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Treewidth and Minimum Fill-in of Weakly Triangulated Graphs

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Abstract

We introduce the notion of "potential maximal clique" of a graph and we use it for computing the treewidth and the minimum fill-in of graphs for which the the potential maximal cliques can be listed in polynomial time. Finally we show how to compute the potential maximal cliques of weakly triangulated graphs.

Keywords: treewidth, minimum fill-in, weakly triangulated graphs, minimal triangulations

Résumé

Nous introduisons la notion de "clique potentielle maximale" d'un graphe et nous l'utilisons pour calculer le treewidth et le minimum fill-in des graphes pour lesquels les cliques potentielles maximales peuvent être énumerées en temps polynômial. Finalement nous montrons comment calculer les cliques potentielles maximales des graphes faiblement triangulés.

Mots-clés: treewidth, graphes triangulés, séparateurs minimaux, triangulations minimales

1 Introduction

The notion of *treewidth* was introduced by Robertson and Seymour in [17]. It plays a major role in graph algorithm design. Indeed, it has been shown that many classical NP-hard problems become polynomial and even linear when restricted to graphs with small treewidth. These algorithms use a tree decomposition or a *triangulation* of the input graph, which is a chordal supergraph, i.e. all the cycles with at least four vertices of the supergraph have a chord. The *minimum fill-in* problem consists in finding a triangulation of a graph such that the number of added edges is minimum. This parameter is used in sparse matrix factorization.

When computing the treewidth or the minimum fill-in, we are looking for triangulations of a graph. In both cases we can restrict to triangulations minimal by inclusion, that we call *minimal triangulations*. Also both problems are NP-complete. Nevertheless, these parameters can be computed in polynomial time for several classes of graphs such as chordal bipartite graphs [11] [3], circle and circular-arc graphs [8] [20] [14], AT-free graphs with polynomial number of separators [13]. Most of these algorithms use the fact that these classes of graphs have a polynomial number of *minimal separators*. It was conjectured in [9] [10] that the treewidth and the minimum fill-in should be tractable in polynomial time for all the graphs having a polynomial number of minimal separators. The conjecture is still open.

In this paper we introduce the notion of "potential maximal clique" and we show that if one can list in polynomial time all the potential maximal cliques of some class of graphs, then the treewidth and the minimum fill-in of those graphs can be computed in polynomial time. This notion is related to the work of [1], from which we can easily deduce that the potential maximal cliques of the previously cited classes of graphs can be listed in polynomial time.

The class of *weakly triangulated graphs*, introduced in [6], is a class of graphs with polynomial number of separators, probably the only one for which the treewith and minimum fill-in problems were still open. We give an algorithm computing the potential maximal cliques of these graphs. Consequently, the treewidth and the minimum fill-in of weakly triangulated graphs are computable in polynomial time.

2 Chordal graphs and minimal separators

Throughout this paper we consider connected, simple, finite, undirected graphs.

A graph H is *chordal* (or *triangulated*) if every cycle of length at least four has a chord. A *triangulation* of a graph $G = (V, E)$ is a chordal graph $H = (V, E')$ such that $E \subseteq E'$. H is a *minimal triangulation* if for any intermediate set E'' with $E \subseteq E'' \subset E'$, the graph (V, E'') is not triangulated.

Another characterization of minimal triangulations is provided in [18] :

Lemma 1 *Let H be a triangulation of a graph G. Then H is a minimal triangulation of G if and only if, for any edge e of $H - G$, the graph $H - e$ is not triangulated.*

A subset $S \subseteq V$ is an *a, b-separator* for two nonadjacent vertices $a, b \in V$ if the removal of S from the graph separates a and b in different connected components. S is a *minimal a, b-separator* if no proper subset of S separates a and b. We say that S is a *minimal separator* of G if there are two vertices a and b such that S is a minimal a, b separator. Notice that a minimal separator can be strictly included in another. We denote by Δ_G the set of all minimal separators of G.

Let G be a graph and S a minimal separator of G. We note $\mathcal{C}_G(S)$ the set of connected components of $G - S$. A component $C \in \mathcal{C}_G(S)$ is *full* if every vertex of S is adjacent to some vertex of C. We denote by $\mathcal{C}_G^*(S)$ the set of all full components of $G - S$. For the following lemma, we refer to [5].

Lemma 2 *A set S of vertices of G is a minimal a, b-separator if and only if a and b are in different full components of S.*

If $C \in \mathcal{C}(S)$, we say that $(S, C) = S \cup C$ is a *block* of S. A block (S, C) is called *full* if C is a full component of S.

Definition 1 *Two separators S and T* cross, *denoted by $S \sharp T$, if there are some distinct components C and D of $G - T$ such that S intersects both of them. If S and T do not cross, they are called* parallel, *denoted by $S || T$.*

It is easy prove that these relations are symmetric (see [15]). Remark that for any couple of parallel separators S and T, T is contained in some block (S, C) of S.

A *clique* of G is a complete subgraph of G. Dirac [4] has shown that all the minimal separators of a chordal graph are cliques. Using the fact that a separator cannot separate two adjacent vertices we deduce the following lemma.

Lemma 3 *Let G be a graph, S a minimal separator and Ω a clique of G. Then Ω is included in some block of S. In particular, the minimal separators of a chordal graph are pairwise parallel.*

Definition 2 *Let G be a graph. The* treewidth *of G, denoted by $tw(G)$, is the minimum, over all triangulations H of G, of $\omega(H) - 1$, where $\omega(H)$ is the the maximum cliquesize of H.*

Definition 3 *The* minimum fill-in *of a graph G, denoted by $mfi(G)$, is the smallest value of $E(H) - E(G)$, where the minimum is taken over all triangulations H of G.*

In other words, computing the treewidth of G means finding a triangulation with smallest cliquesize, while computing the minimum fill-in consists in finding a triangulation with smallest number of edges. In both cases we can restrict our work to minimal triangulations.

Let $S \in \Delta_G$ be a minimal separator. We denote by G_S the graph obtained from G by completing S, i.e. by adding an edge between every pair of non-adjacent vertices of S. If $\Gamma \subseteq \Delta_G$ is a set of separators of G, G_Γ is the graph obtained by completing all the separators of Γ. The results of [12], concluded in [16], establish a strong relation between the minimal triangulations of a graph and its minimal separators.

Theorem 1

- *Let $\Gamma \in \Delta_G$ be a maximal set of pairwise parallel separators of G. Then $H = G_\Gamma$ is a minimal triangulation of G and $\Delta_H = \Gamma$.*
- *Let H be a minimal triangulation of a graph G. Then Δ_H is a maximal set of pairwise parallel separators of G and $H = G_{\Delta_H}$.*

In other terms, every minimal triangulation of a graph G is obtained by considering a maximal set Γ of pairwise parallel separators of G and completing the separators of Γ. The minimal separators of the triangulation are exactly the elements of Γ.

It is important to know that the elements of Γ, who become the separators of H, have strictly the same behavior in H as in G. Indeed, the connected components of $H - S$ are exactly the same in $G - S$, for every $S \in \Gamma$. Moreover, the full components are the same in the two graph: $\mathcal{C}^*_H(S) = \mathcal{C}^*_G(S)$.

3 Potential maximal cliques and maximal sets of neighbor separators

The previous theorem gives a characterization of the minimal triangulations of a graph by means of minimal separators, but it needs a global look over the set of minimal separators. Therefore, it gives no algorithmic information about how we should construct a minimal triangulation in order to optimize its cliquesize or the fill-in. In [1] we introduced the notion of "maximal sets of neighbor separators", and we have shown how these sets are related to the maximal cliques of any triangulation of a graph that we call here "potential maximal cliques". We will give a new tool to recognize the potential maximal cliques of a graph, which will also help to compute all the potential maximal cliques of weakly triangulated graphs.

Definition 4 [1] *A vertex set Ω of a graph G is called a* potential maximal clique *if there is a minimal triangulation H of G such that Ω is a maximal clique of H.*

If K is a vertex set of G, we denote by $\Delta(K)$ the minimal separators of G included in K.

Definition 5 *A family $\mathcal{S}$ of minimal separators of a graph G is called* maximal set of neighbor separators *if there is a potential maximal clique Ω of G such that $\mathcal{S} = \Delta(\Omega)$. We also say that $\mathcal{S}$* borders *Ω in G.*

Definition 6 *Let G be a graph and $\mathcal{S} \subseteq \Delta_G$ a set of pairwise parallel separators such that for any $S \in \mathcal{S}$, there is a block $(S, C(S))$ containing all the separators of $\mathcal{S}$. Suppose that $\mathcal{S}$ ordered by inclusion has no greatest element. We define the* piece between *the elements of $\mathcal{S}$ by*

$$P(\mathcal{S}) = \bigcap_{S \in \mathcal{S}} (S, C(S))$$

[1]Dieter Kratsch suggested us this definition.

Notice that for any $S \in \mathcal{S}$ the block of S containing all the separators of $\mathcal{S}$ is unique : if $T \in \mathcal{S}$ is not included in S, there is a unique connected component of S containing $T - S$.

The two following theorems, proved in [1], allow us to recognize a potential maximal clique of a graph :

Theorem 2 *Let Ω be a vertex set of G and suppose that $\Delta(\Omega)$ has a maximum element S, i.e. every T in $\Delta(\Omega)$ is included in S. Then Ω is a potential maximal clique if and only if Ω is some block (S, C) and $G_S[\Omega]$ is a clique.*

Theorem 3 *Let Ω be a vertex set of G and suppose that $\Delta(\Omega)$ ordered by inclusion has no greatest element. Then Ω is a potential maximal clique if and only if $\Omega = P(\Delta(\Omega))$ and $G_{\Delta(\Omega)}[\Omega]$ is a clique.*

Notice that if we consider a minimal triangulation H of G such that Ω is a maximal clique of H, then the separators of G bordering Ω are also minimal separators of H, as shown in the following lemma :

Lemma 4 *Let H be a minimal triangulation of a graph G and T be a minimal separator of G such that $H[T]$ is a clique. Then T is also a minimal separator of H.*

Proof. We know that $H = G_\Gamma$ for a maximal set of pairwise parallel separators $\Gamma \subseteq \Delta_G$ and that $\Delta_H = \Gamma$ by theorem 1. Let $S \in \Gamma$ be any minimal separator of H. S is also a minimal separator of G. By lemma 3, the clique T will be in some block (S, C) of H. Since the blocks of S in G are the same as in H (theorem 1 and the related remarks) we deduce that S and T are parallel in G. Since Γ is a *maximal* set of pairwise parallel separators, T must be in $\Gamma = \Delta_H$. $\diamond$

We are going to give a strong characterization of potential maximal cliques, which does not use minimal separators. We need some easy observations on theorems 2 and 3.

Corollary 1 *Let Ω be a potential maximal clique of G and let $S \in \Delta(\Omega)$. Then S is strictly contained in Ω and $\Omega - S$ is in a full component of S.*

Proof. Let us prove from theorems 2 and 3 that S is strictly included in Ω. Suppose that $\Delta(\Omega)$ has a greatest element T with respect to inclusion, so we are in the case of theorem 2 and Ω is some block (T, C). Since $S \subseteq T$ and C is not empty, S is strictly contained in $T \cup C = \Omega$. Suppose now that $\Delta(\Omega)$ has no greatest element. Then for any $S \in \Delta(\Omega)$, there is some $T \in \Delta(\Omega)$ with $T - S \neq \emptyset$. Since $T \subseteq \Omega$, we conclude that $\Omega - S \neq \emptyset$. In both cases $S \subset \Omega$.

Let us prove now that $\Omega - S$ is in a full block of S. Consider a minimal triangulation H of G such that Ω is a maximal clique of H. By lemma 4, S is a minimal separator of H. Clearly $\Omega - S$ is in a full component of S in H. Since the full components of S in G are the same as in H, we conclude that $\Omega - S$ is in a full component of S in G. $\diamond$

Corollary 2 *Let Ω be a potential maximal clique of a graph G and let a be any vertex of $V - \Omega$. There is a minimal separator $S \subset \Omega$ that separates a and $\Omega - S$.*

Proof. If we are in the case of theorem 2, $\Delta(\Omega)$ has a greatest element S and $\Omega = (S, C)$. Then clearly S separates any vertex $a \in V - (S, C)$ from any vertex of $C = \Omega - S$. Suppose now that we are under the conditions of theorem 3, so $\Delta(\Omega)$ has no greatest element and $\Omega = P(\Delta(\Omega))$. By definition of $P(\Delta(\Omega))$, since $a \notin P(\Delta(\Omega))$ we deduce that there is some $S \in \Delta(\Omega)$ separating a from $P(\Delta(\Omega)) - S$. $\diamond$

Let now K be a set of vertices of a graph G. We denote by $C_1(K), \ldots, C_p(K)$ the connected components of $G - K$. We denote by $S_i(K)$ the vertices of K adjacent to at least one vertex of $C_i(K)$. When no confusion is possible we will simply speak of C_i and S_i. If $S_i(K) = K$ we say that $C_i(K)$ is a *full component* of K.

Lemma 5 *Let Ω be a potential maximal clique of a graph G and let $\Delta(\Omega)$ be the maximal set of neighbor separators bordering Ω. Then the elements of $\Delta(\Omega)$ are exactly the sets $S_i(\Omega)$.*

Proof. We prove that for any i, $1 \leq i \leq p$, S_i is a minimal a,b-separator for some $a \in C_i$ and $b \in \Omega - S_i$. Corollary 2 tells us that there is some minimal separator $S \in \mathcal{S}$ that separates a from $\Omega - S$; recall that $\Omega - S$ is not empty. Since every vertex in S_i has a neighbor in C_i, if S does not contain a vertex $x \in S_i$, S cannot separate $x \in \Omega - S$ from a so we get $S_i \subseteq S$. By corollary 1, $\Omega - S$ is in a full component of S, and therefore of S_i. Let b be a vertex of $\Omega - S$. Then a and b are in different full components of S_i, so S_i is a minimal a,b-separator by lemma 2. We have to prove now that for any minimal separator $S \subseteq \Omega$, there is some $i, 1 \leq i \leq p$ such that $S = S_i$. We have that $\Omega - S \neq \emptyset$ and $\Omega - S$ is in some full component of S. Let C be another full component of S. Then C is a connected component of $G - \Omega$, let us say C_i. It follows that $S \subseteq S_i$. Let $x \in S_i - S$, since C_i is a full component of S_i, we must have $x \in C$ contradicting $C = C_i$. It remains that $S = S_i$. We conclude that the separators of G included in Ω are exactly the sets S_i. $\diamond$

Remark 1 *Actually, each minimal separator S_i separates any vertex of C_i from any vertex of $G - (S_i, C_i)$.*

We also give a "sufficient condition" to characterize the potential maximal cliques, which is somehow the dual of lemma 5.

Theorem 4 *Let $K \subseteq V$ be a set of vertices. We denote by $\mathcal{S}$ the set of all $S_i(K)$. K is a potential maximal clique if and only if :*

1. *K has no full components.*
2. *$G_{\mathcal{S}}[K]$ is a clique.*

Moreover, $\mathcal{S}$ is the maximal set of neighbor separators bordering K.

Proof. We prove the "only if" part. Suppose that K is a potential maximal clique of G. By lemma 5, the maximal set of neighbor separators bordering K is $\mathcal{S}$. By theorems 2 and 3, K is a clique in the graph $G_{\mathcal{S}}$. It remains to show that K has no full components. Let C_i be any connected component of $G - K$. Then S_i are the neighbors of C_i in K. Since K is a potential maximal clique and S_i is a separator contained in K, we have that S_i is strictly contained in K, by corollary 1. Therefore, C_i is not a full component of K.

We prove now the "if" part. Let us show at first that $S_i, 1 \leq i \leq p$, is a minimal separator. S_i is clearly a separator and C_i is a full component of S_i. Let x be a vertex of $K - S_i$. We show that x belongs to a full component of S_i different from C_i. We denote by C_x the connected component of $G - S_i$ containing x. For any $y \in S_i$, y must have a neighbor in C_x. This is true if x and y are adjacent in G. If x and y are not adjacent, by the second condition of the theorem, x and y belong to a same S_j. C_j being a full component of S_j, there is a path in G connecting x to y entirely contained in C_j except from x and y, we deduce that $C_j \subseteq C_x$. It follows that y has a neighbor in C_x since it has a neighbor in C_j. S_i is a minimal separator of G according to lemma 2.

Now, given two distinct separators S_i and S_j, we have to show that they are parallel. We prove that $K - S_i$ is in a connected component of $G - S_i$. Let $x, y \in K - S_i$. If x and y are adjacent they are clearly in the same component of $G - S_i$. Otherwise, since $G_{\mathcal{S}}[K]$ is a clique, they are in a same S_k, so they are connected via C_k. So S_j intersects only the component of $G - S_i$ containing $K - S_i$ and then $S_i || S_j$. Therefore $\mathcal{S}$ consists in a set of pairwise parallel separators.

We have to show that any separator of G included in K is an element of $\mathcal{S}$. Consider a minimal triangulation H of G such that all the elements of $\mathcal{S}$ are separators of H. We know that K is a clique in H. Now let $U \subseteq K$ be any minimal separator of G. Notice that U must be strictly included in K, otherwise K would have two full components in G contradicting our choice of K. Clearly U is a clique in H, so by lemma 4, it is a minimal separator of H. Since K is a clique in H, it must be included in some full block of U. Let (U, C) be another full block of U in H, and consequently in G. We have that C is a connected component of $G - U$ and U separates C and $K - U$. We deduce that C is also a connected component of $G - K$, let us say C_i. By definition of S_i, we have $U \subseteq S_i$. Suppose there exists a vertex $x \in S_i - U$, since x has a neighbor in C, the connected component C of $G - U$ would contain x contradicting $C = C_i$. So we have $U = S_i$ and $U \in \mathcal{S}$.

We want to prove that $\mathcal{S}$ satisfies the conditions of theorems 2 or 3. Remark that for any $y \in V - K$, y is in some connected component C_i of $G - K$ and the separator $S_i \in \mathcal{S}$ separates y from $K - S_i$. Suppose now that $\mathcal{S}$ has an element S, maximum by inclusion. Let (S, C) be the block of S containing K. By the previous remark, for any $y \in V - K$, S separates y and $K - S$, so $y \notin (S, C)$. It remains that $(S, C) = K$, so $\mathcal{S}$ satisfies all the conditions of theorem 2. Now if $\mathcal{S}$ does not have an element maximum by inclusion, clearly K is contained in the piece between the separators of $\mathcal{S}$ in G. By the previous remark, $P_G(\mathcal{S})$ does not contain any $y \in V - K$, so $K = P_G(\mathcal{S})$ and therefore we are under the conditions of theorem 3. It remains that $\mathcal{S}$ forms a maximal set of neighbor separators of G, bordering the set K. $\diamond$

4 Triangulating blocks

In this section we prove that the potential maximal cliques of a graph are sufficient to compute its treewidth and its minimum fill-in.

Let $B = (S, C)$ be a block of the graph G. The graph $R(S, C) = G_S[S \cup C]$ is called the *realization* of the block B. The following lemma, proved in [13], gives a relation between minimal triangulation of a graph and minimal triangulations of some block realizations.

Lemma 6 *Let $S \in \Delta_G$ and let $C_1, C_2, \ldots, C_p$ be the connected components of $G[V-S]$. Suppose that H_i is a minimal triangulation of $R(S, C_i)$ for any $i, 1 \leq i \leq p$. Then the graph $H = (V, E(H))$ with $E(H) = \bigcup_{i=1}^{p} E(H_i)$ is a minimal triangulation of G.*

Conversely, let H be a minimal triangulation of G with $S \in \Delta_H$. Then $H[S \cup C]$ is a minimal triangulation of the realization $R(S, C)$ for each component C of $G[V-S]$.

This gives an equation for computing the treewidth and the minimum fill-in of a graph (see [13] for a proof) :

Corollary 3 *Let G be a non-complete graph. Then*

$$tw(G) = \min_{S \in \Delta(G)} \max_{C \in \mathcal{C}(S)} tw(R(S, C))$$

$$mfi(G) = \min_{S \in \Delta(G)} (fill(S) + \sum_{C \in \mathcal{C}(S)} mfi(R(S, C)))$$

where $fill(S)$ is the number of non-edges of S.

We give a version of lemma 6 using potential maximal cliques instead of minimal separators.

Theorem 5 *Let H be a minimal triangulation of G and let Ω be a maximal clique of H. For each connected component $C_i, 1 \leq i \leq p$ of $G - \Omega$, let S_i be the vertices of Ω having a neighbor in C_i. Then $H_i = H[S_i \cup C_i]$ are minimal triangulations of the realizations $R(S_i, C_i)$.*

Conversely, let Ω be a potential maximal clique of G. For each connected component $C_i, 1 \leq i \leq p$ of $G - \Omega$, let H_i be a minimal triangulation of $R(S_i, C_i)$. Then $H = (V, E(H))$ with $E(H) = \bigcup_{i=1}^{p} E(H_i) \cup \{\{x, y\} | x, y \in \Omega\}$ is a minimal triangulation of G.

Proof. For proving the first part, notice that by theorem 4, (S_i, C_i) are blocks of G and of H. Then by lemma 6, $H[S_i \cup C_i]$ are minimal triangulations of $R(S_i, C_i)$.

We prove now the second part. Let us prove that H is a triangulated graph. Suppose that H has a chordless cycle of length at least four. Since Ω is a clique, the cycle is not contained in Ω, so it has a vertex a in some component C_i. Since $H[S_i \cup C_i]$ is chordal, the cycle is not contained in (S_i, C_i). Then at least two non consecutive points of the cycle, say b and c, are in S_i. So the cycle has a chord, namely the edge $\{b, c\}$.

Let us prove now that H is a minimal triangulation of G. Let $e = \{x, y\}$ be an edge of $H - G$. Suppose at first that x and y are not both contained in Ω. Then x and y are in some H_i. Since e is not in Ω, e is not in S_i. It remains that e is an edge of $H_i - R(S_i, C_i)$. By lemma 1, $H_i - e$ is not chordal, so $H - e$ is not chordal. Suppose now that $x, y \in \Omega$. Then x, y must be in some S_i, since $H[\Omega] = G_{\mathcal{S}}[\Omega]$ and $e \in H - G$. Consider two full components C and D of $H - S_i$ and a shortest path from x to y in each of them. The two paths form a cycle of length at least four, which is a chordless cycle of $H - e$.

By lemma 1, H is a minimal triangulation of G. $\diamond$

We want to give a caracterization of the minimal triangulations of a realization $R(S,C)$ using the potential maximal cliques Ω with $S \subset \Omega$ and $\Omega \subseteq (S,C)$ and the minimal triangulations of the realizations of some blocks (S_i, C_i), strictly included in (S,C). This will express the treewidth and the minimum fill-in of a realization from the treewidth, respectively the minimum fill-in of realizations of smaller blocks. This parameters will be computed by dynamic programming on blocks.

The minimal triangulations of the realizations of non-full blocks are easy reducible to the case of full blocks. For the following lemma, see for example [2] :

Lemma 7 *Let (S,C) be a non-full block of G and let S^* be the set of vertices of S having some neighbor in C. Then (S^*,C) is a full block of G and a super-graph H of $R(S,C)$ is a minimal triangulation of $R(S,C)$ if and only if $H[S^* \cup C]$ is a minimal triangulation of $R(S^*,C)$.*

We conclude :

Corollary 4 *Let (S,C) be a non-full block of G and let S^* be the vertices of S adjacent in G to at least one vertex of C. Then*

$$tw(R(S,C)) = \max(|S| - 1, tw(R(S^*,C)))$$

$$mfi(R(S,C)) = fill(S) + mfi(R(S^*,C))$$

It remains to express the treewidth and the minimum fill-in of realizations of full blocks from realizations of smaller blocks.

Lemma 8 *Let $R(S,C)$ be the realization of some full block (S,C) and let $H(S,C)$ be a minimal triangulation of $R(S,C)$. Then there is a maximal clique Ω of $H(S,C)$ such that $S \subset \Omega$ and Ω is a potential maximal clique of G.*

Proof. Clearly S is a clique in $H(S,C)$. Consider a maximal clique Ω of $H(S,C)$, containing S. We have that $S \neq \Omega$ because C is a full component of S in G, so also in $H(S,C)$ and, by theorem 4, a maximal clique of a triangulated graph has no full component. It remains to show that Ω is a potential maximal clique of G. By lemma 6, there is a minimal triangulation H of G such that $H[S \cup C] = H(S,C)$. The components of S in H are the same as in G. Since S separates C from $V - (C \cup S)$, Ω is a maximal clique of H so by definition it is a potential maximal clique of G. ⋄

Notice that we proved that any maximal clique of $H(S,C)$ is a potential maximal clique of G and, moreover, it is a maximal clique of any minimal triangulation H with $H[S \cup C] = H(S,C)$.

By theorem 5 and lemma 8 we have proved:

Theorem 6 *Let (S,C) be a full block of a graph G. Then $H(S,C)$ is a minimal triangulation of $R(S,C)$ if and only if there is a potential maximal clique $\Omega \subseteq (S,C)$ of G such that $S \subset \Omega$ and $H(S,C) = (S \cup C, E(H))$ with $E(H) = \bigcup_{i=1}^{p} E(H_i) \cup \{\{x,y\} | x,y \in \Omega\}$, where H_i are minimal triangulations of $R(S_i,C_i)$.*

Corollary 5 *Let (S, C) be a full block of G. Then*

$$tw(R(S,C)) = \min_{S \subset \Omega \subseteq (S,C)} \max(|\Omega| - 1, tw(R(S_i, C_i)))$$

$$mfi(R(S,C)) = \min_{S \subset \Omega \subseteq (S,C)} fill(S) + \sum mfi(R(S_i, C_i))$$

We give a sketch of the algorithm computing, by standard dynamic programming techniques, the treewidth and the minimum fill-in of a gaph by using its potential maximal cliques.

The algorithm computes first all the blocks (S, C) and sorts them by the number of vertices. For each block (S, C), we compute all the potential maximal cliques Ω s.t. $S \subset \Omega \subseteq (S, C)$. For each potential maximal clique Ω, we keep the blocks (S', C') with $S' \subset \Omega$ and $C' \cap \Omega = \emptyset$. Finally, for each block (S, C), taken in increasing order, for each potential maximal clique Ω with $S \subset \Omega \subseteq (S, C)$, we compute the treewidth, respectively the minimum fill-in of $R(S, C)$ from the treewidth, respectively minimum fill-in of $R(S', C')$ with $(S', C') \subset (S, C)$ and $S' \subset \Omega$. The whole algorithm can be implemented in $\mathcal{O}(bpn + re)$ time, where n and e are the number of vertices, respectively of edges of G and r, b and p are the number of minimal separators, blocks, respectively potential maximal cliques of G. The algorithm is clearly polynomial in n and p.

5 Weakly triangulated graphs

We consider now two non-adjacent vertices x, y of G. Let G' be the graph obtained from G by adding the edge $\{x, y\}$. We will show in this section that the potential maximal cliques of G can be computed from the minimal separators of G and the potential maximal cliques of G'. We will use this technique to compute all the potential maximal cliques of any weakly triangulated graph.

Let once again Ω be a potential maximal clique clique of G bordered by the maximal set of neighbor separators $\mathcal{S} = \Delta(\Omega)$. Let $C_1, \ldots, C_p$ be the connected components of $G - \Omega$ and let S_i be the set of vertices of Ω having at least a neighbor in C_i. We want to describe the behaviour of Ω and $\mathcal{S}$ in the graph G'. We deduce directly from theorem 4 :

Lemma 9

1. *If $x, y \in \Omega$ or there is an $i, 1 \leq i \leq p$ such that $x, y \in C_i$ or $x \in S_i$ and $y \in C_i$ then Ω is a potential maximal clique of G' bordered by $\mathcal{S}$.*

2. *If $y \in C_1$, $x \in \Omega - S_1$ and $\Omega \neq S_1 \cup \{x\}$, then Ω is a potential maximal clique of G' bordered by the set $\mathcal{S}'$ whose elements are $S_1 \cup \{x\}, S_2, \ldots, S_p$.*

3. *If $x \in C_1$, $y \in C_2$ and $\Omega \neq S_1 \cup S_2$ then Ω is a potential maximal clique of G' bordered by the set $\mathcal{S}'$ whose elements are $S_1 \cup S_2, S_3, \ldots, S_p$.*

It follows directly :

Theorem 7 *Let Ω be a potential maximal clique of G bordered by a maximal set of neighbor separators $\mathcal{S}$. Let x, y be two non-adjacent vertices of G and let $G' = G \cup \{x, y\}$. Two cases are possible :*

1. *Ω can be written as $S_1 \cup \{x\}$, $S_1 \cup \{y\}$ or $S_1 \cup S_2$, where S_1, S_2 are minimal x, y-separators of G.*
2. *Ω is a potential maximal clique of G', bordered by a maximal set $\mathcal{S}'$ of neighbor separators of G', easily computable from $\mathcal{S}$.*

The weakly triangulated graphs were introduced in [6]. A graph G is called *weakly triangulated* if neither G nor its complement $\overline{G}$ have an induced cycle with more than four vertices. This class contains the chordal graphs, the chordal bipartite graphs and the distance hereditary graphs.

We denote by $N(x)$ the neighbors of the vertex x. We say that two vertices x, y of a graph G form a *two-pair* if their common neighbors $N(x) \cap N(y)$ form an x, y-separator. The following theorem was proved in [7]:

Theorem 8 *If G is a weakly triangulated graph, then every induced subgraph of G that is not a clique contains a two-pair.*

Spinrad and Sritharan give in [19] an algorithm recognizing the weakly triangulated graphs, based on the following theorem:

Theorem 9 *Let $G = (V, E)$ be a graph and let $\{x, y\}$ be a two-pair of G. Let $G' = (V, E')$ be the graph obtained from G by adding the edge $\{x, y\}$. Then G is weakly triangulated if and only if G' is weakly triangulated.*

Notice that a clique is a weakly triangulated graph. The recognition algorithm considers an input graph G and, while G has a two-pair $\{x, y\}$, it adds the edge between x and y to G. At the end of the loop, either G became a clique, in which case the initial graph was weakly triangulated by theorem 9, or G is not a clique and it has no two-pair, in which case the input graph could not be weakly triangulated by theorems 8 and 9.

We denote by $\overline{e}$ the number of edges of $\overline{G}$. Let now $G = (V, E)$ be a weakly triangulated graph and let $f_1 = \{x_1, y_1\}, \ldots, f_p = \{x_{\overline{e}}, y_{\overline{e}}\}$ be the edges added to G by the recognition algorithm in this order. In particular, $(V, E \cup \{f_1, \ldots, f_{\overline{e}}\})$ is a clique. We denote by G_i the graph $(V, E \cup \{f_1, f_2, \ldots, f_i\})$, with $0 \leq i \leq \overline{e}$ (so $G_0 = G$ and $G_{\overline{e}}$ is a clique). We will describe the minimal separators, respectively the maximal sets of neighbor separators of G_i using the minimal separators, respectively the maximal sets of neighbor separators of G_{i+1}, for any $i < \overline{e}$.

Theorem 10 *Let G be a non complete weakly triangulated graph and let $\{x, y\}$ be a two-pair of G. Let S_{xy} be the set $N(x) \cap N(y)$. We take G' the graph obtained from G by adding the edge $\{x, y\}$. Then $\Delta_G \subseteq \Delta_{G'} \cup \{S_{xy}\}$.*

Proof. Clearly S_{xy} is a minimal x, y-separator of G, by definition of a two-pair. Let S be a minimal separator of G.

Suppose at first that S separates x and y, and let C_x, C_y be the connected components of $G - S$ containing x, respectively y. If both C_x and C_y are full components of S in $G - S$, S is a minimal x, y-separator. Notice that $N(x) \subset C_x \cup S$ and $N(y) \subset C_y \cup S$. We deduce that $N(x) \cap N(y) \subseteq S$, and since $N(x) \cap N(y) = S_{xy}$ is a x, y-separator in G by definition of a two-pair, it remains that $S = S_{xy}$. Consider now that C_x and C_y are not both full components of S in G. The connected components of $G' - S$ are the same as in

G, excepting C_x and C_y which form a unique component $C_x \cup C_y$ in G'. If S had in G two full components D and E different from C_x and C_y, then D and E are full components of S in G', so S is a minimal separator of G'. Otherwise, S has a unique full component D in G different from C_x and C_y, and one of C_x, C_y must be full in G. Therefore, D and $C_x \cup C_y$ are full components of S in G', so S is a minimal separator of G'.

If S does not separate x and y, the connected components of $G' - S$ are the same as in $G - S$, so S has two full components in G'. Consequently, S is a minimal separator of G'. $\diamond$

So for any $i < p$, G_i has at most one more minimal separator than G_{i+1}. We deduce:

Corollary 6 [2] *A weakly triangulated graph G has at most $\overline{e}$ minimal separators, where $\overline{e}$ is the number of edges of $\overline{G}$.*

We can conclude directly from theorem 7 and lemma 6 that all the potential maximal cliques and all the maximal sets of neighbor separators of a weakly triangulated graph can be computed in polynomial time. But we can refine the results of the third case of lemma 9.

Lemma 10 *Let G be a graph, let x, y be a two-pair of G and let Ω be a potential maximal clique of G such that x and y are in different conected components of $G - \Omega$. Then Ω is a potential maximal clique of $G' = G \cup \{x, y\}$.*

Proof. We use the fact that if x, y is a two-pair of a graph G, then $S_{xy} = N(x) \cap N(y)$ is the only x, y-minimal separator of G. Let C_1 and C_2 the connected components of $G - \Omega$ containing x, respectively y. Like previously, let S_1 and S_2 the sets of vertices of Ω having a neighbor in C_1, respectively C_2. We want to prove that both S_1 and S_2 contain S_{xy} and at least one of them is equal to S_{xy}. By remark 1, S_1 separates x and y, so it must contain $N(x) \cap N(y) = S_{xy}$. The same holds for S_2. Suppose that there is some vertex $a \in S_1 - S_{xy}$ and some vertex $b \in S_2 - S_{xy}$. Since a has a neighbor in C_1, a is in the connected component $G - S_{xy}$ containing x. Also b is in the connected component of $G - S_{xy}$ containing y. Then Ω intersects different connected components of $G - S_{xy}$, whic is, by corollary 1, a contradiction to the fact that $S_{xy} \subset \Omega$. It remains that $S_1 = S_{xy}$ or $S_2 = S_{xy}$. Therefore, $S_1 \cup S_2$ is equal to S_1 or S_2, and consequently $\Omega \neq S_1 \cup S_2$ by corollary 1. By the third part of lemma 9, Ω will be a potential maximal clique of G'. $\diamond$

We deduce from theorem 7 and lemma 10 :

Corollary 7 *Let Ω be a potential maximal clique of G bordered by a maximal set of neighbor separators $\mathcal{S}$. Let x, y be a two-pair of G and let $G' = G \cup \{xy\}$. Let $S_{xy} = N(x) \cap N(y)$. Two cases are possible :*

1. *Ω can be written as $S_{xy} \cup \{x\}$ or $S_{xy} \cup \{y\}$.*
2. *Ω is a potential maximal clique of G', bordered by a maximal set $\mathcal{S}'$ of neighbor separators of G', easily computable from $\mathcal{S}$.*

[2] Ton Kloks independently discovered this result (unpublished).

Corollary 8 *A weakly triangulated graph G has at most $2\overline{e}+1$ potential maximal cliques.*

Proof. We consider the sequence of graphs $G_0 = G, G_1, \ldots, G_{\overline{e}}$ previously defined. Since G_{i+1} is obtained from G_i by adding an edge between a two-pair, by corollary 7 G_i has at most two more potential maximal cliques than G_{i+1}. Clearly $G_{\overline{e}}$, which is a clique, has a unique potential maximal clique. ⋄

Corollary 9 *The treewidth and the minimum fill-in of weakly triangulated graphs can be computed in polynomial time.*

The complexity of computing all potential maximal cliques is $\mathcal{O}(n^2\overline{e})$. It is sufficient to use the weakly triangulated graphs recognition algorithm of [19] to generate the two-pairs $\{x_i, y_i\}$ and the sets $S_{x_i y_i}$. This takes $\mathcal{O}(n^2\overline{e})$ time. One can check in $\mathcal{O}(n^2)$ time if a set is a potential maximal clique, using theorem 4. So the list of all potential maximal cliques is computable in $\mathcal{O}(n^2\overline{e})$.

For weakly triangulated graphs, the number of minimal separators is $r = \mathcal{O}(\overline{e})$, the number of blocks is $b = \mathcal{O}(n\overline{e})$ and the number of potential maximal cliques is $p = \mathcal{O}(\overline{e})$. According to the complexity of the algorithm of section 4, the treewidth and the minimum fill-in are computable in $\mathcal{O}(n^2\overline{e}^2)$ time.

6 Conclusions

We have introduced the notion of potential maximal clique. This leads to compute the treewidth and the minimum fill-in for natural classes of graphs, having a polynomial number of minimal separators, in a unified manner. Moreover, we solved the two problems for a new class of graphs, namely the weakly triangulated graphs. Once again these graphs have few separators.

Our technique allows to approach the conjecture of Bodlaender et al. which claims that the treewidth and the minimum fill-in should be polynomially tractable for the whole class of graphs having a polynomial number of minimal separators. A way to prove this conjecture would be to establish the following result: does there exists a polynomial P such that for any graph G, one can compute a sequence of graphs $G_0 = G, G_1, \ldots, G_p$ where G_p is a clique, G_{i+1} is obtained from G_i by adding an edge and, for all $i, 1 \leq i \leq p$, $|\Delta_{G_i}| \leq P(|\Delta_G|, |V(G)|)$?

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