Aspects of Computer Science (STACS 2000)*, volume 1770 of *Lecture Notes in Computer Science*, pages 503–515. Springer-Verlag, 2000.
7. H. Broersma, T. Kloks, D. Kratsch, and H. Müller. A generalization of AT-free graphs and a generic algorithm for solving triangulation problems. In *Workshop on Graphs WG'98*, volume 1517 of *Lecture Notes in Computer Science*, pages 88–99. Springer-Verlag, 1998.
8. M. C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs.* Academic Press, New York, 1980.
9. T. Kloks, D. Kratsch, and H. Müller. Approximating the bandwidth for asteroidal triple-free graphs. *Journal of Algorithms*, 32:41–57, 1999.
10. T. Kloks, D. Kratsch, and J. Spinrad. On treewidth and minimum fill-in of asteroidal triple-free graphs. *Theoretical Computer Science*, 175:309–335, 1997.
11. A. Parra and P. Scheffler. Characterizations and algorithmic applications of chordal graph embeddings. *Discrete Appl. Math.*, 79(1-3):171–188, 1997.
12. N. Robertson and P. Seymour. Graphs minors. II. Algorithmic aspects of tree-width. *J. of Algorithms*, 7:309–322, 1986.

A Difficult case

We consider here the case when $X = (T \cap C) \cup U$ has strictly more that k vertices. Our goal is to show that also in this case there is a minimal separator Y of $G' = G_{\{S,C\}}[S \cup C]$ such that $|Y| \leq k$.

The next few lemmas will construct the vertex set Y and will allow us to prove that Y is a minimal separator of G' (proposition 10).

Lemma 2. *There is a minimal separator T' of H such that T' crosses X.*

Proof. This is equivalent to say that X is not a clique of H.

Any clique of H has at most $k+1$ vertices, so the assertion is true if $|X| > k+1$. Suppose that $|X| = k+1$. If X is a clique in H, then X is a maximal clique of H. We say that a vertex set Ω is a *potential maximal clique* of a graph if Ω forms a maximal clique in some minimal triangulation of that graph. In our particular case, X is a potential maximal clique of G. We proved in [5], theorem 4, that if Ω is a potential maximal clique of G, then $G \backslash \Omega$ has no full component associated to Ω. We show now that $G \backslash X$ has a full component associated to X, contradicting the fact X is a potential maximal clique of G.

Let E' be the unique component associated to T that intersects U (recall that $T \| U$) and let E be a full component associated to T, different from E'. Since $X = (T \cap C) \cup U$, X does not intersect E, so E is connected in $G \backslash X$. Let D be a full component associated to S, different from C. Notice that $X \cap D = \emptyset$, because $X \subseteq (S, C)$. Thus, D is connected in $G \backslash X$. We prove now that $D \cup E$ is contained in a full component associated to X. Let us show that there are two adjacent vertices $x \in E$ and $y \in D$. Since $S \# T$, there is at least one vertex $x \in S \cap E$. Clearly, $x \notin X$. But D is a full component associated to S, so the vertex x has a neighbor y in D. It follows that $D \cup E$ is contained in some connected component F of $G \backslash X$. Each vertex of S has a neighbor in D and each vertex of T has a neighbor in E, so each vertex of X has a neighbor in F.

We have proved that F is a full component associated to X in G, so X is not a maximal clique of H. Therefore, there is a minimal separator T' of H such that T' crosses X. □

Notation 1 *Henceforth we denote by E' the unique component associated to T that intersects U. Like previously, E will be a full component associated to T, different from E'.*

Remark 2. Notice that E' is also the unique component associated to T that intersects X, respectively T' (see figure 2).

Lemma 3. *Every vertex of $T_C = T \cap C$ has a neighbor in $S \cap E$.*

Proof. This comes directly from corollary 2. □

Lemma 4. *T' crosses S.*

Proof. The claim is clearly true if T' separates two wertices of $U \cap S$.

Suppose now that T' does not cross $U \cap S$. In particular, T' does not cross $X \backslash T$. Since $T \| T'$, T' cannot separate two vertices of $T \cap X$. But T' separates two vertices of $X = (T \cap X) \cup (X \backslash T)$, so the only possibility is that T' separates a vertex x of $T \cap X$ and a vertex y of $X \backslash T$. Let x' be a vertex of $S \cap E$, adjacent to x (x' exists by lemma 3). We show that T' separates x' and y. Indeed, $x' \notin T'$ (T' is contained in the block (T, E')) and T' separates x and y. Since x and x' are adjacent, T' also separates x' and y. But $x', y \in S$, so T' crosses S. □

Lemma 5. *We put $T_C = T \cap C$ and $T'_C = T' \cap C$. Then both $T_C \backslash T'_C$ and $T'_C \backslash T_C$ are non-empty.*

Proof. Recall that T was chosen among the minimal separators of H crossing S such that $T \cap C$ is maximal by inclusion. Since T' is a minimal separator of H crossing S, we have that T'_C cannot properly contain T_C.

We show that $T'_C \backslash T_C \neq \emptyset$. Let $C(U)$ be a full component associated to U, different from the one intersecting S. In particular, T does not intersect $C(U)$ (T and S are contained in a same one-block associated to U). Notice also that $C(U) \subset C$. Let us prove that T' contains at least one vertex of $C(U)$, so $T'_C \backslash T_C \neq \emptyset$. We will consider two cases, when $T' \| U$, respectively $T' \# U$.

First case: $T' \| U$. Since $T' \| T$ and T' crosses $X = T_C \cup U$, we have that T' separates some $x \in T_C \backslash T'$ and some $y \in U \backslash T'$. Let y' be a neighbor of y in $C(U)$ (y' exists, because $C(U)$ is a full component associated to U). We shall see that $y' \in T'$. Indeed if $y' \notin T'$, since T' separates x and y and since y' is adjacent to y, T' also separates x and y'. By lemma 3, x has a neighbor in $S \cap E$, so both x and y' have neighbors in $S \backslash T'$, contradicting proposition 7.

Second case: $T' \# U$. T' intersects all the full components associated to U, so T' intersects $C(U)$.

We conclude that $T' \cap C(U) \neq \emptyset$, so $\in T'_C \backslash T_C$ contains at least one vertex. Since T_C is not properly contained in T'_C, we deduce that $T_C \backslash T'_C$ is non-empty. □

Remark 3. Since $T \not\subseteq T'$ and $T' \not\subseteq T$, the piece between T and T' exists.

Notation 2 *We will denote by F' the component associated to T' that intersects T and by F a full component associated to T', different from F' (see figure 2). We denote $B = P(T, T')$ and $B_C = B \cap C$.*

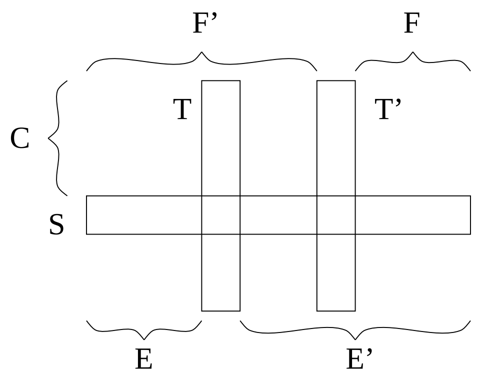

Fig. 2. Difficult case – notations

Lemma 6. *Every vertex of T'_C has a neighbor in $S \cap F$.*

Proof. The minimal separators S and T' cross. Clearly $C \backslash T' \neq \emptyset$, because $|C| > k+1$ and $|T'| \leq k$. By proposition 6, the vertex set $S \cup T'_C$ is a block $P(S, U')$ of G, with $U' = N(C \backslash T')$.

Let us show that U' intersects F'. Let a be a vertex of $T_C \backslash T'_C$. By lemma 3, a has a neighbor a' in $S \cap E$. Since $a' \in U' = N(C \backslash T')$, it follows that U' intersects E, and therefore U' intersects $F' \supset E$. F is a full component associated to T', different from F'. By applying corollary 2 to S and T', we obtain that $T' \cap C$ is contained in $N(S \cap F)$. □

Lemma 7. *Let G be an AT-free graph and (S, C) be a full block of G. Consider two minimal separators Q and Q' of G, both crossing S and such that $P(Q, Q')$ exists. Then $P(Q, Q') \cap C = (Q \cup Q') \cap C$.*

In particular, $B_C = T_C \cup T'_C$.

Proof. Suppose there is a vertex $x \in P(Q, Q') \cap C$, which is not contained in $Q \cap C$ or in $Q' \cap C$. Let D be a full component associated to S, different from C. Consider a vertex $y \in D$ and let S_{xy} be the unique minimal x, y-separator contained in $N(x)$. We prove that Q, Q' and S_{xy} form an asteroidal triple of separators.

We show first that S_{xy} is contained in $P(Q,Q')$. Let $C_x(Q)$ be the connected component of $G\backslash Q$ that contains x. In particular, Q' intersects C_x, otherwise Q separates x and Q', contradicting $x \in P(Q,Q')$. Let $C_x(Q')$ be the connected component associated to Q' such that $x \in C_x(Q')$ – so Q intersects $C_x(Q')$. We deduce that $P(Q,Q') = (Q, C_x(Q)) \cap (Q', C_x(Q'))$. We prove now that S_{xy} is contained in both $(Q, C_x(Q))$ and $(Q', C_x(Q'))$. Recall that S_{xy} is included in the neighborhood of x. Since $x \in C_x(Q)$ and the neighborhood of $C_x(Q)$ is contained in Q, we have $S_{xy} \subseteq N(x) \subseteq (Q, C_x(Q))$. For similar reasons, $S_{xy} \subseteq (Q', C_x(Q'))$. We conclude that S_{xy} is included in $P(Q,Q')$.

Let us prove that Q, Q' and S_{xy} do not satisfy any of the three conditions of proposition 5: S_{xy} does not contain Q or Q', S_{xy} is not contained in Q or Q', and S_{xy} does not separate Q and Q'. We show that S_{xy} does not contain Q or Q'. Notice that $x \in C$ and $S = N(C)$, so $S_{xy} \subseteq (S, C)$. But both Q and Q' have vertices in D, so $Q \not\subseteq S_{xy}$ and $Q' \not\subseteq S_{xy}$. We prove now that S_{xy} does not separate Q and Q'. Since $S_{xy} \subseteq (S, C)$, we deduce that D is contained in a connected component of $G\backslash S_{xy}$. In particular, S_{xy} does not separate the vertices of $Q \cap D$ and of $Q' \cap D$, so S_{xy} does not separate Q and Q'. It remains to prove that S_{xy} is not contained in Q (the case $S_{xy} \not\subseteq Q'$ is similar). Suppose that $S_{xy} \subset Q$. By construction of S_{xy}, S_{xy} separates x and the vertices of $D \cap Q'$. Since $S_{xy} \| Q'$, all the vertices of $Q'\backslash S_{xy}$ are in a same component associated to S_{xy}. Thus, S_{xy} separates x and every vertex of $Q'\backslash S_{xy}$. Then also $Q \supset S_{xy}$ separates x and $Q'\backslash Q$. This contradicts $x \in P(Q,Q')$. □

Corollary 4. *$B_C = T_C \cup T'_C$ is contained in the neighborhood of $S \cap (E \cup F)$.*

Proof. This comes directly from lemmas 7, 3 and 6. □

Lemma 8. *Let Z be a minimal separator of H, such that Z separates two vertices $x \in T$ and $y \in T'$. Then $Z \subseteq P(T,T')$ and Z crosses S.*

Proof. Since Z, T and T' are minimal separators of H, they are pairwise parallel. Let us show that Z is in the piece between T and T'. Recall that $P(T,T') = (T, E') \cap (T', F')$. Suppose that $Z \not\subseteq P(T,T')$, for example $Z \not\subseteq (T, E')$. In particular, Z does not intersect E', so E' is contained in a connected component $C_{E'}$ of $G\backslash Z$. We prove that T is contained in the one-block $(Z, C_{E'})$. Indeed, $x \in T$ has a neighbor in the full component E' associated to T, so $x \in C_{E'} \cup N(C_{E'}) \subseteq (Z, C_{E'})$. But $x \notin Z$, so $x \in C_{E'}$. It follows that T is contained in $(Z, C_{E'})$. Thus, $P(T,T') \subseteq (T, E') \subseteq (Z, C_{E'})$, contradicting the fact that Z separates $x, y \in P(T,T')$.

We prove now that Z separates E and F. Since E is a full component associated to T, the vertex x has a neighbor in E. Notice that $Z \subseteq P(T,T')$ does not intersect E. It follows that x and E are in a same connected component C_x of $G\backslash Z$. For the same reason, y and F are in a same connected component C_y of $G\backslash Z$. Since Z separates x and y, Z also separates E and F, in particular Z separates $S \cap E$ and $S \cap F$. We conclude that Z crosses S. □

Lemma 9. *B_C is a clique of H.*

Proof. Suppose there is a minimal separator Z of H separating two vertices of B_C. Since Z is parallel to both T and T' and $B_C = T_C \cup T'_C$, it follows that Z separates two vertices $a \in T_C\backslash T'_C$ and $b \in T'_C\backslash T_C$. On one hand, this implies that $Z \subseteq P(T,T')$ and Z crosses S. On the other hand, both a and b have neighbors in $S\backslash P(T,T')$, by corollary 4. This contradicts the proposition 7. □

Lemma 10. *Let Q and Q' be two minimal separators crossing S. Suppose that $B_C \subseteq P(Q,Q') \subseteq P(T,T')$ and B_C is not contained in Q nor in Q'. Let $Y = B_C \cup N_G(C\backslash B_C)$. Then $Y \subset P(Q,Q')$. In particular, $Y \subset P(T,T')$.*

Proof. It is sufficient to show that $N_G(C\backslash B_C)$ is contained in $P(Q,Q')$. Suppose there is a vertex $x' \in N_G(C\backslash B_C)\backslash P(Q,Q')$ and let x be a neighbor of x' in $C\backslash B_C$. Let C_x the connected component of $G\backslash P(Q,Q')$ containing x and x' and let $S_x = N(C_x)$. According to remark 1, $P(Q,Q')$ is a block and the minimal separator S_x bordering $P(Q,Q')$ is contained in Q or in Q'. Suppose w.l.o.g. that $S_x \subseteq Q$, so Q separates C_x and $P(Q,Q')$. Let y be a vertex of $B_C\backslash Q$. Clearly, Q separates x and

y. By corollary 4, y has a vertex y' in $S\backslash P(Q,Q')$. Hence, both x and y have neighbors in $S\backslash Q$. But Q separates x and y, contradicting proposition 7.

Thus, $Y \subseteq P(Q,Q')$. It remains to show that the inclusion is strict On one hand, $Q\#S$ so Q intersects a full component D associated to S, different from C. On the other hand, Y is contained in $C \cup S$. Hence, Y is strictly contained in $P(Q,Q')$. □

Proposition 10. *Let $B_C = P(T,T') \cap C$ and $Y = B_C \cup N_G(C\backslash B_C)$. Then Y is a minimal separator of the graph $G' = G_{\{S,C\}}[S \cup C]$.*

Proof. The proof is very similar to the proof of proposition 8. We will show that $C\backslash Y$ and $S\backslash Y$ are full components associated to Y in G'.

By lemma 9, $B_C = Y \cap C$ is a clique of H, so $|Y \cap C| \leq k+1$. Since $|C| > k+1$, $C\backslash Y$ is not empty. We show that $N_{G'}(C\backslash Y) = Y$. C is a clique of G', so $N_{G'}(C\backslash Y) \cap C = Y \cap C$. Notice that $N_{G'}(C\backslash Y) \cap S = N_G(C\backslash Y) \cap S = Y \cap S$. Thus, $N_{G'}(C\backslash Y) = Y$.

By lemma 10, $Y \subseteq P(T,T')$, so $S\backslash Y$ contains $S_E = S \cap E$ and $S_F = S \cap F$. Hence, $S\backslash Y$ is not empty. Let D be the connected component of $G'\backslash Y$ containing the clique $S\backslash Y$. Since $V(G') = (S\backslash Y) \cup Y \cup (C\backslash Y)$, we have $D = S\backslash Y$. By the fact that S is a clique of G', we have that $S \cap Y$ is contained in $N_{G'}(S\backslash Y)$. By corollary 4 we know that $Y\backslash S$ is contained in $N_G(S_E \cup S_F)$, in particular $Y\backslash S \subseteq N_{G'}(S\backslash Y)$. Thus, $N_{G'}(S\backslash Y) = Y$.

So $C\backslash Y$ and $S\backslash Y$ are two full components associated to Y in G', and by lemma 1 Y is a minimal separator of G'. □

We want to show now that $|Y| \leq k$. Actually, we will prove that Y is a strictly included in a clique of H (theorem 3). We know by lemma 10 that Y is strictly contained in $P(T,T')$. In particular, if $P(T,T')$ is a clique of H, we have that $|Y| \leq k$.

Suppose now that $P(T,T')$ is not a clique of H. Therefore, there is a minimal separator Z of H, crossing $P(T,T')$. We will show that either $Y \subset P(Z,T)$ or $Y \subset P(Z,T')$. We recursively iterate this process until $P(Z,T)$ or $P(Z,T')$ becomes a clique of H. Since a clique of H has at most $k+1$ vertices, we will conclude that $|Y| \leq k$.

Remark 4. From now on, the only assumptions we need about the minimal separators T and T' are:

1. T and T' cross S,
2. both $T_C\backslash T'_C$ and $T'_C\backslash T_C$ are non-empty,
3. $B_C = T_C \cup T'_C$.

We not only prove that $B_C \subset P(Z,T)$ or $B_C \subset P(Z,T')$, we also show that Z and T (respectively Z and T') satisfy the same assumptions as T and T' (cf. proposition 12). Thus, we will be able to reiterate the process with Z and T (respectively Z and T') instead of T and T'.

We are going to prove that $B_C = Y \cap C$ is contained in $P(Z,T)$ or in $P(Z,T')$ (proposition 11).

Lemma 11. *If $Z \subset T$, then Z separates T and T'.*

Proof. Since T and T' are parallel to Z, they are respectively contained in some blocks (Z,C_T) and $(Z,C_{T'})$ associated to Z. Suppose that Z does not separate T and T', so $C_T = C'_T$. Let us prove that $P(T,T') = (T,E') \cap (T',F')$ is contained in (Z,C_T), contradicting Z crosses $P(T,T')$. We have $T \subseteq (Z,C_T)$, so it is sufficient to show that $E' \subseteq C_T$. Clearly E' does not intersect $Z \subset T$, so E' is contained in a connected component of $G\backslash Z$. Let y be a vertex of $T'\backslash T$. From $T' \subseteq (T,E')$, we deduce $y \in E'$. From $T' \subseteq (Z,C_T)$ and $y \notin Z$ we deduce $y \in C_T$. It follows that $E' \subseteq C_T$.

Hence, $(T,E') \subseteq (Z,C_T)$, contradicting the fact that Z crosses $P(T,T') \subseteq (T,E')$. □

Lemma 12. *Z crosses S.*

Proof. Since $Z \subseteq P(T,T')$ and T,T' and Z do not form an asteroidal triple of separators, we have by proposition 5 that Z contains T or T', Z is contained in T or T', or Z separates T and T'. We consider each of these cases.

Suppose that Z contains T. Since T crosses S, we have that S separates two vertices of $T \subseteq Z$, and therefore S crosses Z. The case $Z \supseteq T'$ is similar. Now if Z is contained in T or T', we have by lemma 11 that Z separates T and T'. It remains to prove that if Z separates T and T', then Z crosses S. This comes directly from lemma 8. □

Remark 5. B_C is not contained in any minimal separator of H crossing S. In particular, B_C is not contained in Z.

Proof. Recall that T was chosen among the minimal separators of H crossing S such that $T \cap C$ is maximal by inclusion. The conclusion follows directly from the fact that B_C strictly contains T_C. □

Proposition 11. *$B_C \subseteq P(T,Z)$ or $B_C \subseteq P(Z,T')$.*

Proof. According to remark 5, there is a vertex $b \in B_C \backslash Z$. Since $B_C = T_C \cup T'_C$, we have $b \in T_C$ or $b \in T'_C$. Suppose w.l.o.g. that $b \in T'_C$. We prove first that $P(Z,T')$ exists. Since $b \in T' \backslash Z$, it remains to show that $Z \not\subseteq T'$. Since Z crosses $P(T,T')$, we have $Z \neq T'$. Suppose that $Z \subset T'$. By lemma 11, Z separates T and T'. Let $a \in T_C \backslash T'_C$, in particular $a \in T \backslash Z$. We have that Z separates $a, b \in B_C$, contradicting the fact that B_C is a clique.

We have proved that $P(Z,T')$ exists. Let us show that it contains B_C. Let $C_{T'}$ be the connected component of $G \backslash Z$ which intersect T' – so $P(Z,T') = (Z, C_{T'}) \cap (T', F')$. We have $b \in B_C \cap C_{T'}$. Since B_C is a clique of H, Z cannot separate two vertices of B_C. Thus, B_C is contained in some one-block associated to Z, so the only choice is $B_C \subseteq (Z, C_{T'})$. Moreover $B_C \subseteq P(T,T') \subseteq (T',F')$, so $B_C \subseteq P(Z,T')$. □

Proposition 12. *Suppose that $B_C \subseteq P(Z,T')$. We denote $Z_C = Z \cap C$. Then:*

1. *Z and T' cross S.*
2. *$B_C = Z_C \cup T'_C$.*
3. *both $Z_C \backslash T'_C$ and $T'_C \backslash Z_C$ are non-empty.*

Proof. Clearly, both Z and T' cross S. By lemma 7 we have that $P(Z,T') \cap C = Z_C \cup T'_C$. It follows that $B_C \subseteq Z_C \cup T'_C$. Notice now that $P(T,T') \cap C = B_C \subseteq P(Z,T') \cap C$, so we have $B_C = Z_C \cup T'_C$.

Since $B_C = Z_C \cup T'_C$ and B_C is not contained in Z_C or T'_C (cf. remark 5), we conclude that $Z_C \backslash T'_C \neq \emptyset$ and $T'_C \backslash T_C \neq \emptyset$. □

Theorem 3. *There is a clique $\Omega = P(T_1,T_2)$ of H, such that $Y = B_C \cup N_G(C \backslash B_C)$ is strictly contained in Ω.*

Proof. Let us prove first that there is a maximal clique $\Omega = P(T_1,T_2)$ of H such that both T_1 and T_2 cross S and B_C is contained in Ω. The assertion is clearly true if $P(T,T')$ is a clique of H. Otherwise, there is a $Z \in \Delta_H$ crossing $P(T,T')$. We have proved that B_C is contained in $P(Z,T)$ or in $P(Z,T')$. Suppose w.l.o.g. $B_C \subseteq P(Z,T')$. We already pointed out in remark 4 that we can reiterate the process with Z,T' instead of T,T'. Since $P(Z,T')$ is strictly contained in $P(T,T')$, we will finally obtain a clique $\Omega = P(T_1,T_2)$ containing B_C and such that T_1 and T_2 cross S.

By lemma 10 we have that Y is strictly contained in $\Omega = P(T_1,T_2)$. So $|Y| \leq |\Omega| - 1 \leq k$. □

From proposition 10 and theorem 3 we deduce:

Corollary 5. *Y is a minimal separator of $G' = G_{\{S,C\}}[S \cup C]$, of size at most k.*

Using corollary 3, we conclude that in this case we can also compute the minimal separator U such that $|U| \leq k$ and $|P(S,U)| \leq |S| + k$, which proves theorem 2.