Disproving the Normal Graph Conjecture

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- Example: K_k is perfect.

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- Theorem (SPGT, Chudnovsky, Robertson, Seymour, Thomas 2006)

  G is perfect iff G does not contain an induced odd cycle of length at least 5 or the complement of an induced odd cycle of length at least 5.

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- For G perfect: $\omega(G^n) = \omega(G)^n$

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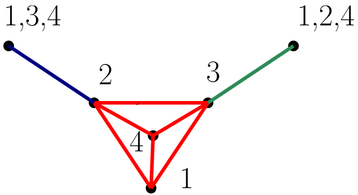
- 

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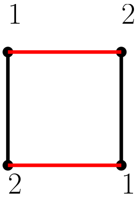
- 

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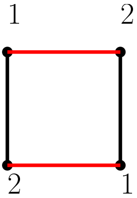
- 

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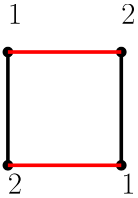
- 

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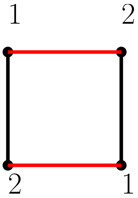
- 

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- No other minimal non-normal graphs known.

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- Conjecture (DeSimone, Körner, '99)
  A graph with no $C_5, C_7, \bar{C}_7$ as induced subgraph is normal.

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- The proof is probabilistic.

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- Remove one vertex from each of $2n^{0.7}$ cycles to destroy all cycles length < 8.

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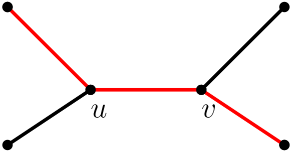
- Degrees: **Whp** almost every vertex has degree $\Omega(n^{0.1})$

Overview of the proof

- G girth 8 and normal, then we may assume the clique cover induces vertex-disjoint stars.



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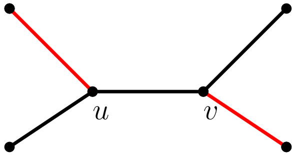
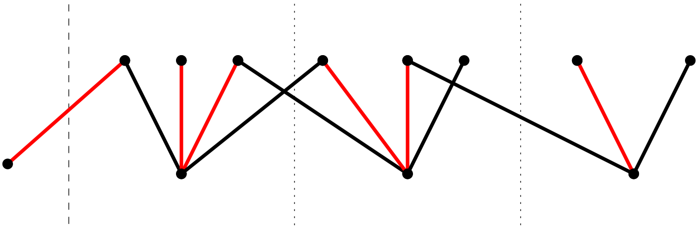


Obtaining a large independent set



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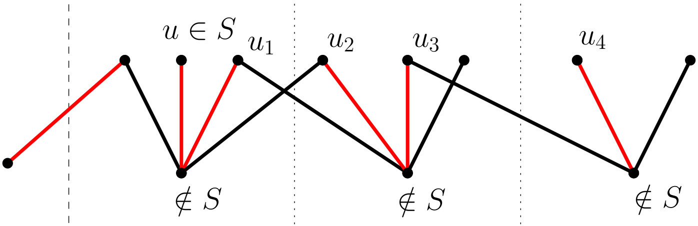

Thank You