Subdivision of oriented cycles in oriented graphs

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Orientation and colouring

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Theorem (Bondy)

D is strong, $\chi(D) = k \Rightarrow D$ has a directed cycle with at least k vertices.

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Lemma (Gyárfás, Thomassen)

There exists oriented graph D_i such that $\chi(D_i) \geq i$ and every cycle in D_i has at least four blocks.

Conjecture (L. Addario-Berry et al.)

For all k and l, there exists an integer $\alpha(k,l)$ such that :
$\chi(D) > \alpha(k,l)$ and D **strong** *$\Rightarrow$ D contains an oriented cycle with two blocks, one of length at least k, and the other at least l.*

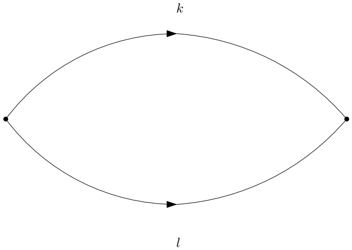

Figure : Cycle with two blocks.

This can be seen as finding a subdivision of $C(k,l)$ the cycle with one block of size k and one block of size l.

Results

We showed the following:

Theorem

$\alpha(1,2) = \alpha(2,2) = \alpha(1,3) = 4$, $\alpha(2,3) = 5$ *and* $\alpha(3,3) \leq 7$.

Where $\alpha(k,l)$ is the smallest integer such that:
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Let D be a 2-strong digraph. If $\chi(D) \geq (k+l-2)(k-1)+2$, then D contains a subdivision of $C(k,l)$.

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A *handle decomposition* of D starting at $v \in V(D)$ is a triple $(v, (h_i)_{1 \leq i \leq p}, (D_i)_{0 \leq i \leq p})$, where $(D_i)_{0 \leq i \leq p}$ is a sequence of strongly connected digraphs and $(h_i)_{1 \leq i \leq p}$ is a sequence of handles such that:

- $V(D_0) = \{v\}$,
- for $1 \leq i \leq p$, h_i is a handle of D_i and D_i is the (arc-disjoint) union of D_{i-1} and h_i, and
- $D = D_p$.

Handle decomposition

Figure : $D_0 = \{v_0\}$

Handle decomposition

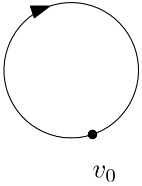

Figure : $D_1 = D_0 \cup h_1$

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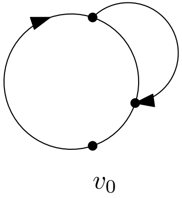

Figure : $D_2 = D_1 \cup h_2$

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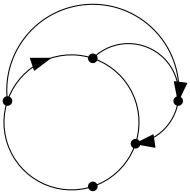
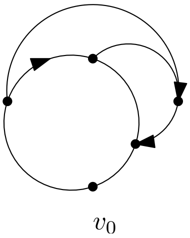


Figure : $D_3 = D_2 \cup h_3$

Handle decomposition



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Every strong digraph admits a handle decomposition.

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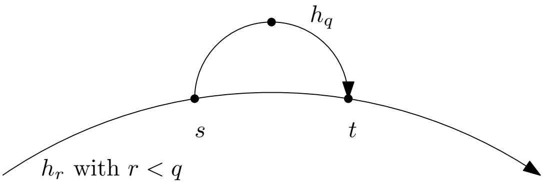

Figure : h_q is an (s,t) dipath

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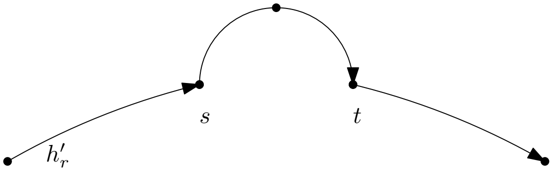

Figure : h'_r is longer than h_r

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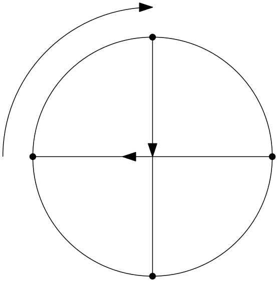

Figure : 2 crossing chords

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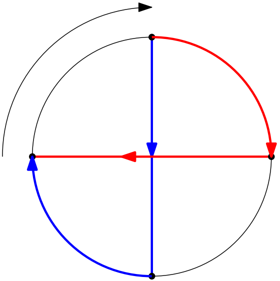

Figure : A subdivision of $C(2,2)$

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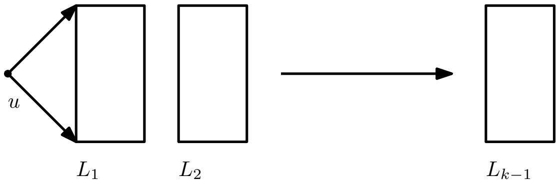
Assume $k > l$ and pick u any vertex. Let $L_i = \{v | \text{dist}_D(u,v) = i\}$.
If $v \in L_k$ we have a subdivision of $C(k,k)$. If $L_k = \emptyset$,
$V(D) = \{u\} \cup L_1 \cup \cdots \cup L_{k-1}$.



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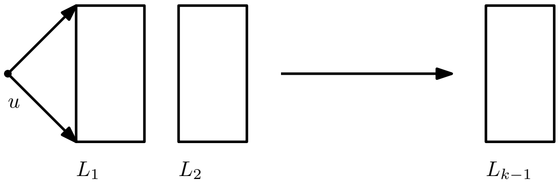

$\chi(D) \leq 1 + \sum_i \chi(L_i)$, so there exists i, with $\chi(L_i) \geq (k+l-1)$.

Because $\chi(L_i) \geq (k+l-1)$, there exists a path with two blocks, one of length k, the other of length $l-1$.

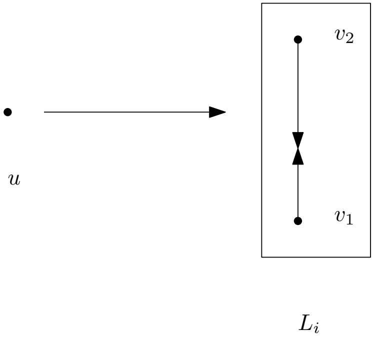

Figure : Path of two blocks in L_i

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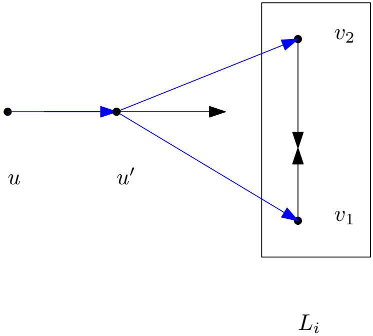

Figure : Subdivision of $C(k, l)$

Open questions

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- *Which connectivity condition is necessary.*
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Proposition

$\chi(D) > 12$ and D strong $\Rightarrow$ D contains a subdivision of a cycle with 4 blocks.

Thank you!