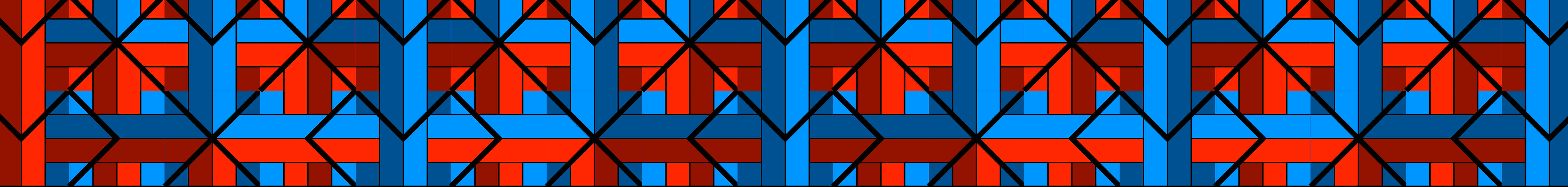
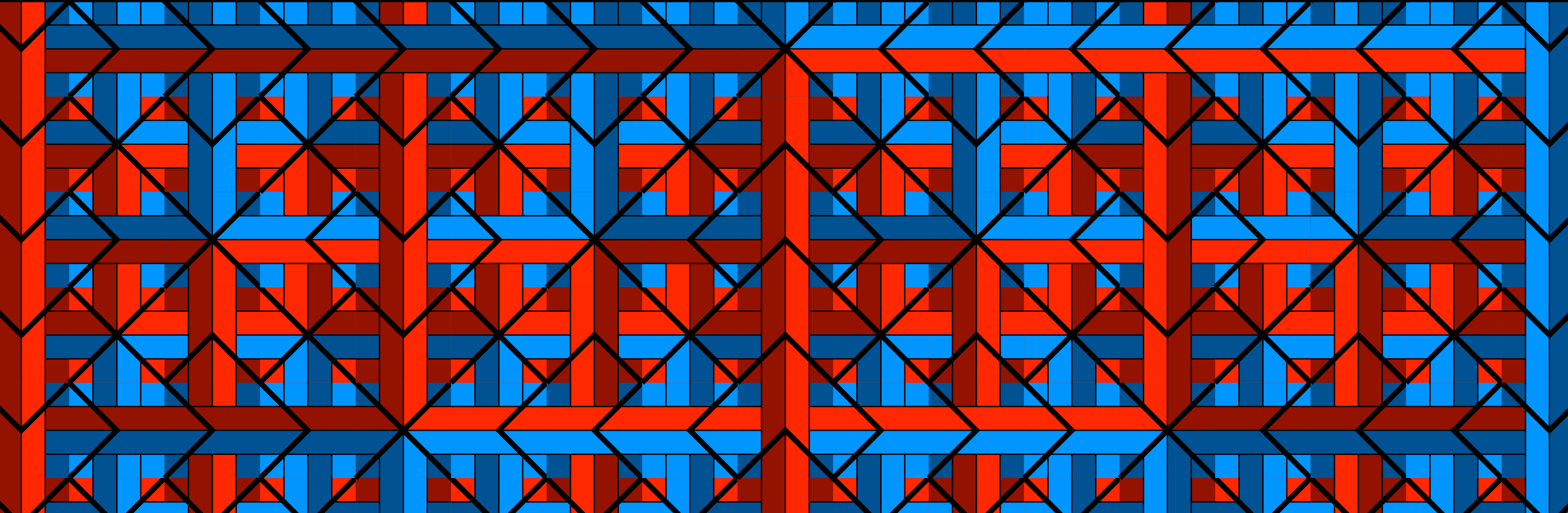


Two-by-two Substitution Systems and the Undecidability of the Domino Problem

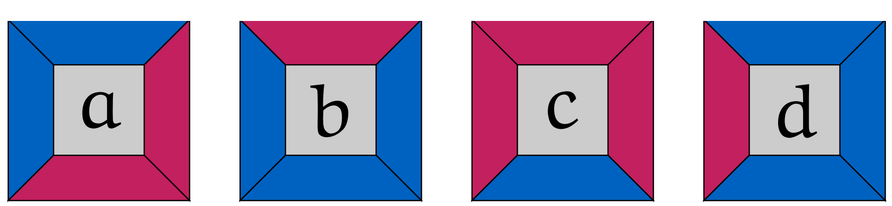
N. Ollinger (LIF, Aix-Marseille Université, CNRS, France)

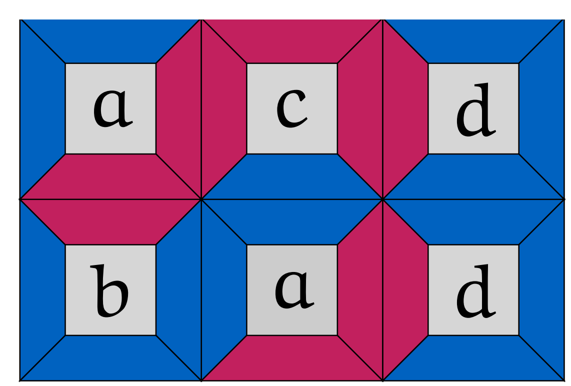


The Domino Problem

"Assume we are **given a finite set of square plates** of the same size with **edges colored**, each in a different manner. Suppose further there are infinitely many copies of each plate (plate type). We are **not permitted to rotate or reflect a plate**. The question is to find an effective procedure by which we can **decide**, for each given finite set of plates, **whether we can cover up the whole plane** (or, equivalently, an infinite quadrant thereof) **with copies of the plates** subject to the restriction that **adjoining edges must have the same color**."

(Wang, 1961)





Bits of History: the AEA case

- 1961. Wang and Büchi consider the classical decision problem for AEA first-order formulae. Wang introduces **DP**. Büchi introduces the **Immortality Problem**.

- 1961. The AEA *entscheidungsproblem* is proved undecidable by Kahr-Moore-Wang by considering a simpler problem with dominos.

- 1964. Berger, a student of Wang, proves the undecidability of **DP** providing as a corollary a new proof of the undecidability for AEA.

- 1965. Hooper, a student of Wang, proves the undecidability of **IP** providing as a corollary a new proof of the undecidability for AEA.

Existing Constructions

- Two main type of related activity in the literature:
  (1) construct aperiodic tile sets (small ones);
  (2) give a full proof of the undecidability of **DP** (implies (1)).

- From 104 tiles (Berger, 1964) to 13 tiles (Čulik, 1996) aperiodic sets.

- Seminal self-similarity based proofs:
  - Berger, 1964 *(20426 tiles, a full PhD thesis)*
  - Robinson, 1971 *(56 tiles, 17 pages, long case analysis)*
  - Durand *et al*, 2007 *(Kleene's fixpoint existence argument)*

- Tiling rows seen as transducer trace based proof:
  - Kari, 2007 *(affine maps, short concise proof, reduction from* ***IP****)*

Motivation for a New Construction

- Undecidability of **DP** proof technic proved useful for variations of **DP** for undecidability in dynamical systems like Cellular Automata: Kari, 1989, 1992, Durand, 1994, Mazoyer and Rapaport, 1997.

- Teaching the proofs in a uniform textbook framework:
  - Affine technics have dependencies on **IP**, also they have not yet been adapted to all the constructions.
  - Self-similarity constructions have big tile sets and/or complex proofs involving tedious case-based reasoning.

- We propose a new self-similarity based construction building on classical proof schemes with concise arguments and few tiles.

Au menu

1. Patterns, colorings, and tilings
2. 2x2 substitutions
3. A Set of 104 tiles
4. This set is aperiodic
5. Enforcing any substitution (sketch)
6. Undecidability of the Domino Problem (sketch)

This is a mix of different tools and ideas from:

[Berger 64] *The Undecidability of the Domino Problem*

[Robinson 71] *Undecidability and nonperiodicity for tilings of the plane*

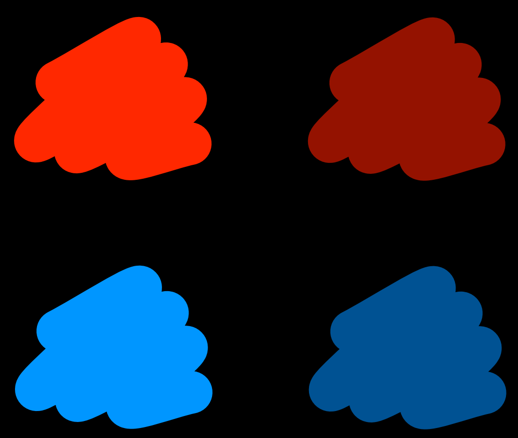
[Grünbaum Shephard 89] *Tilings and Patterns, an introduction*

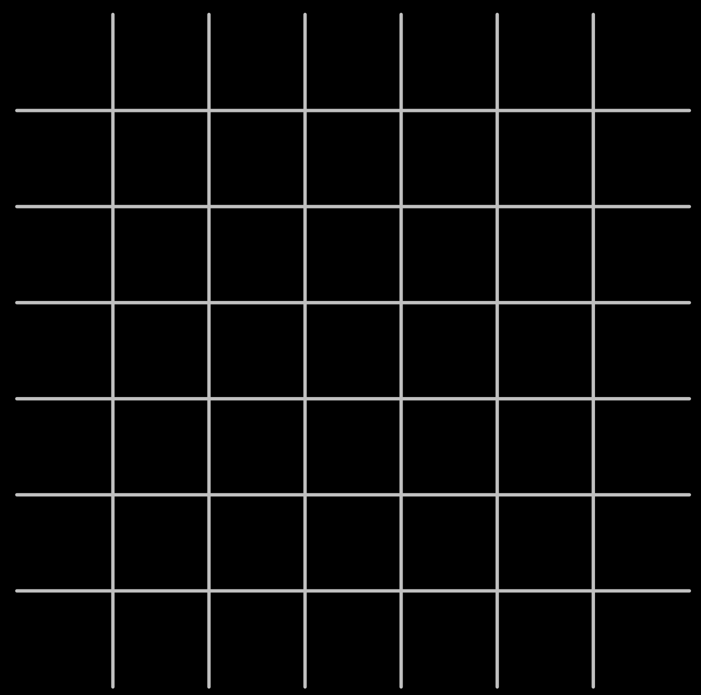
[Durand Levin Shen 05] *Local rules and global order, or aperiodic tilings*

Patterns, colorings and tilings

Foreword

- Formal stuff stand on white background.
- Pictures stand on black background.
- This talk is very geometrical: we draw pictures on the Euclidian plane.
- Letters might be represented as colors or even as colored pictures.





Patterns

A *pattern* $\mathcal{P}$ is a subset of $\mathbb{Z}^2$

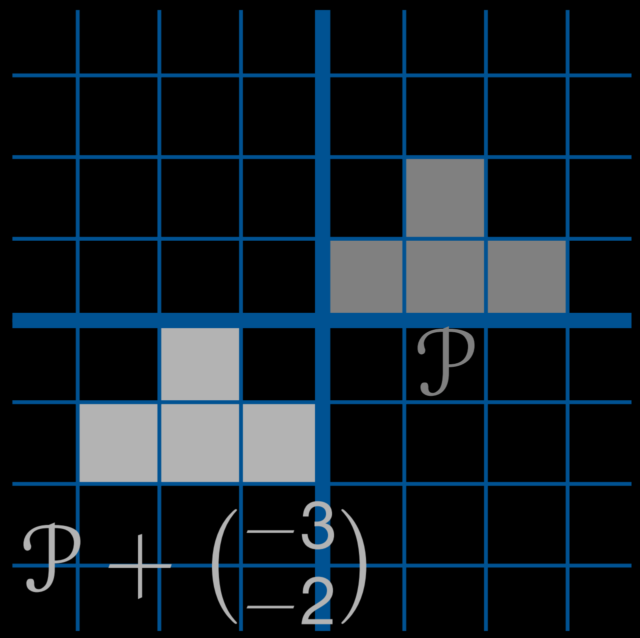
$$\mathcal{P} + u = \{z + u | z \in \mathcal{P}\}$$

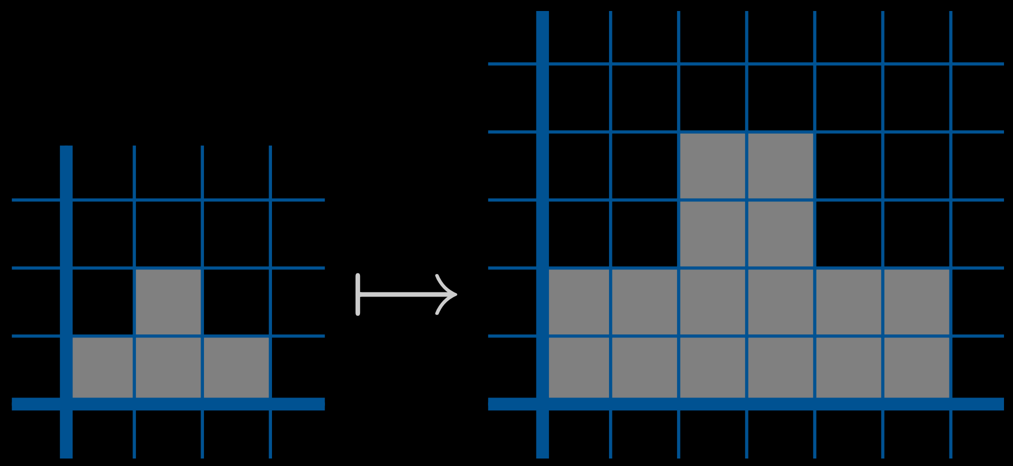
Powers of two patterns $\boxplus^i$

$$\boxplus^i = \left\{x \in \mathbb{Z} \middle| 0 \leqslant x < 2^i\right\}^2$$

Two-by-two *scaling* $\square(\mathcal{P})$

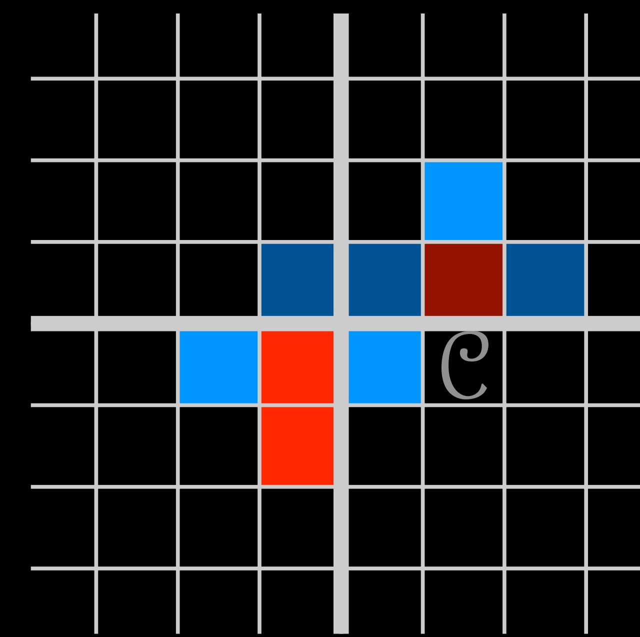
$$\square(\mathcal{P}) = \{2z + c \mid z \in \mathcal{P}, c \in \boxplus\}$$





Colorings

A *coloring* $\mathcal{C} : \mathcal{P} \to \Sigma$ covers a pattern $\mathcal{P}$ with letters of a finite alphabet Σ.

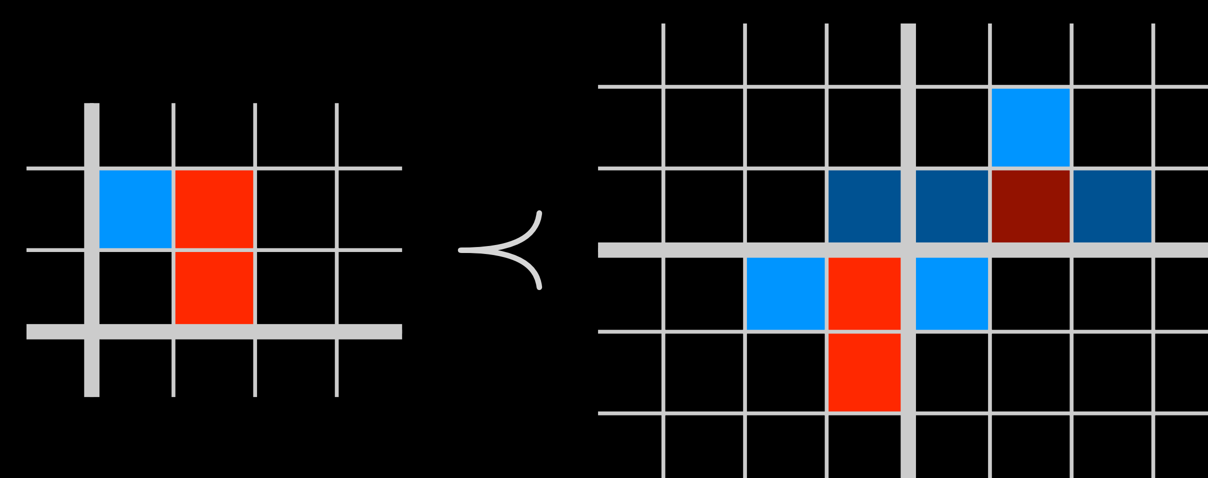


$\mathfrak{u} \cdot \mathcal{C}$ is the *translation* of $\mathcal{C}$, by a vector $\mathfrak{u} \in \mathbb{Z}^2$, satisfying:

$$\forall z \in \mathcal{P}, \quad \mathfrak{u} \cdot \mathcal{C}(z + \mathfrak{u}) = \mathcal{C}(z)$$

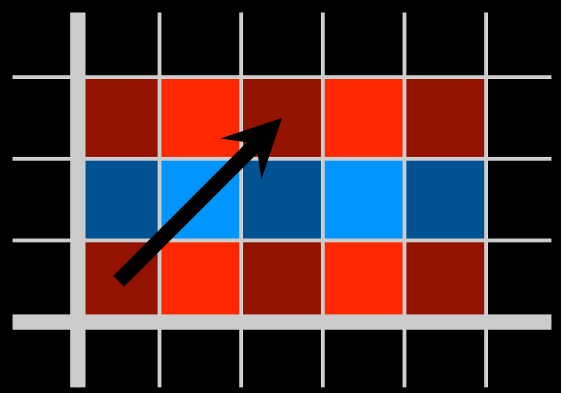
$\mathcal{C}$ *occurs* in $\mathcal{C}'$, denoted $\mathcal{C} \prec \mathcal{C}'$ if

$$\exists \mathfrak{u} \in \mathbb{Z}^2, \quad \mathcal{C} = (\mathfrak{u} \cdot \mathcal{C}')_{|\text{Sup}(\mathcal{C})}$$



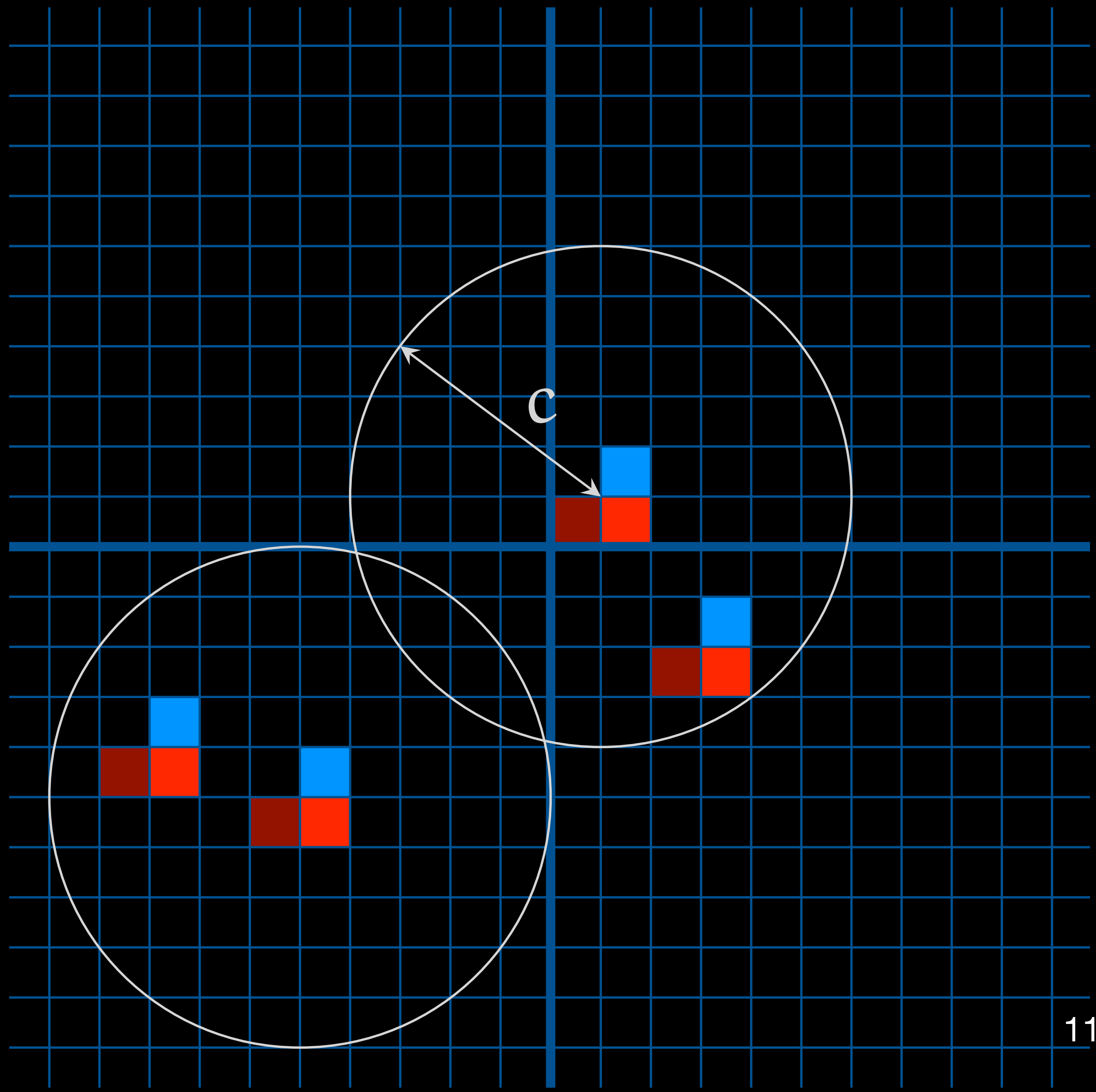
(Quasi-)Periodicity

$\mathcal{C}$ is *periodic* with period $\mathfrak{u}$ if
$\mathcal{C}_{|\mathcal{P}} = (\mathfrak{u} \cdot \mathcal{C})_{|\mathcal{P}}$
where $\mathcal{P} = \text{Sup}(\mathcal{C}) \cap \text{Sup}(\mathfrak{u} \cdot \mathcal{C})$.



A set of colorings is *aperiodic* if all its element have no periodicity vector but 0.

$\mathcal{C}$ is *quasi-periodic* if
$\forall \mathcal{P} \subseteq_{\text{finite}} \text{Sup}(\mathcal{C})\ \exists c\ \forall \mathfrak{u}$
$(\mathfrak{u} \cdot \mathcal{C})_{|\mathcal{P}} = \mathcal{C}_{|\mathcal{P}} \rightarrow \exists v \leqslant c$
$v \neq 0 \ \wedge\ (v \cdot \mathfrak{u} \cdot \mathcal{C})_{|\mathcal{P}} = \mathcal{C}_{|\mathcal{P}}$



Subshifts

Let X be the set of colorings with support $\mathbb{Z}^2$. Endow X with the product topology of the discrete topology on Σ.

This topology is compatible with the metric d defined for all colorings $\mathcal{C}, \mathcal{C}' \in \mathrm{X}$ by $\mathrm{d}(\mathcal{C}, \mathcal{C}') = 2^{-\min\{|z|,\ \mathcal{C}(z) \neq \mathcal{C}'(z)\}}$.

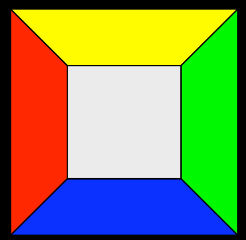
Such topology is compact and perfect. A subset of X both topologically closed and closed by translations is a *subshift* from symbolic dynamics.

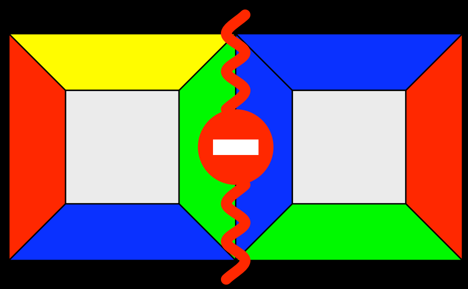
Tilings are the *subshifts of finite type*: subshifts defined by a finite set of forbidden words.

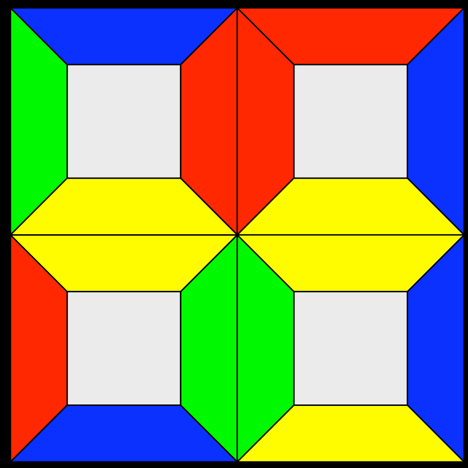
Wang tiles

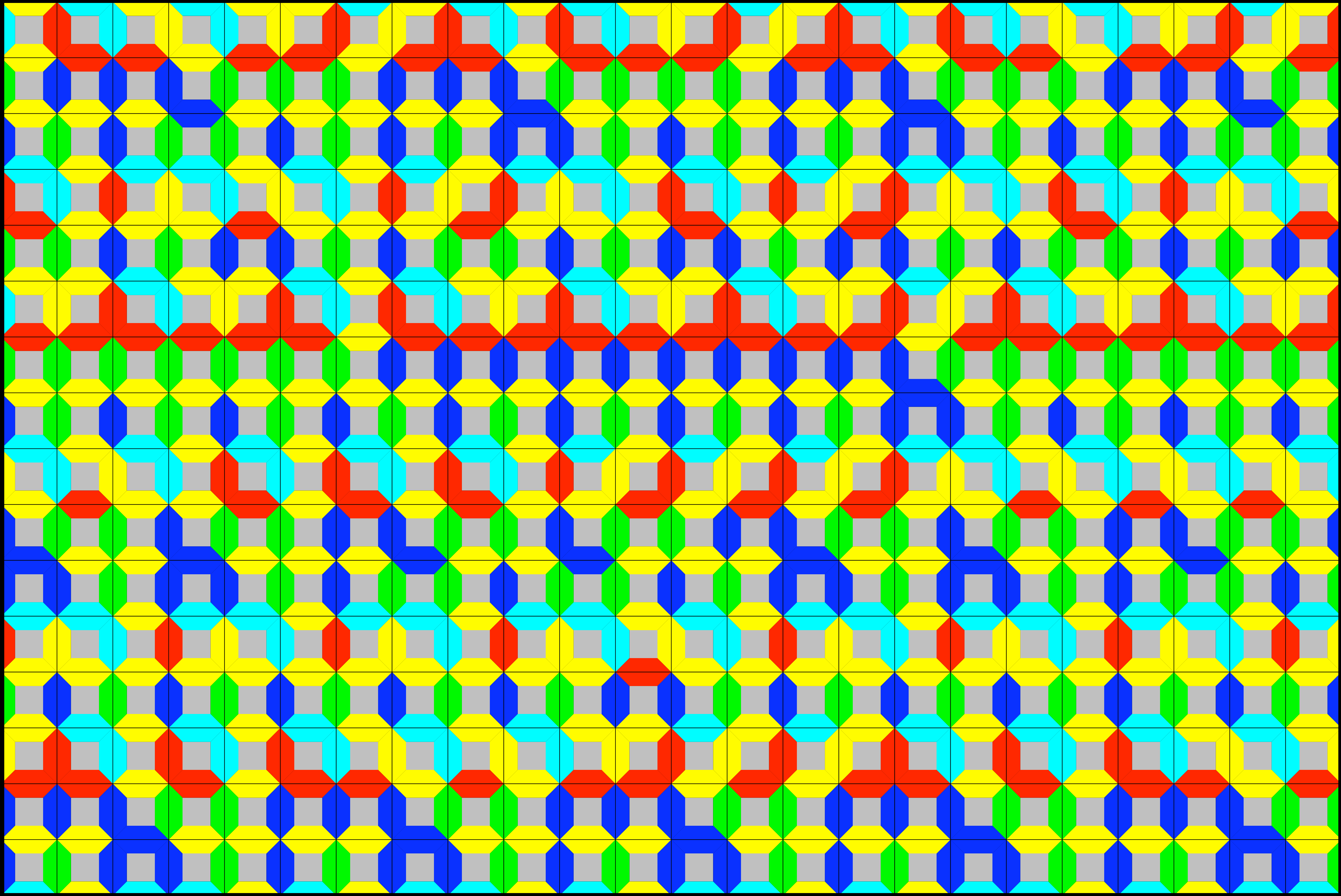
- A Wang tile is a unit square with colored edges.
- A tile set is a finite set of Wang tiles.
- Tiling is done with translations only (no rotation) by matching colors along edges.

$$\tau \subseteq C^4$$









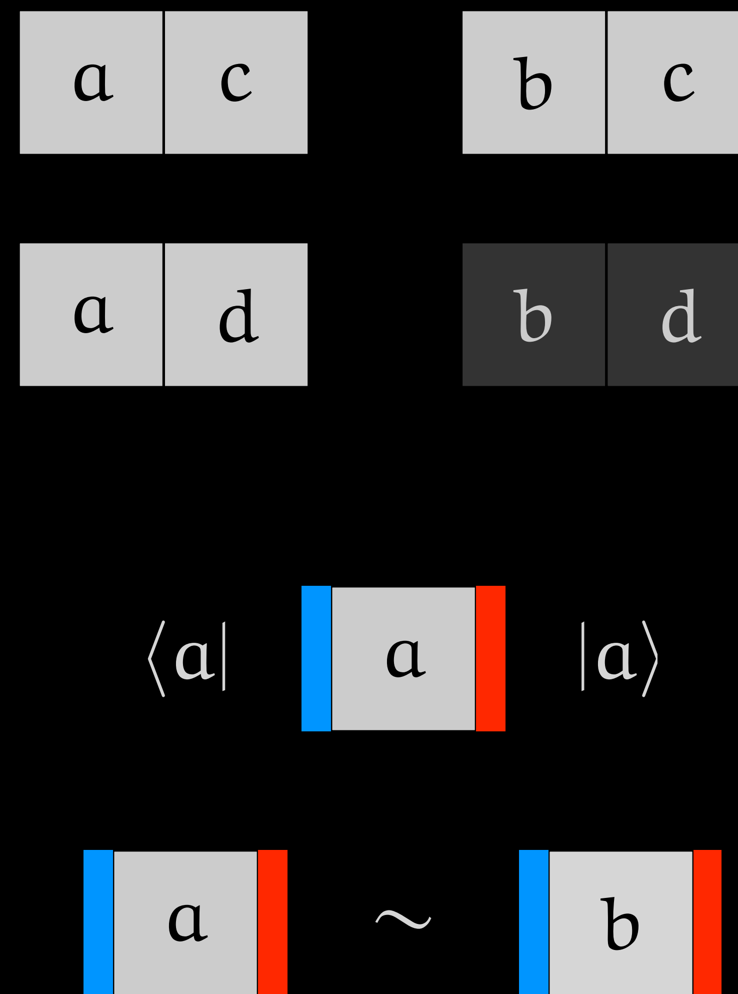
Tiling the whole plane with Wang tiles

Domino Relation

A *domino relation* $\mathcal{R} \subseteq Y \times Y$:

$\forall a, b, c, d \in Y^4$

$a\mathcal{R}c \wedge a\mathcal{R}d \wedge b\mathcal{R}d \rightarrow b\mathcal{R}c$



Right color: $(a, b) \sim_{\mathcal{R}} |a\rangle$

Left color: $(a, b) \sim_{\mathcal{R}} \langle b|$

Degenerated pair:
$\langle a| = \langle b| \wedge |a\rangle = |b\rangle$

A *tile set* τ is a triple $(T, \mathcal{H}, \mathcal{V})$

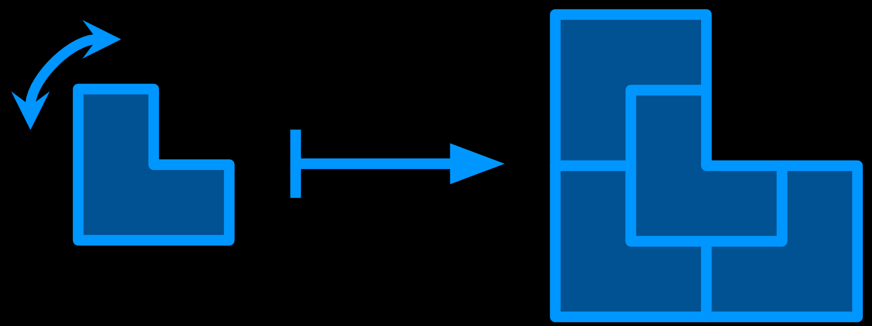
Aperiodicity and **DP**

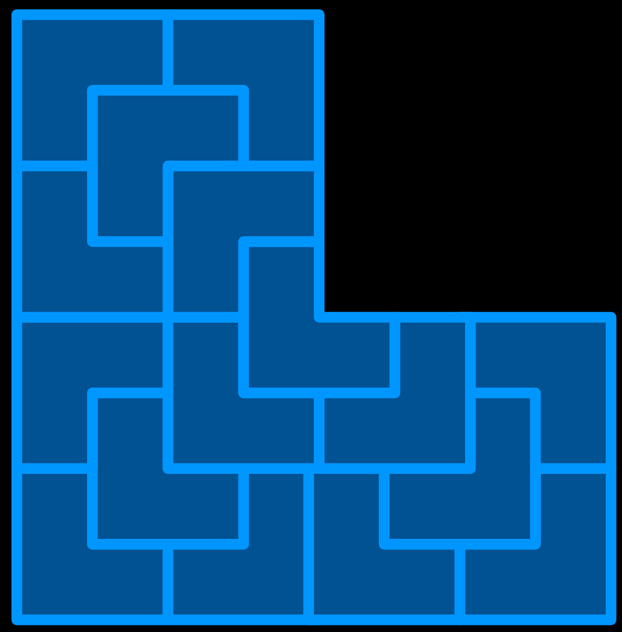
- If a tile set admits a periodic tiling then it admits a biperiodic tiling.
- Biperiodic tilings are recursively enumerable.
- If a tile set does not tile the plane, there exists a bound on the size of square patterns it can tile.
- Every tile set admits a quasi-periodic tiling.
- We will construct an aperiodic tile set that produces only quasi-periodic tilings.
- In dimension 1, the simplest non periodic biinfinite words are sturmian words. Substitutions is a classical tool to generate them.

2×2 substitutions

Substitutions

- Geometric substitutions provide a convenient recursive way to define aperiodic colorings of the plane.
- Subtle geometrical arguments are required: discuss dissection, inflation, scaling factor, etc.
- The classical L (or chair) substitution.





2×2 Substitutions

- A 2×2 substitution system maps a finite alphabet to 2×2 squares of letters on that alphabet.

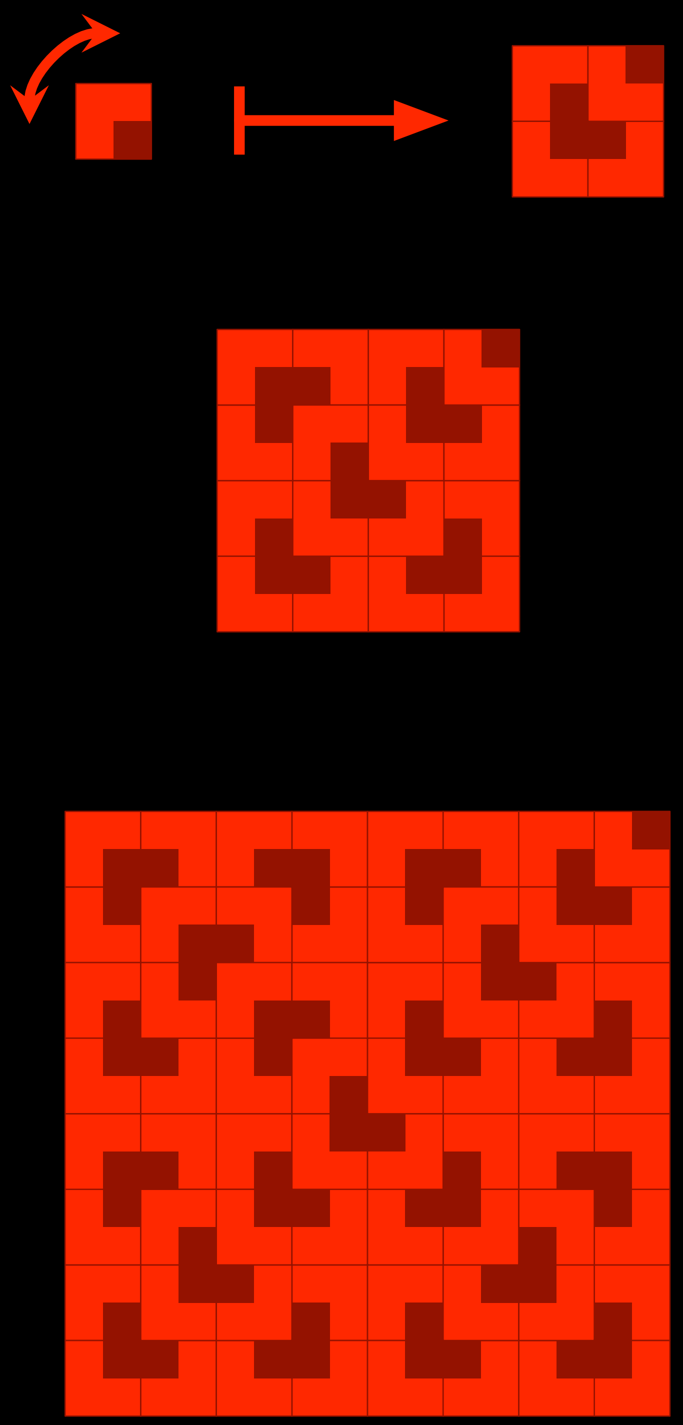
$$s : \Sigma \to \Sigma^{\boxplus}$$

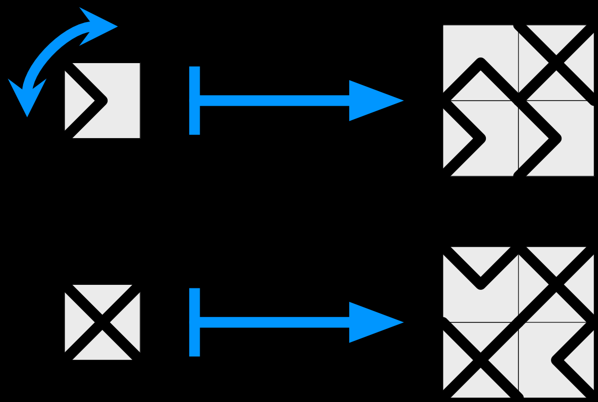
- The substitution is iterated to generate bigger squares.

$$S : \Sigma^{\mathcal{P}} \to \Sigma^{\square(\mathcal{P})}$$

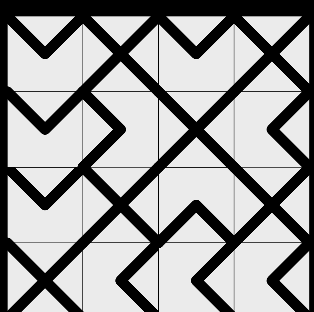
$$\forall z \in \mathcal{P}, \forall c \in \boxplus,$$
$$S(\mathcal{C})(2z + c) = s(\mathcal{C}(z))(c)$$

$$S(u \cdot \mathcal{C}) = 2u \cdot S(\mathcal{C})$$

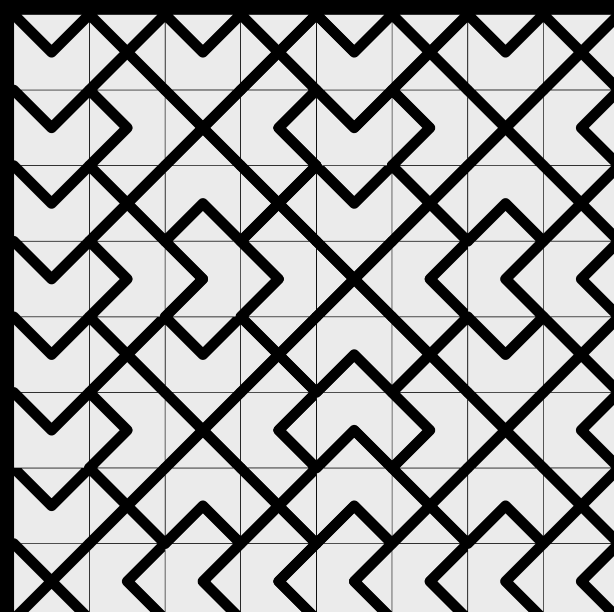




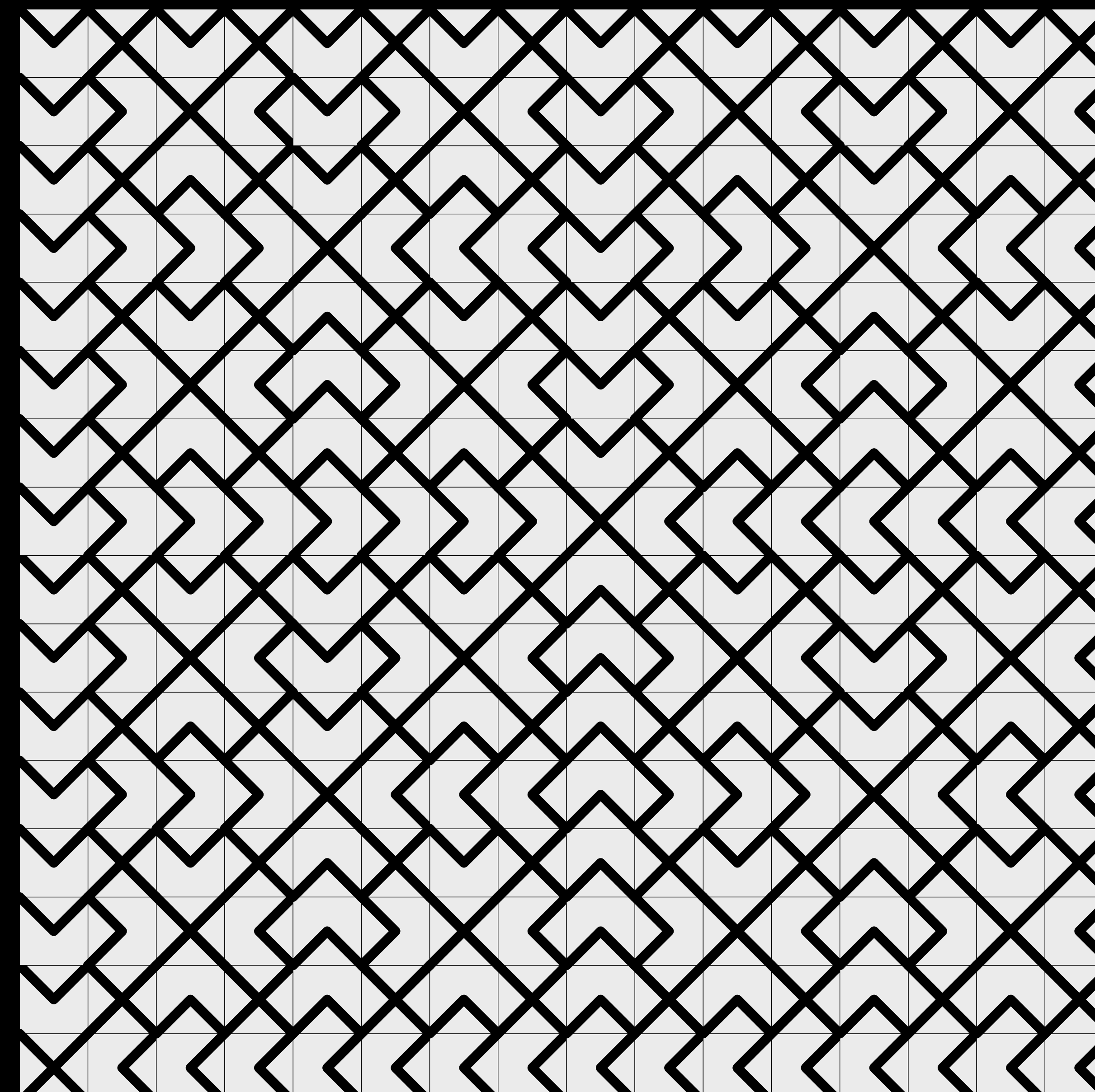
(5 letters)



$s^2(\boxtimes)$



$s^3(\boxtimes)$



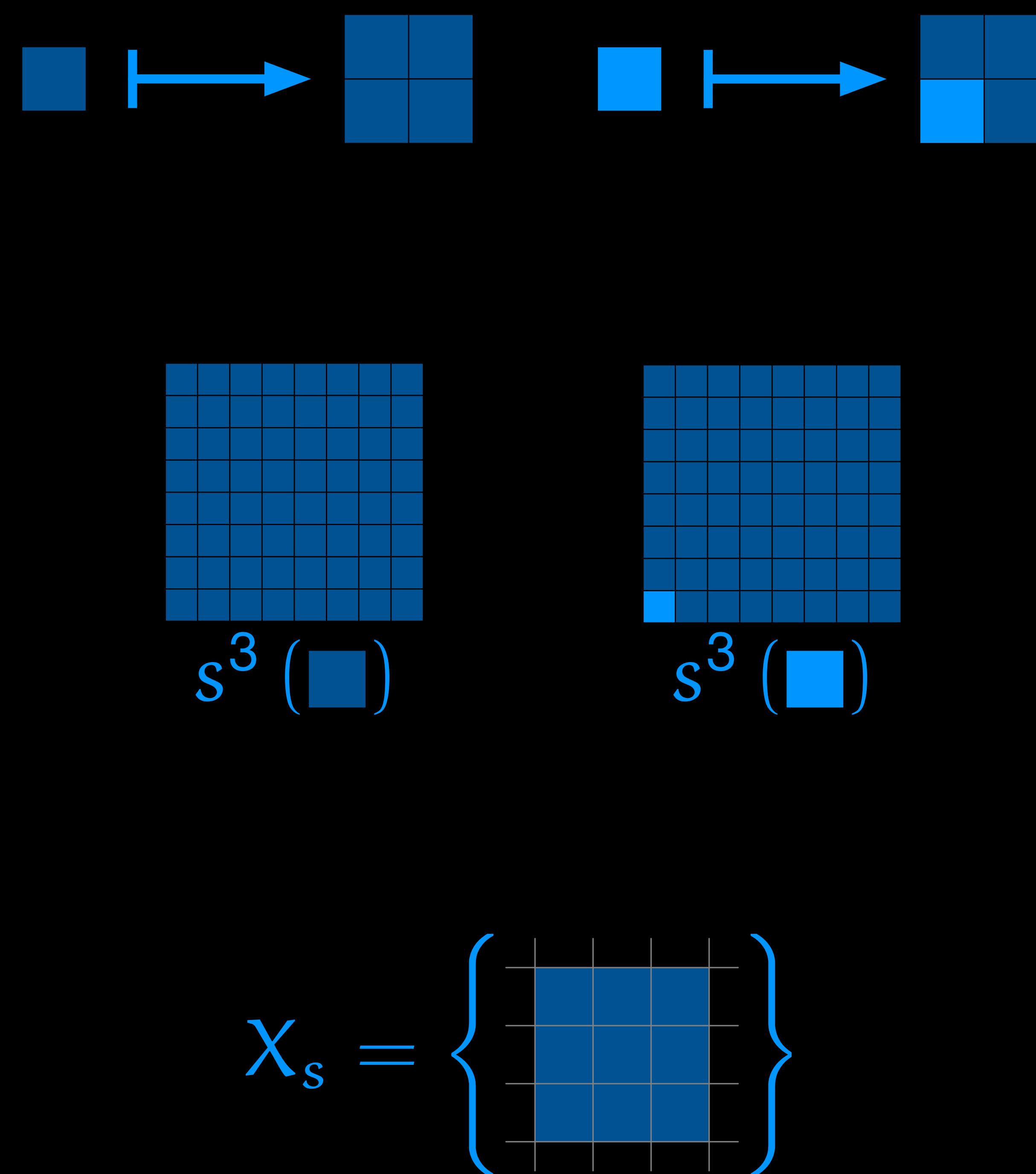
$s^4(\boxtimes)$

Another L-style substitution

Pattern closure?

- What is a coloring of the plane generated by a substitution?
- One possibility is to consider a *closure* of generated patterns up to translations.
- Difficult to check...

$\mathcal{C} \in X_s$ if
$\forall \mathcal{C}' \prec_{\text{finite}} \mathcal{C}\ \exists a, i\ \mathcal{C}' \prec S^i(a)$



Limit set!

- The global map of a substitution is continuous.

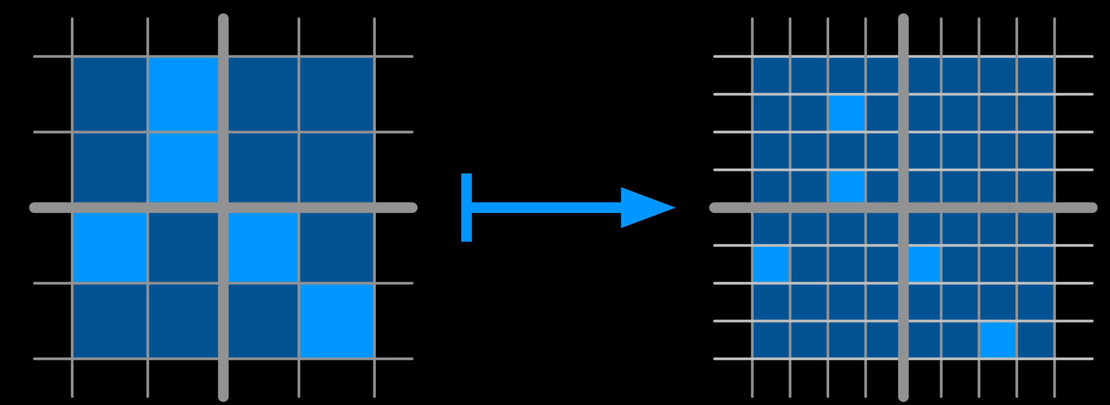
$$X = \Sigma^{\mathbb{Z}^2}$$

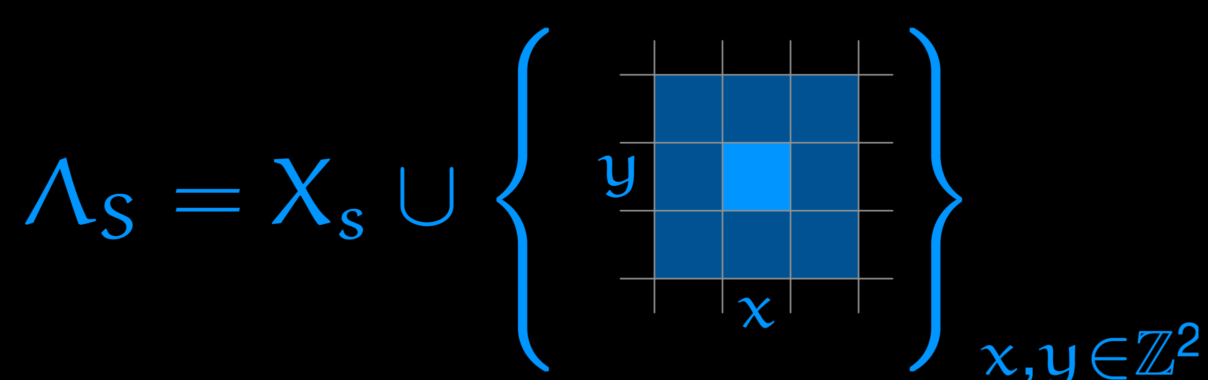
- Take the limit set of iterations of the global map closed up to translations.

- More colorings!

$$\Lambda_S = \bigcap_n \Lambda_S^n \text{ where } \Lambda_S^0 = \Sigma^{\mathbb{Z}^2}$$

$$\Lambda_S^{n+1} = \{u \cdot S(\mathcal{C}) \mid \mathcal{C} \in \Lambda_S^n, u \in \boxplus\}$$





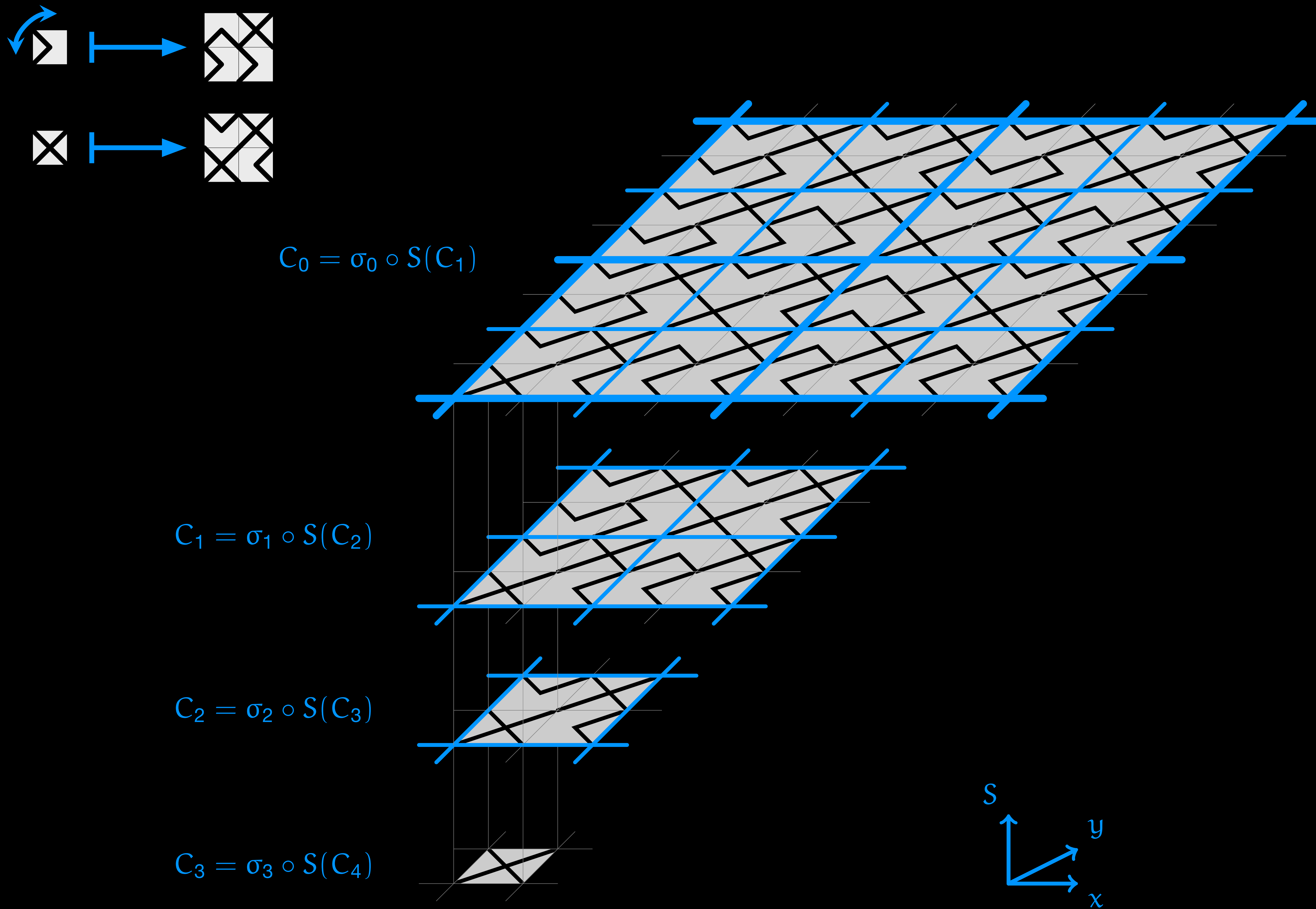
History

An *history* for a coloring $\mathcal{C} \in X$ is a sequence $(\mathcal{C}_i, u_i) \in (X \times \boxplus)^{\mathbb{N}}$ such that $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_i = u_i \cdot S(\mathcal{C}_{i+1})$, for all $i \in \mathbb{N}$.

Proposition 1. The set Λ_S is precisely the set of colorings admitting histories.

$\supseteq$ $\mathcal{C}_0 \in \Lambda_S$

$\subseteq$ Compacity argument



History of a coloring in the limit set

Stories

The *story* at position $z \in \mathbb{Z}^2$ for an history $(\mathcal{C}_i, u_i) \in (X \times \boxplus)^{\mathbb{N}}$ is the sequence $(a_i, v_i) \in (\Sigma \times \boxplus)^{\mathbb{N}}$ such that, for all $i \in \mathbb{N}$, $a_i = s(a_{i+1})(v_i)$ and $a_i = \mathcal{C}_i(z_i)$ where $z_i \in \mathbb{Z}^2$ is the only position such that z is an element of the pattern $\mathcal{P}_i = \boxplus^i - \sum_{j=0}^{i-1} 2^j u_j - 2^i z_i$.

Finding stories of neighbors starting from a story works like incremeting an odometer.

Proposition 2. Every history can be reconstructed from 1,2, or 4 of its stories.

Unambiguity and aperiodicity

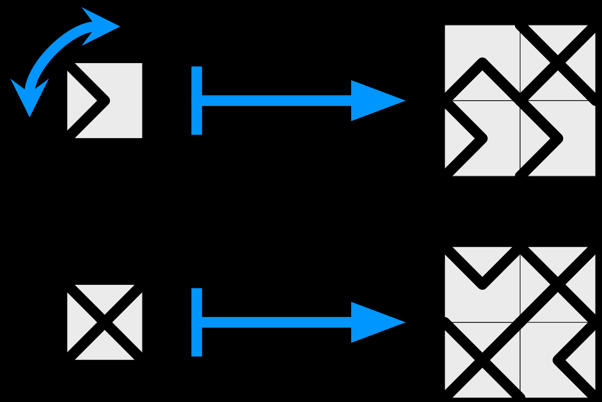
A substitution is *aperiodic* if its limit set Λ_S is aperiodic.

A substitution is *unambiguous* if, for every coloring $\mathcal{C}$ from its limit set Λ_S, there exists a unique coloring $\mathcal{C}'$ and a unique translation $u \in \boxplus$ satisfying $\mathcal{C} = u \cdot S(\mathcal{C}')$.

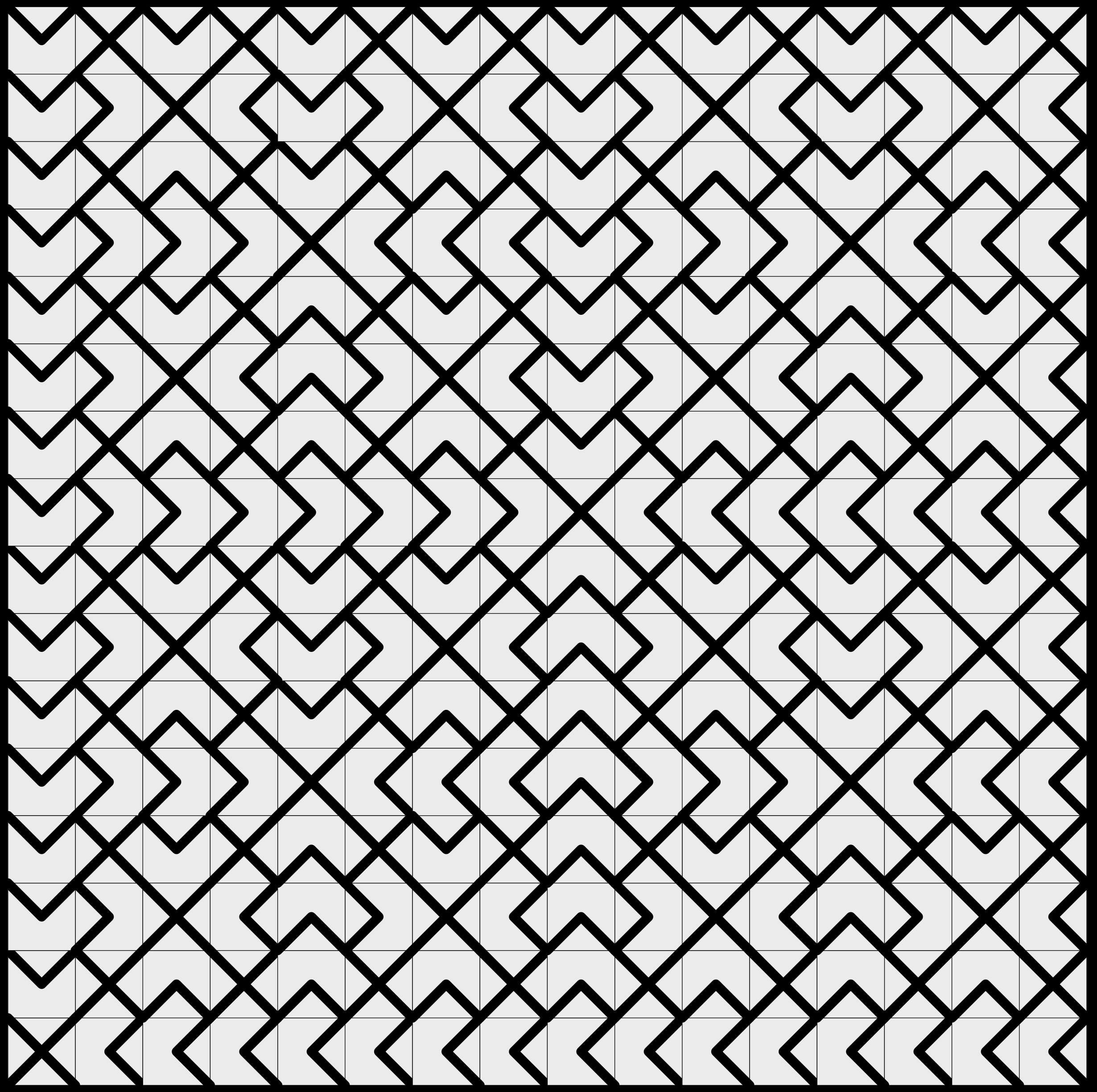
Every unambiguous substitution admits a unique history.

Proposition 3. Unambiguity implies aperiodicity.

Idea. Consider a periodic coloring with minimal period p, construct one of period $p/2$.



is unambiguous

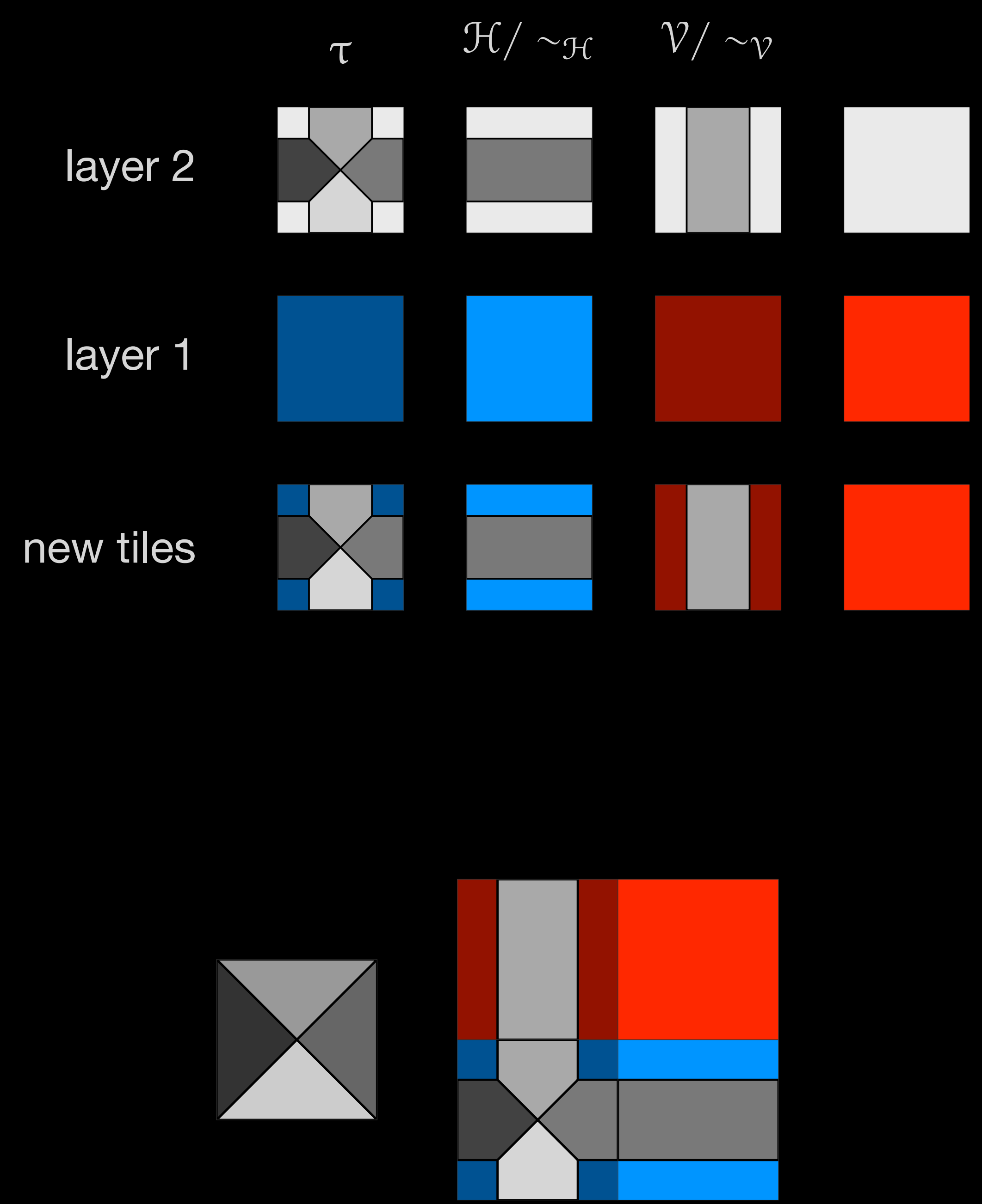


A set of 104 tiles

Tiling Coding

A tile set $(\mathrm{T}', \mathcal{H}', \mathcal{V}')$ *codes* a tile set $(\mathrm{T}, \mathcal{H}, \mathcal{V})$, according to a *coding rule* $\mathrm{t} : \mathrm{T} \to \mathrm{T}'^{\boxplus}$ if t is injective and

$$X_{\tau'} = \{\mathrm{u} \cdot \mathrm{t}(\mathcal{C}) | \mathcal{C} \in X_\tau, \mathrm{u} \in \boxplus\}.$$



Aperiodicity

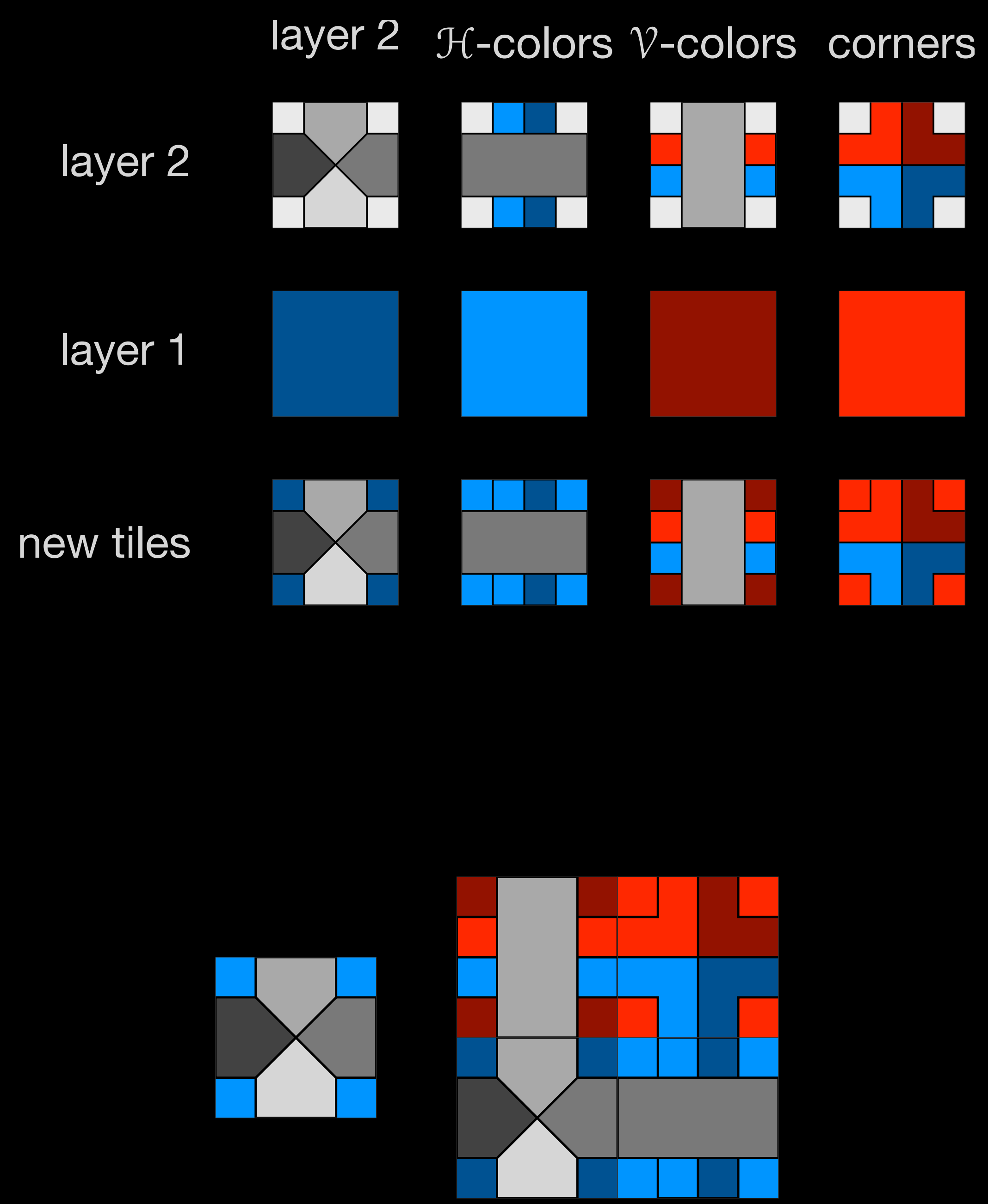
A tile set $(T, \mathcal{H}, \mathcal{V})$ *codes* a substitution $s : T \to T^{\boxplus}$ if it codes itself according to the coding rule s.

Proposition 4. A tile set both admitting a tiling and coding an unambiguous substitution is aperiodic.

Idea. $X_\tau \subseteq \Lambda_S$ and $X_\tau \neq \emptyset$.

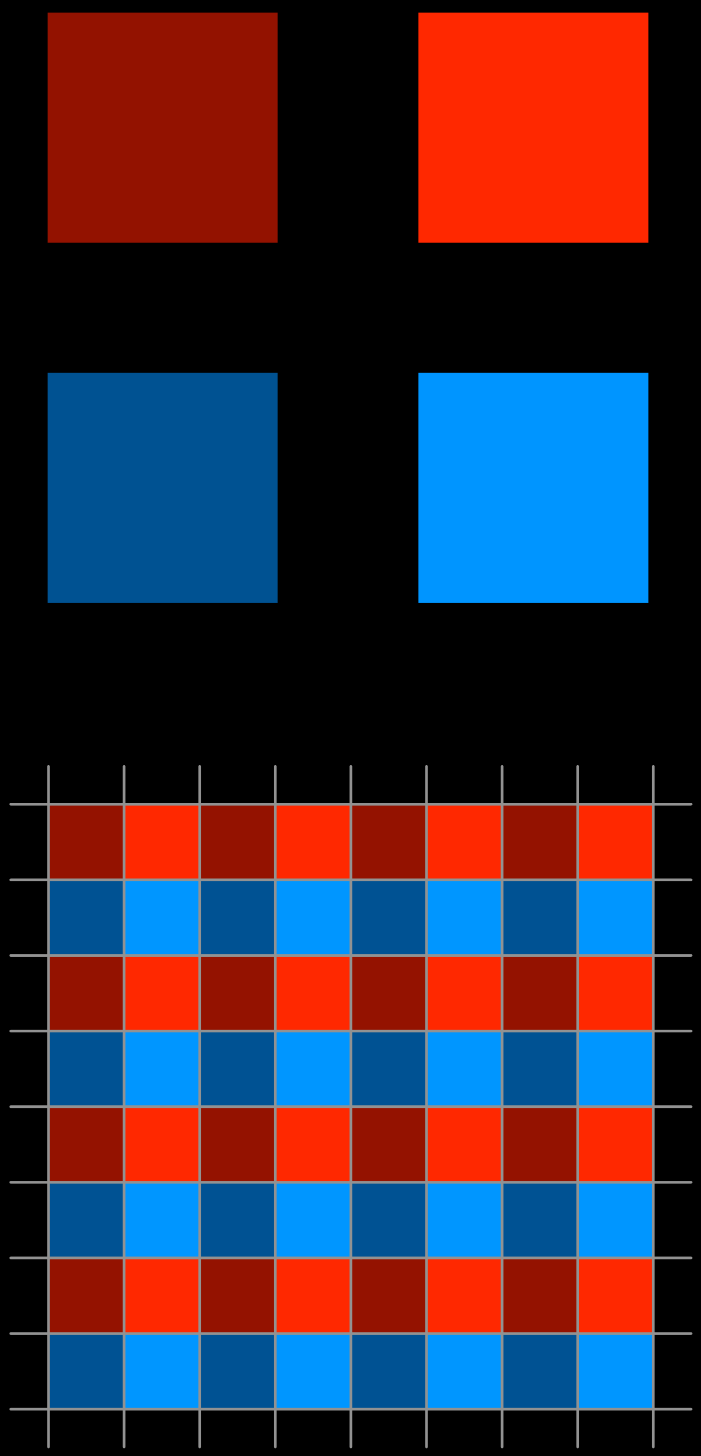
Coding Scheme 2

- Better scheme: not strictly increasing the number of tiles.
- **Problem.** it cannot encode any layered tile set, constraints between layer 1 and layer 2 are checked edge by edge.
- **Solution.** add a third layer with one bit of information per edge.



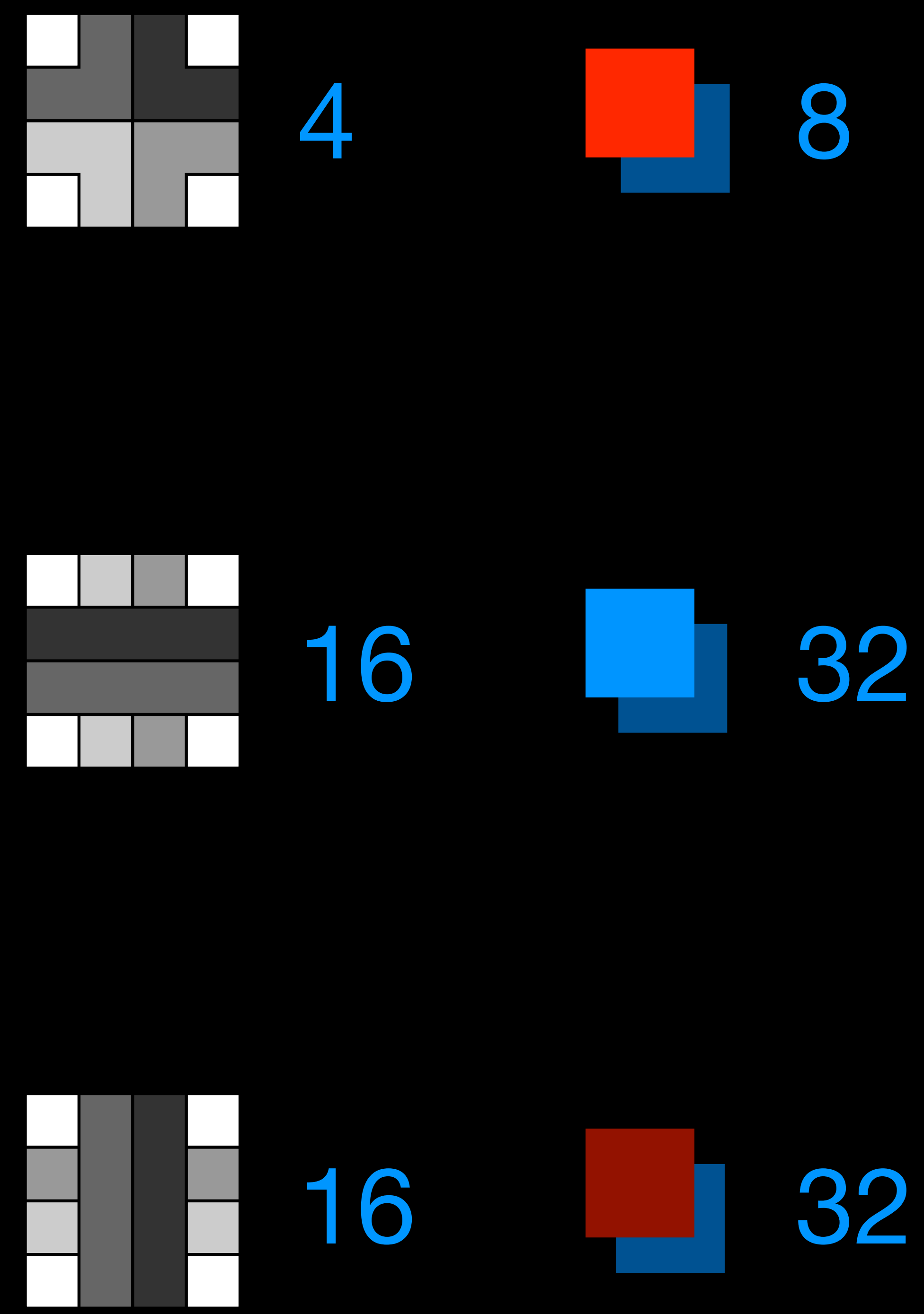
Layer 1. initialization

- 4 tiles to group tiles 2×2.
- Alternate red/blue vertically.
- Alternate light/dark horizontally.
- Simple and very constrained matching rule.



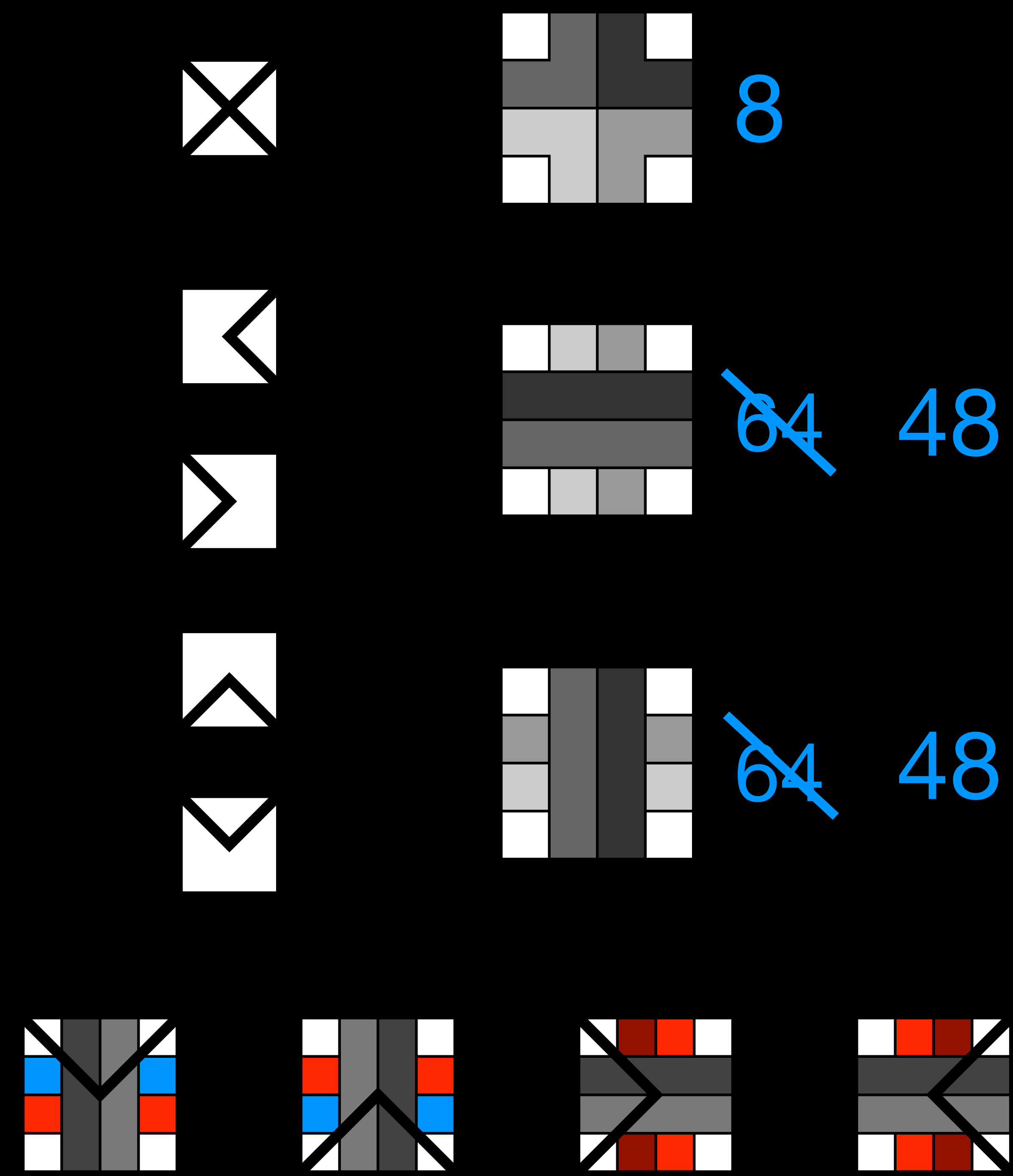
Layer 2. grids

- Tiles carry wires, two on each edge.
- Three kinds of tiles: X, H, V.
- Pairs of wires should be compatible.
- Matching rule: wire colors should match.
- Restrict layer 1 by kind.



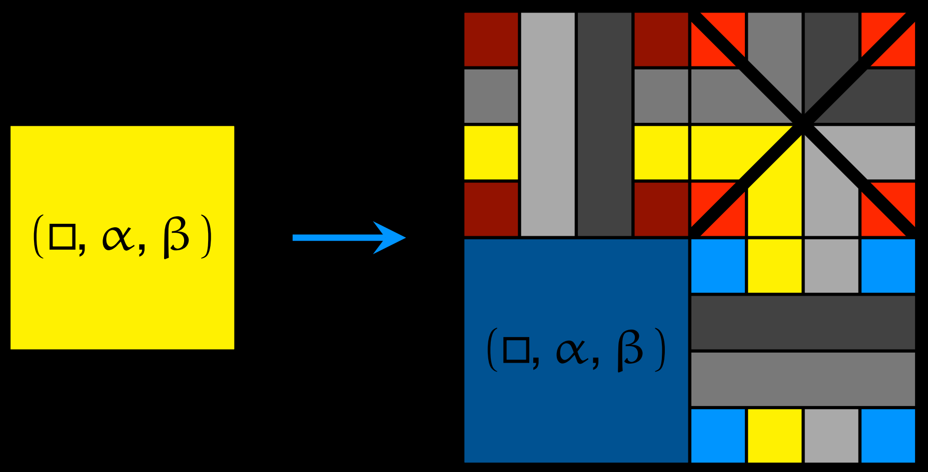
Layer 3. corner-*ifier*

- One more bit of information per edge to find corners.
- 5 different tiles with arrows.
- Matching rule: arrows should match along edges.
- Restrict arrows by kind.
- **Important:** red wire is a one-way lane.

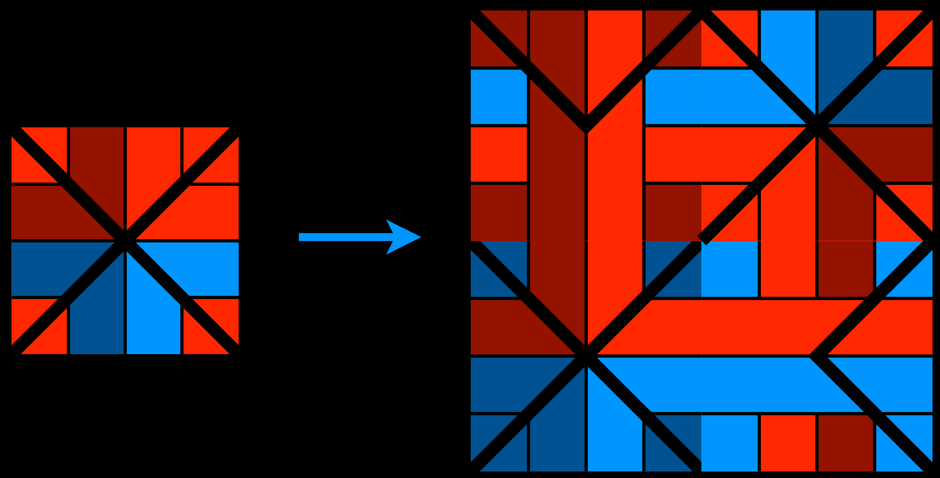


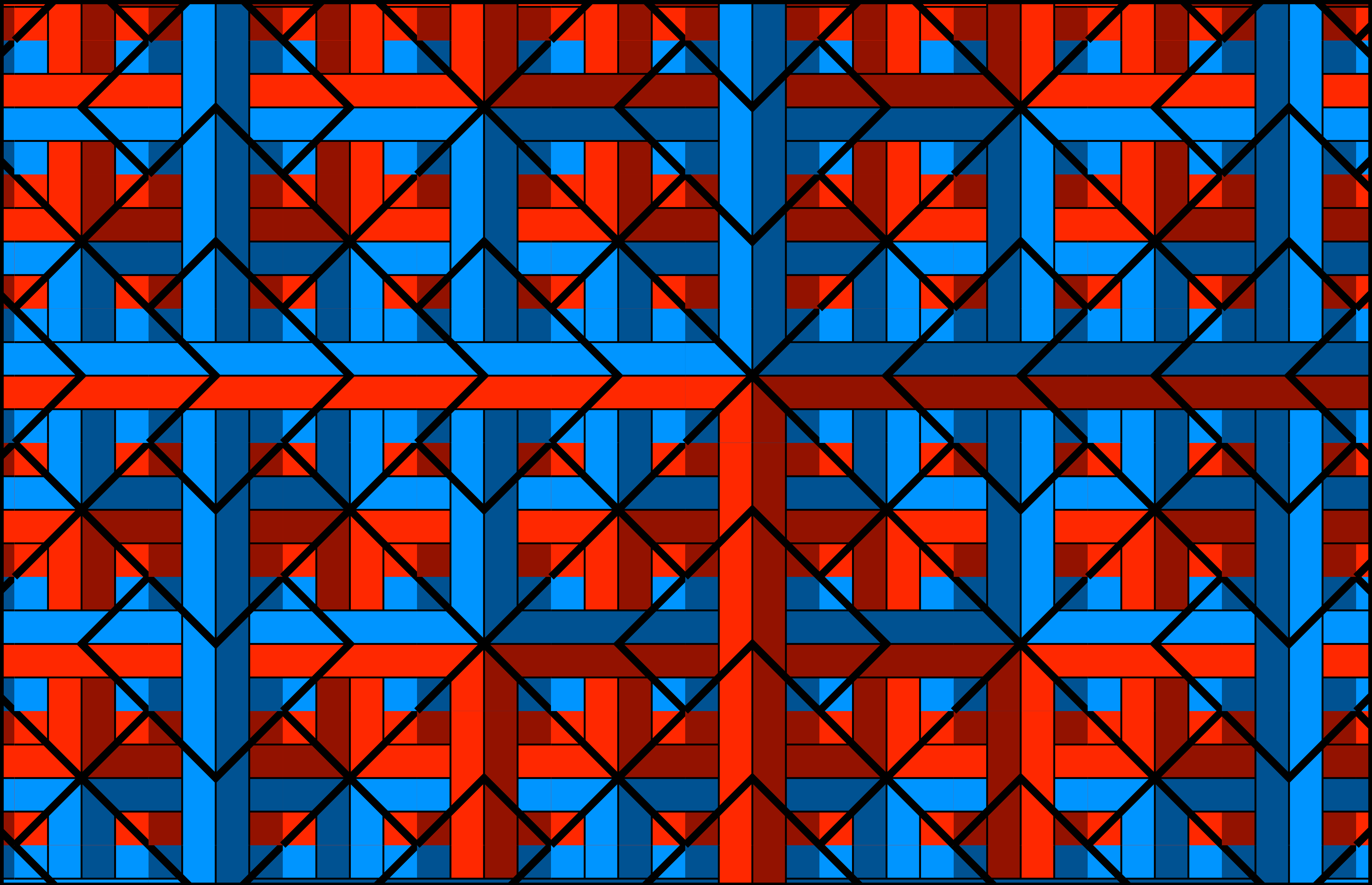
Canonical substitution

- Copy the tile in the SW corner but for layer 1.
- Put the only possible X in NE that carry layer 1 of the original tile on SW wire.
- Propagate wires colors.
- Let H et V tile propagate layer 3 arrows.
- The substitution is injective.

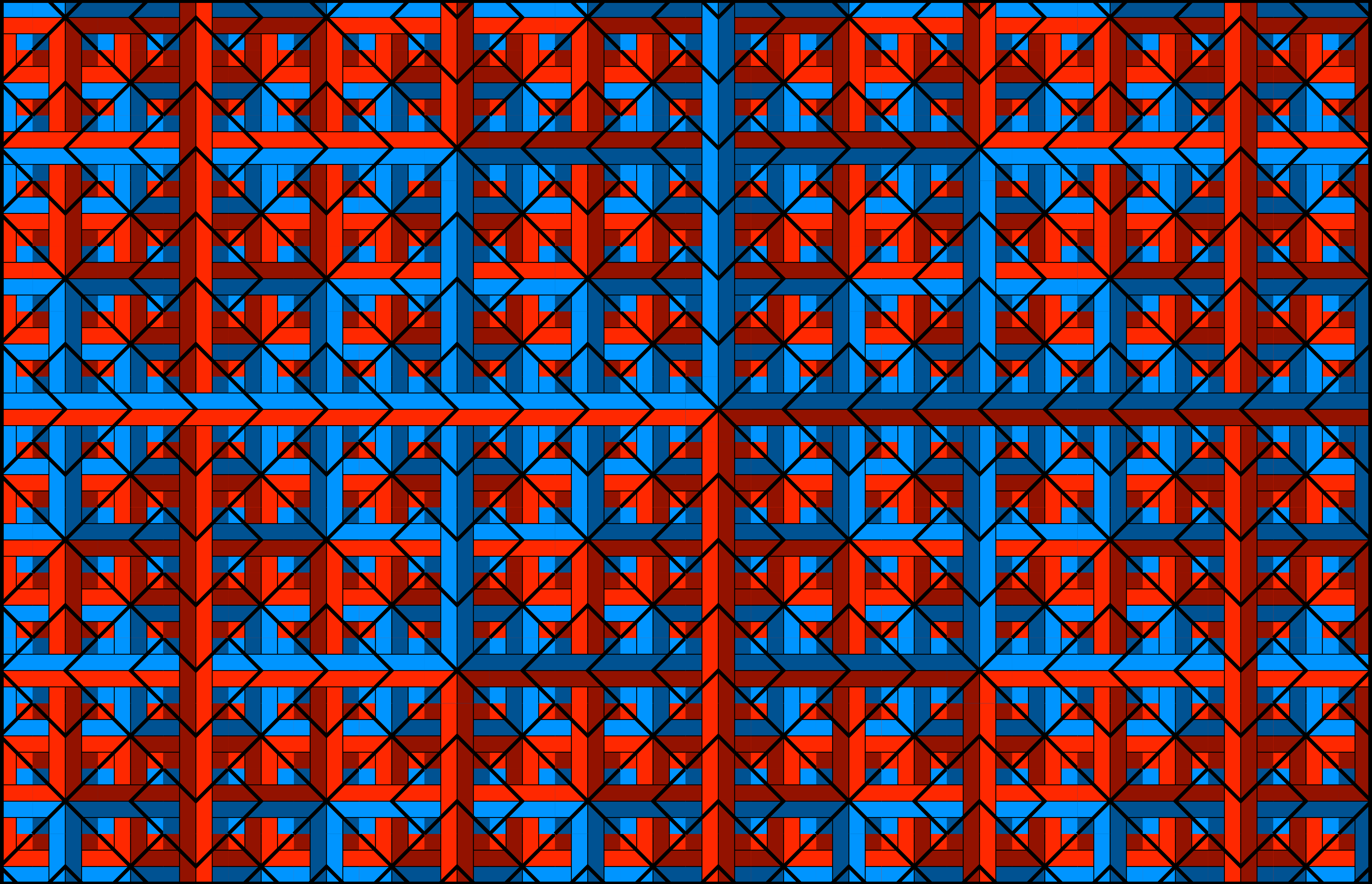


example:





The tileset can tile the whole plane...



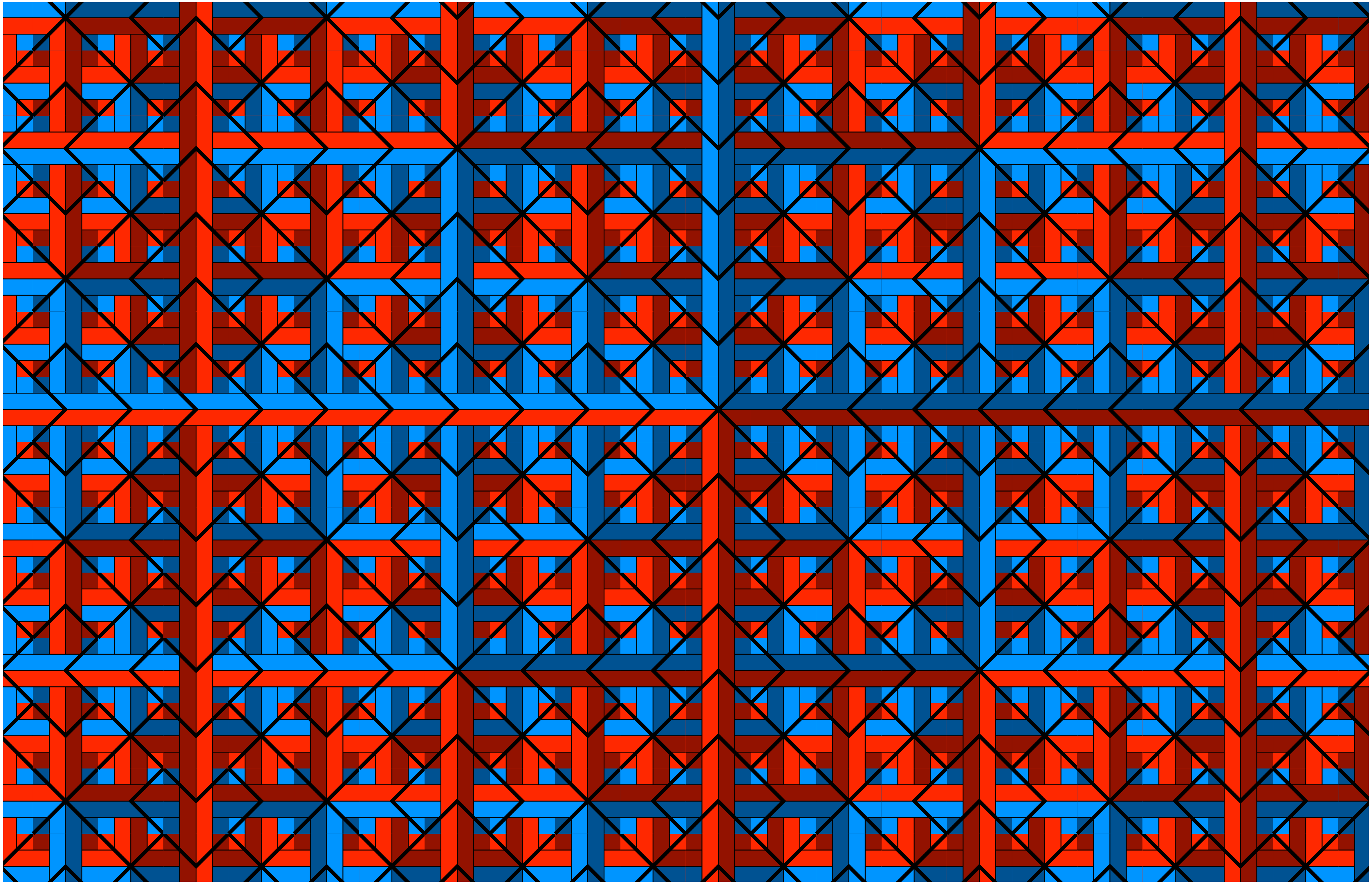
...extend *ad lib*!

This tile set is aperiodic

Roadmap of the proof

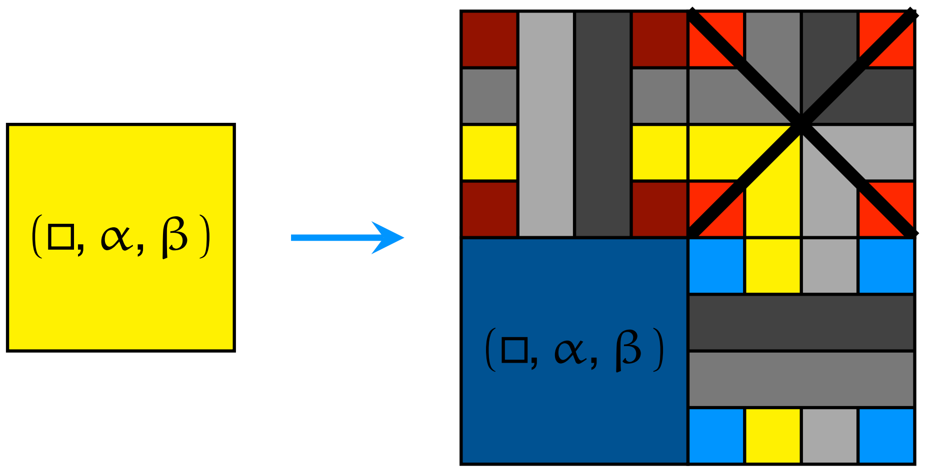
1. The tile set admits a tiling;
2. The substitution is unambiguous;
3. The tile set codes the substitution.
4. Apply proposition 4 to conclude.

The tile set admits a tiling

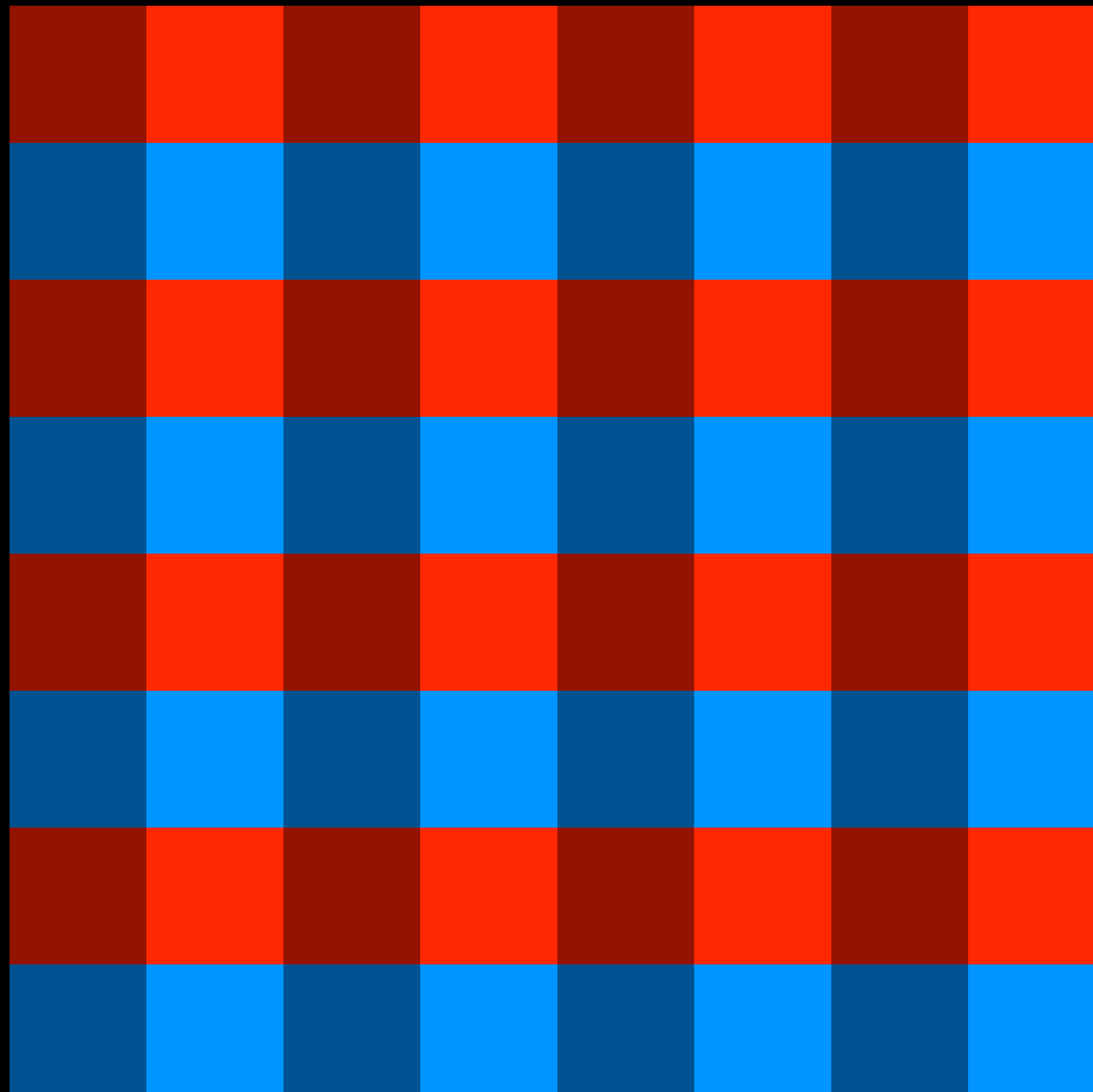


The substitution is unambiguous

- It is injective and the projectors have disjoined images.



each tiling from this set is an image of the canonical substitution.



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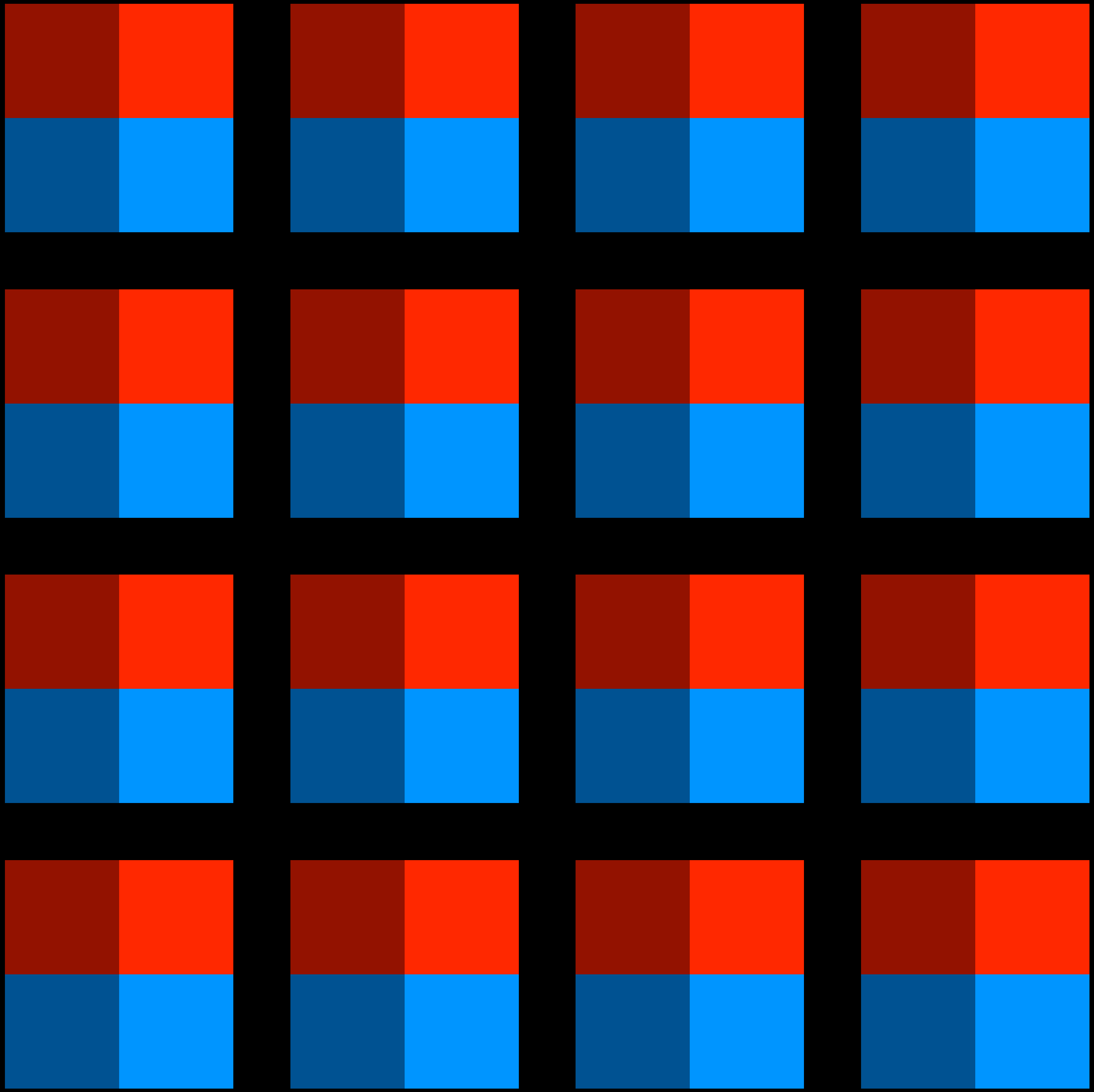

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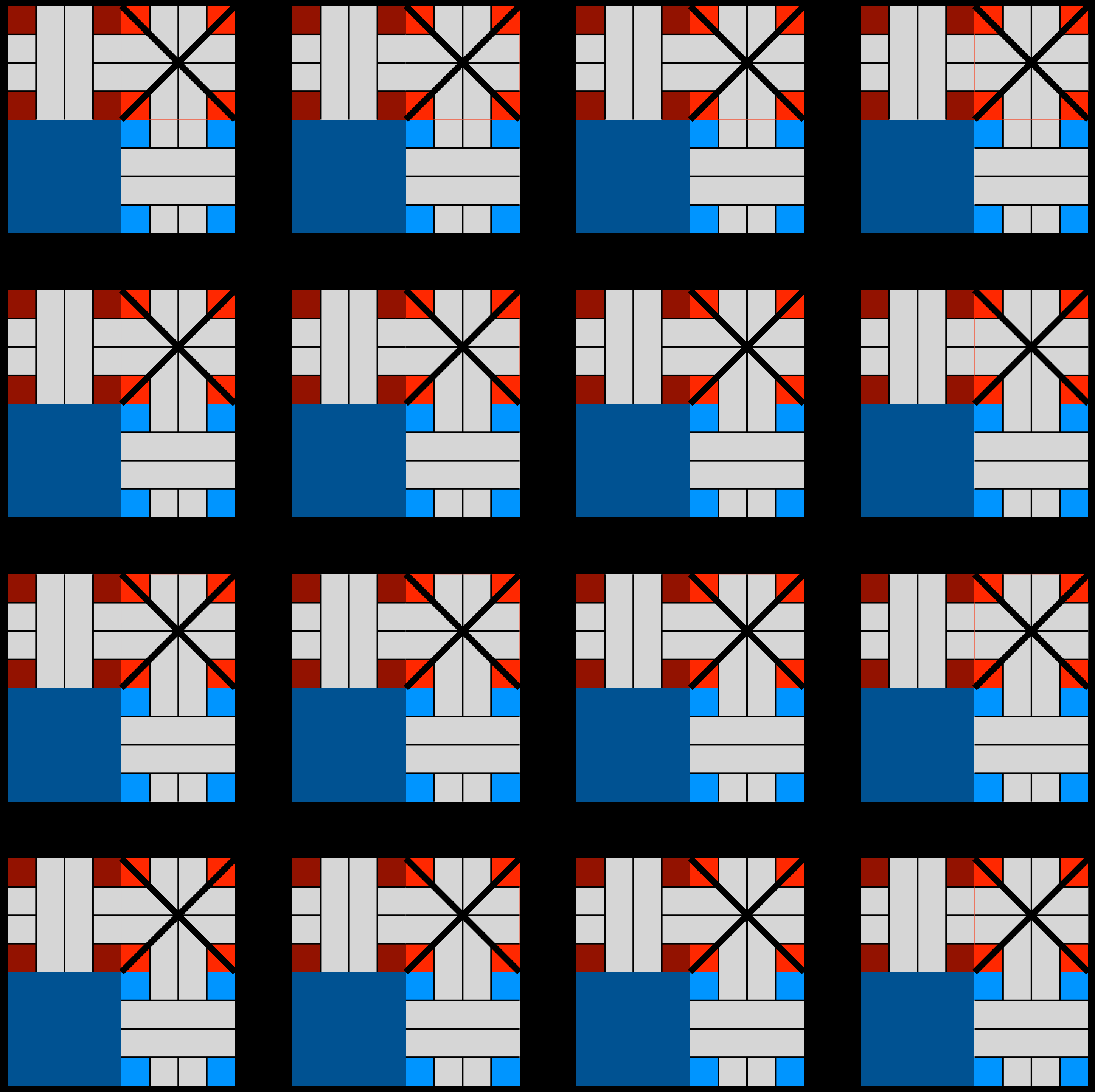

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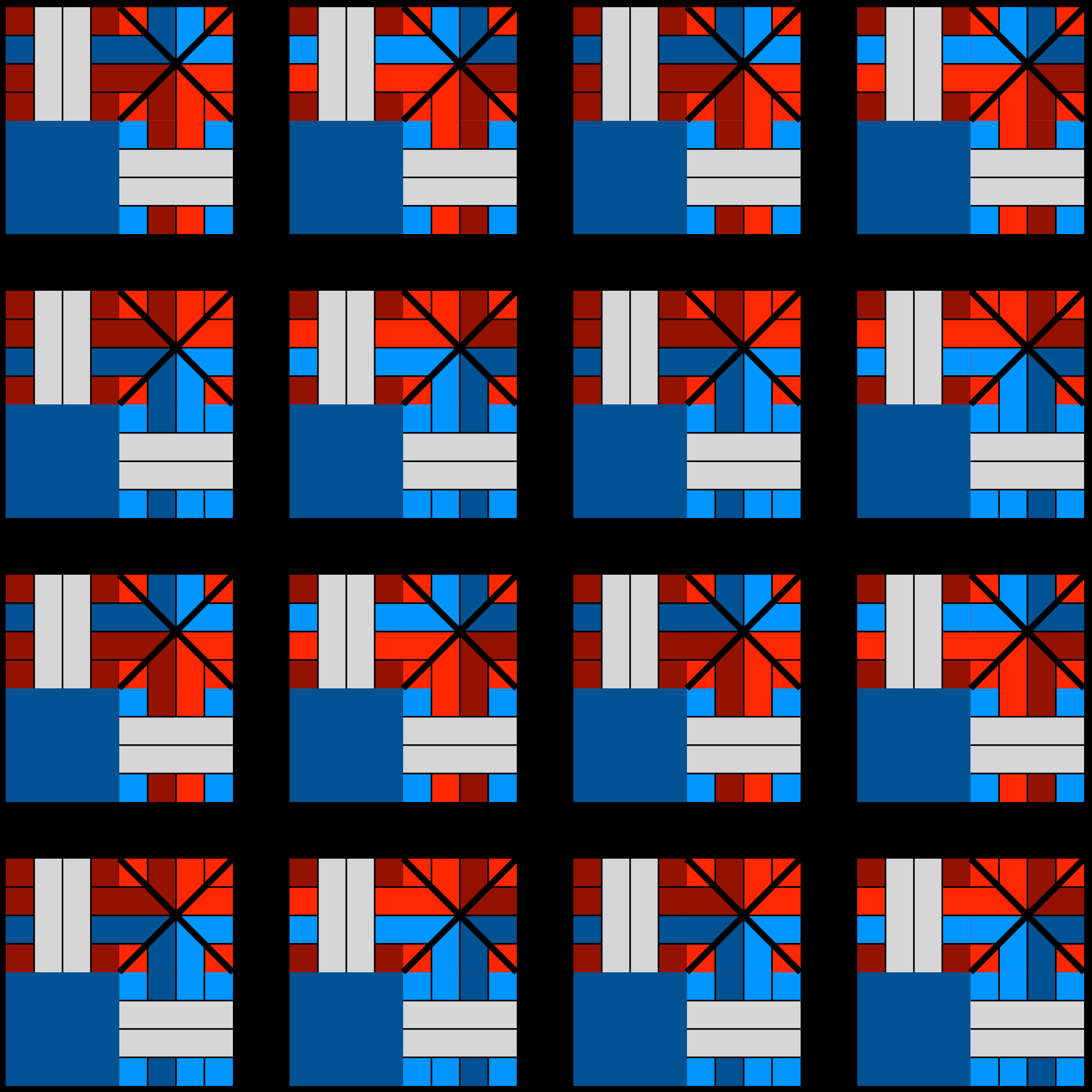

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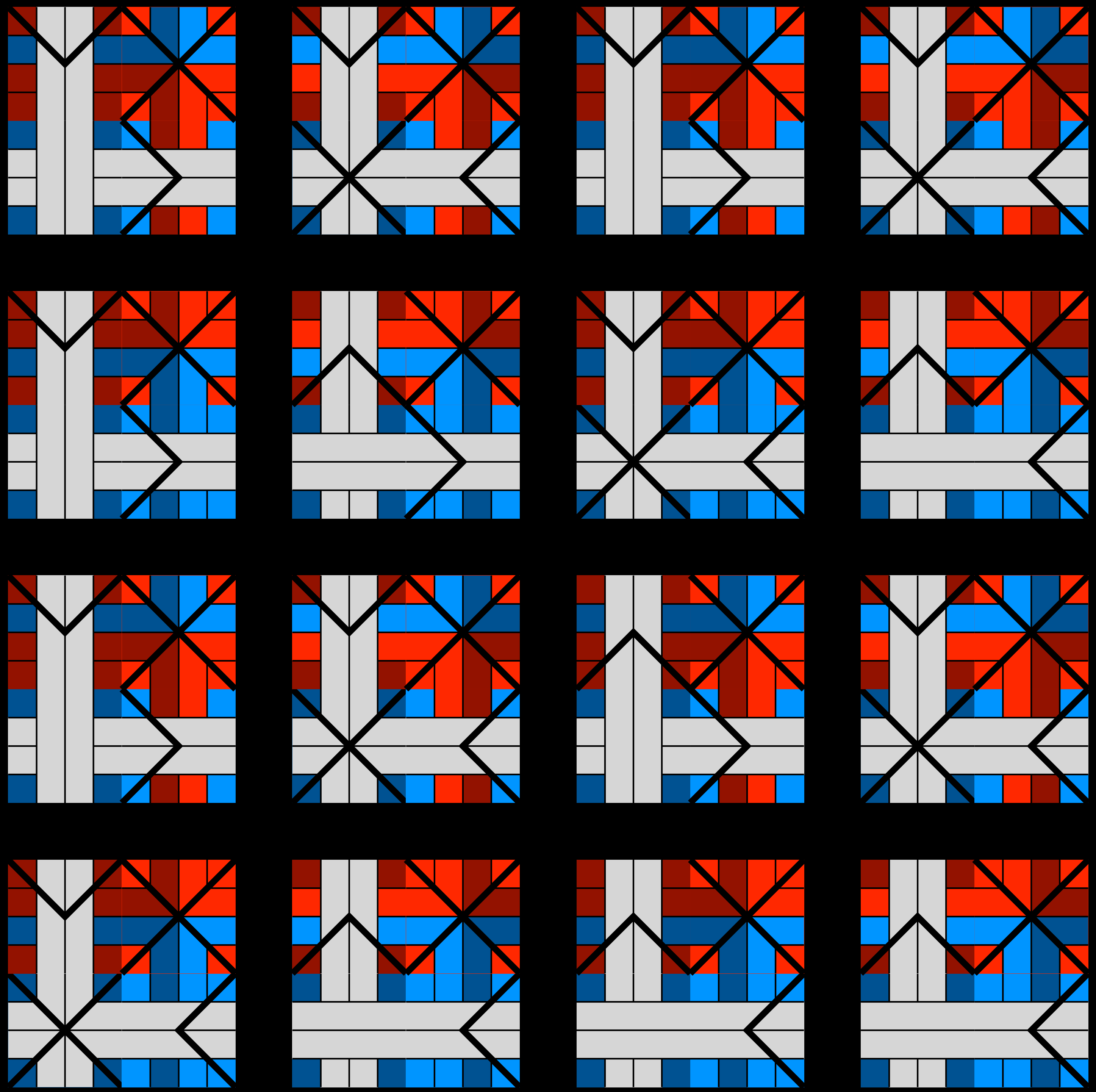

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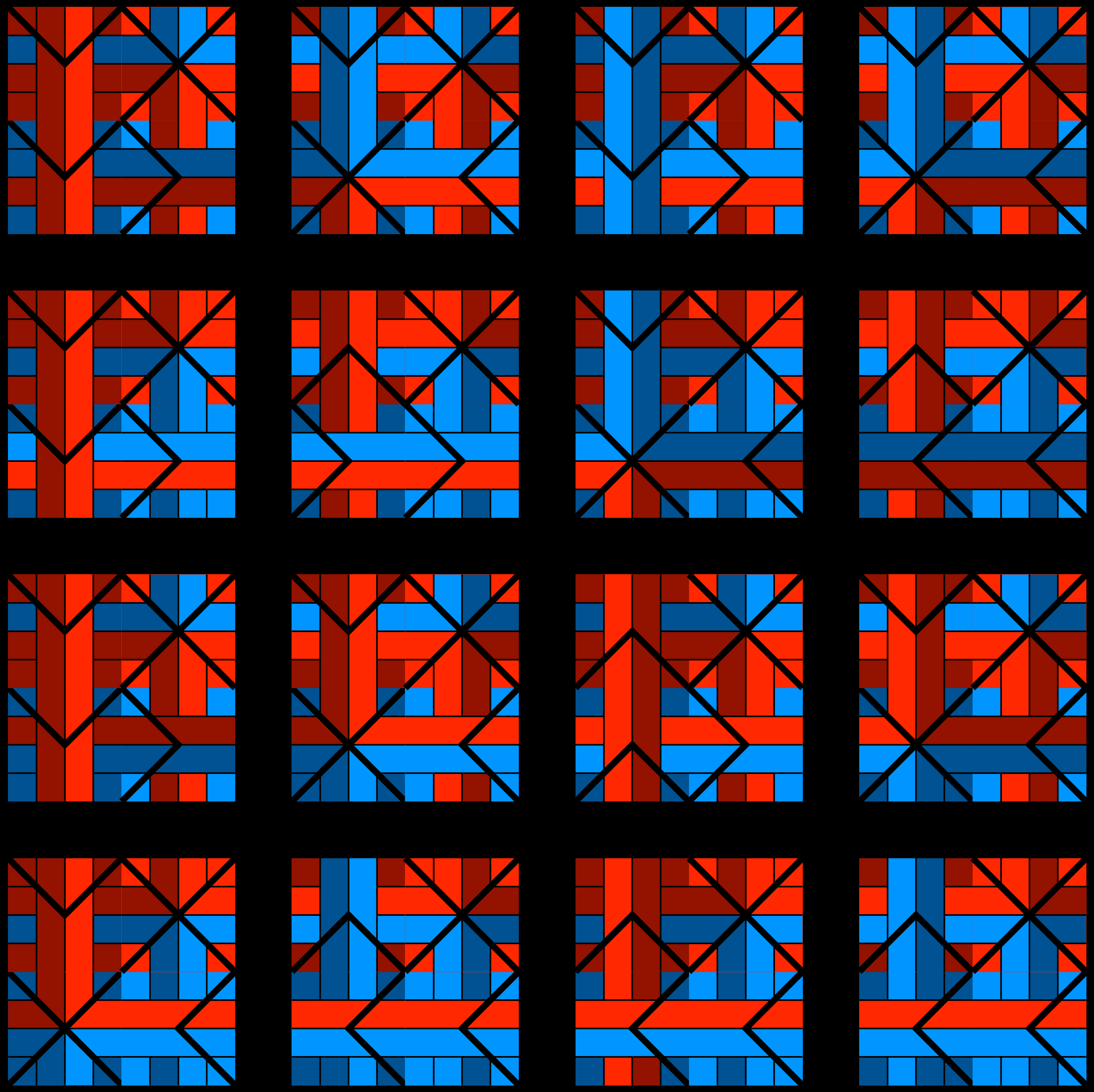

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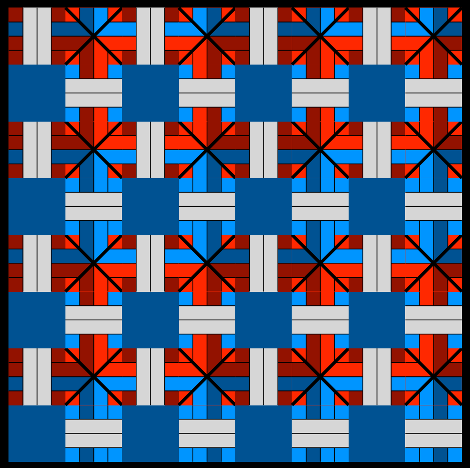
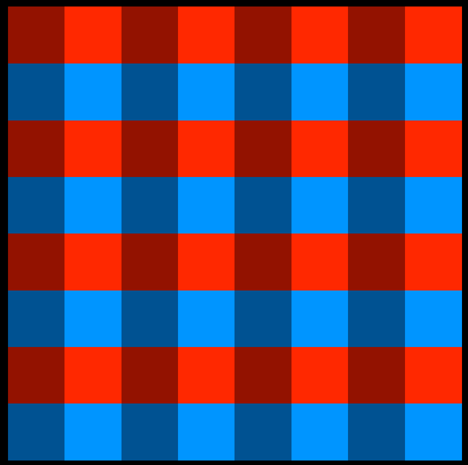
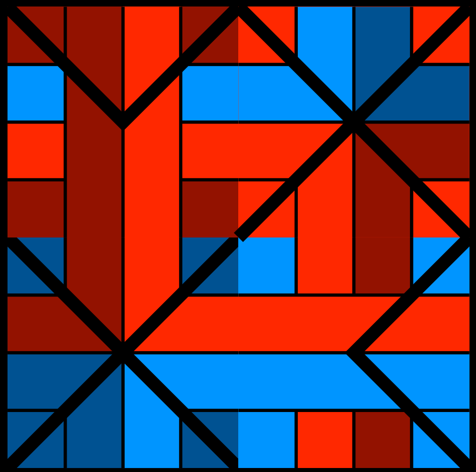
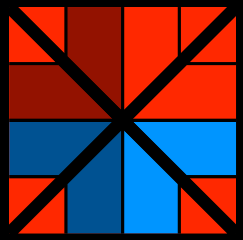




Layer 1

we just remove wires and squarifiers that propagate





Layers 2 & 3

the preimage of each tiling is a (valid) tiling.

Comparing to Literature

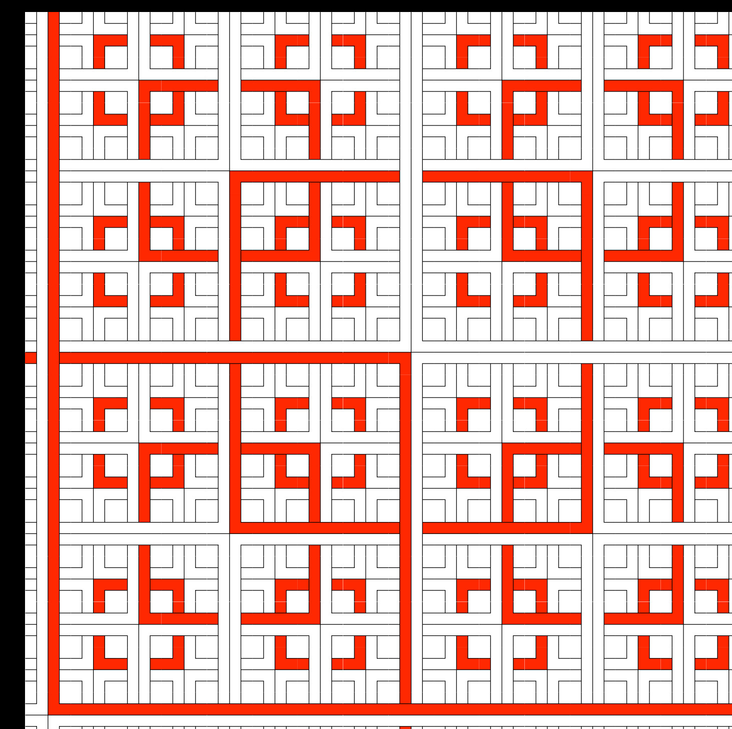
- Although it is no well known, Berger PhD dissertation already contains an aperiodic of 104 tiles. This tile set **does not** appear in the AMS memoir. The tile set also has 3 layers, the first two being isomorphic. The third level is different leading to a different set of tilings and more tedious proof.

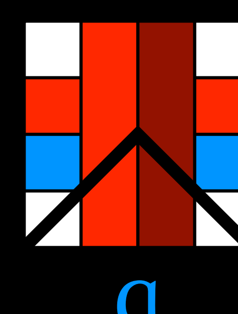
- The well known 56 tiles set of Robinson can be found by merging all three colors different from light red into white: this way you obtain 56 tiles isomorphic to Robinson's ones. The set generates more tilings, leading to subtle synchronization (alignment) problems.

Enforcing any substitution

Encode substitutions

- Select the red color of wires: the sequence of squares encode a tree.
- Put a letter on each square.
- Apply the substitution rule to go from one square to the next one.
- Up to projection the substitution limit set is encoded in the set of tilings.





$$b = s(a)\begin{pmatrix}1\\1\end{pmatrix}$$

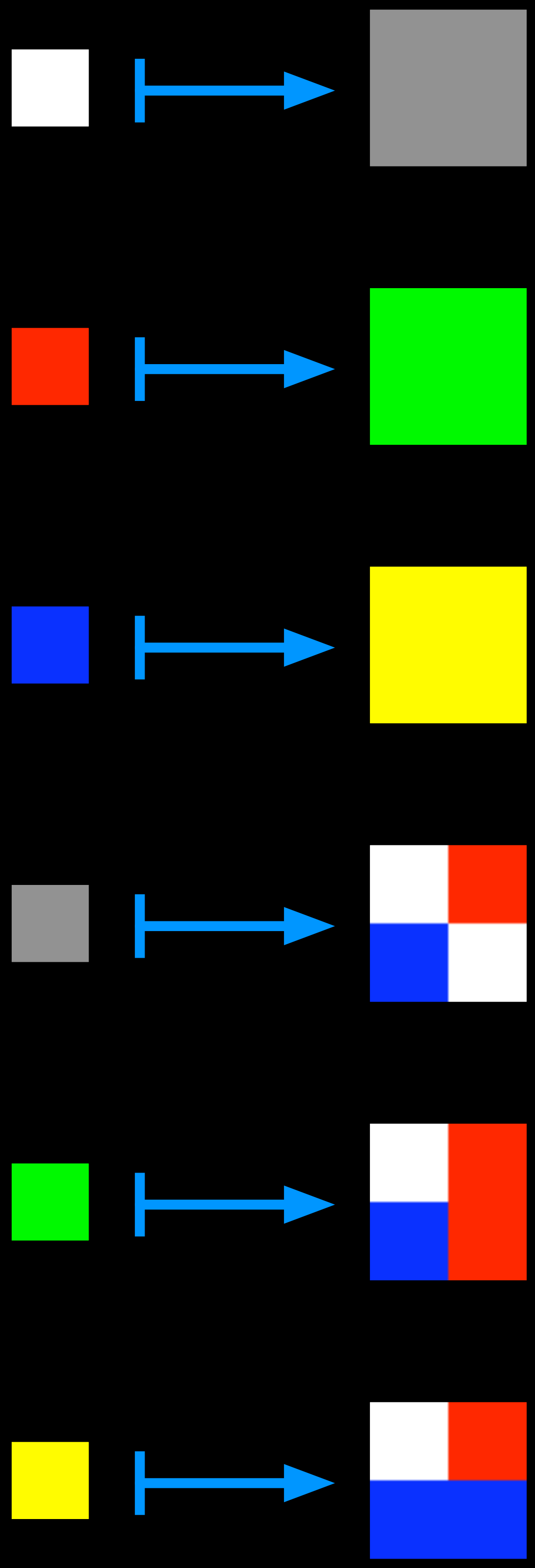
Substitution enforcement

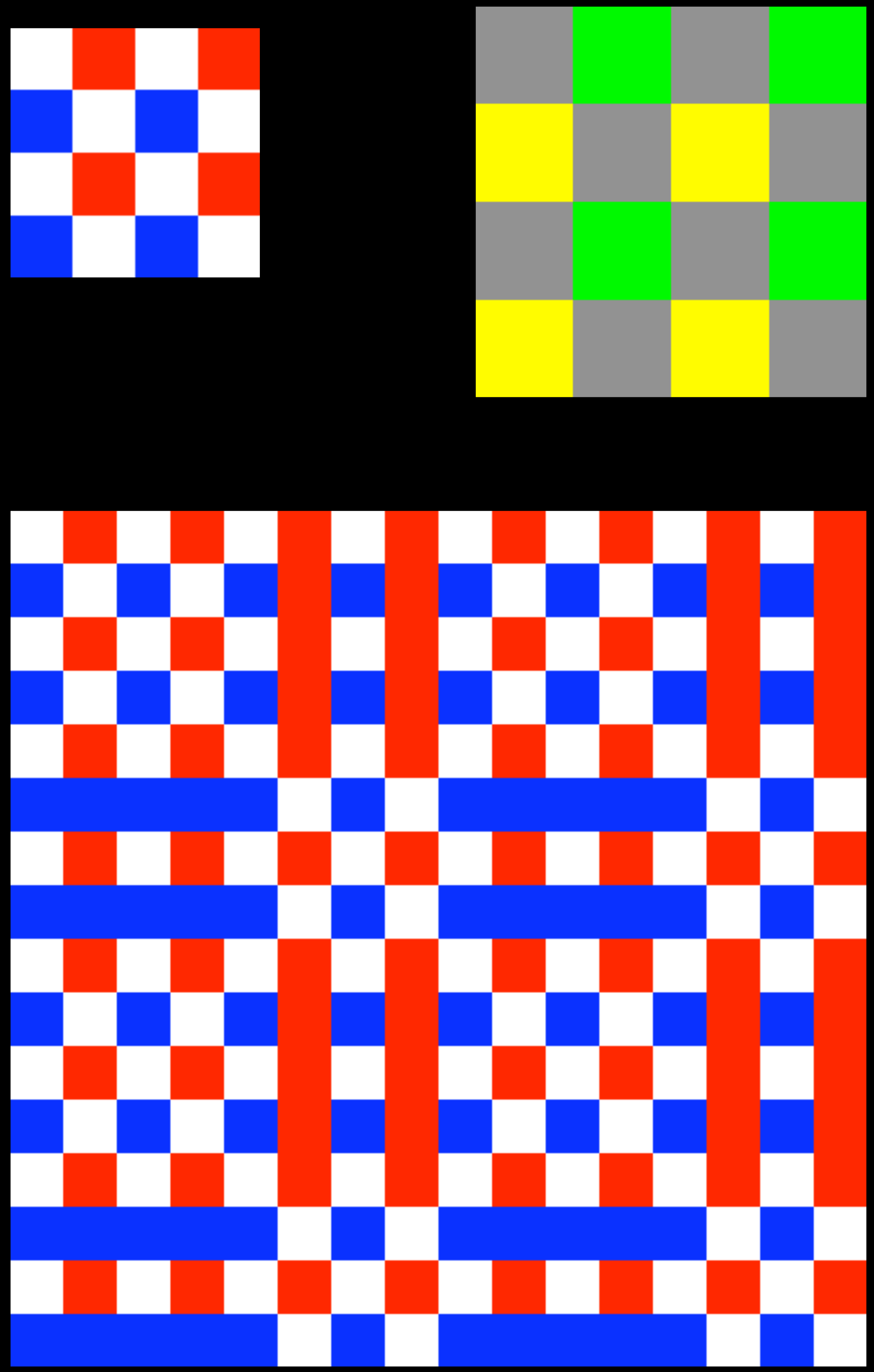
Let π map every tile of $\tau(s')$ to $s'(a)(u)$ where a and u are the letter and the value of $\boxplus$ on layer 1.

Theorem 2. Let s' be any substitution system. The tile set $\tau(s')$ enforces s': $\pi\left(X_{\tau(s')}\right) = \Lambda_{S'}$.

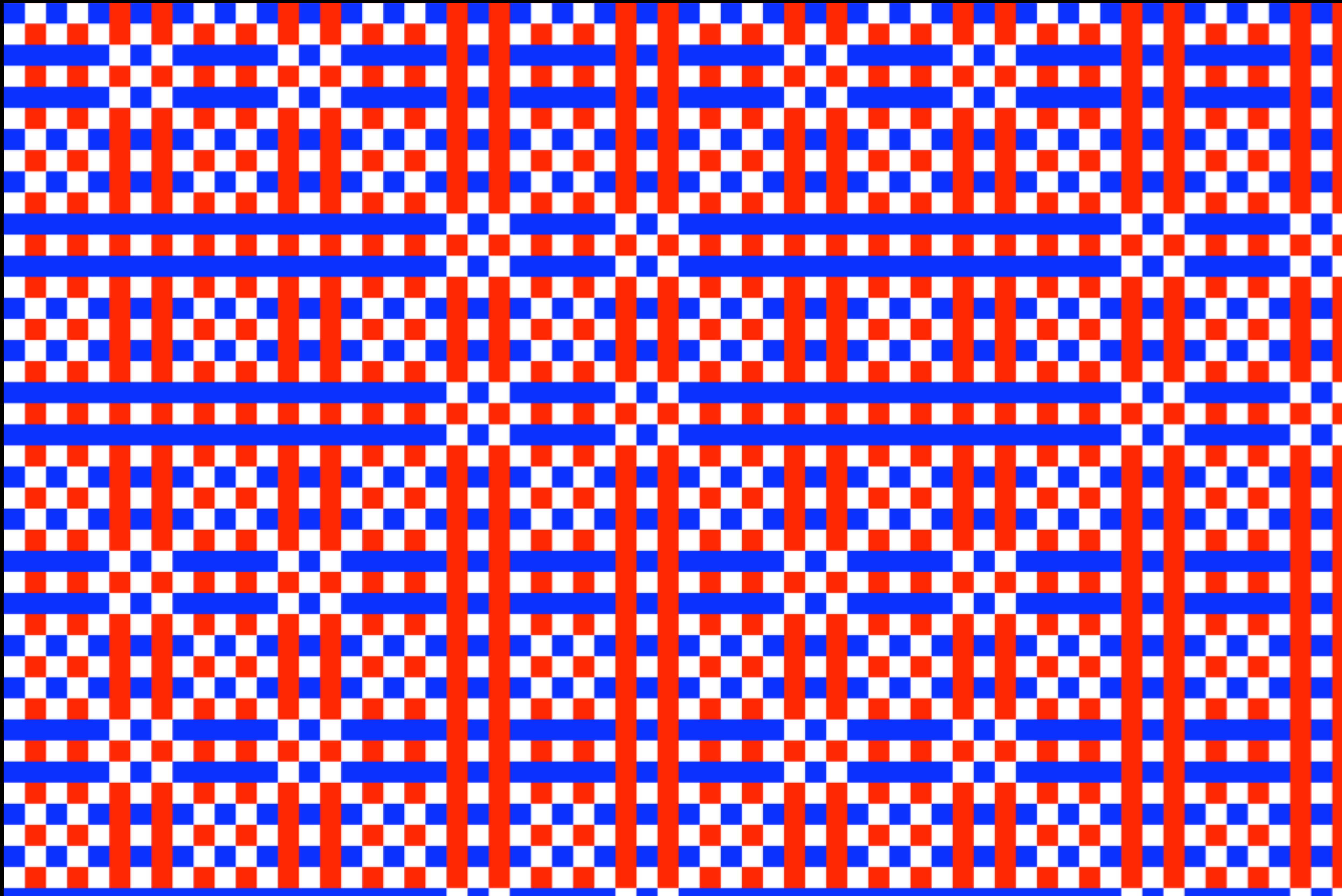
Idea. Every tiling of $\tau(s')$ codes an history of S' and every history of S' can be encoded into a tiling of $\tau(s')$.

Undecidability of the Domino Problem





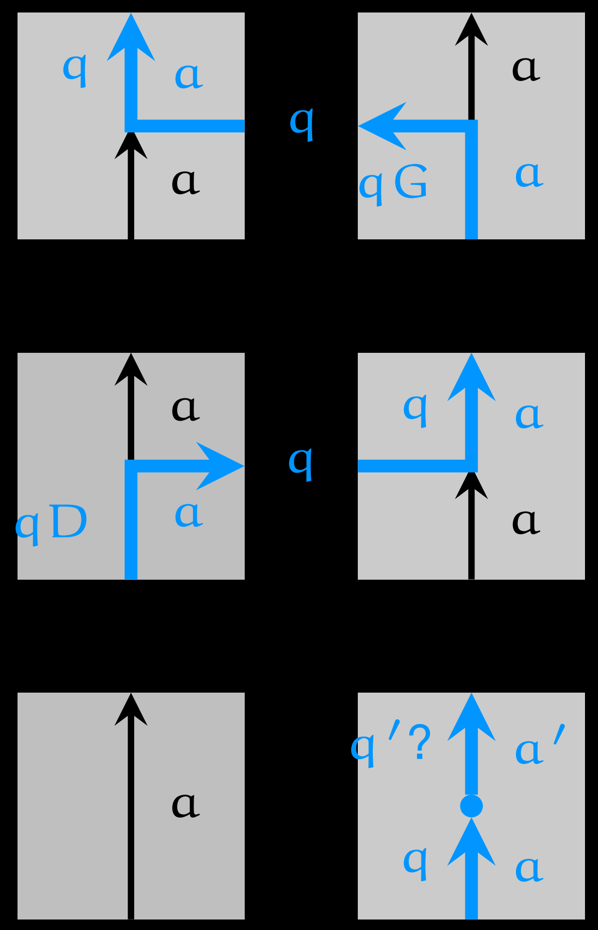
The O2 substitution



O2 iteration

Turing computation

- Turing machines can be simulated with tilesets.
- Put an initial computation at the SW corner of each O2 computation square.
- Remove the halting state.
- The tileset tiles the plane iff the Turing machine does not halt.



Thank you for your attention