List-coloring in claw-free perfect graphs

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Joint-work with Sylvain Gravier and Frédéric Maffray

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04/11/15

Coloring

Given a graph G, a (proper) k-coloring of the vertices of G is a mapping $c : V(G) \to \{1, 2, \ldots, k\}$ for which every pair of adjacent vertices x, y satisfies $c(x) \neq c(y)$.

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Chromatic number

The chromatic number of G, denoted by $\chi(G)$, is the smallest integer k such that G admits a k-coloring.

List-coloring

- Let G be a graph. Every vertex $v \in V(G)$ has a list $L(v)$ of prescribed colors, we want to find a proper vertex-coloring c such that $c(v) \in L(v)$.

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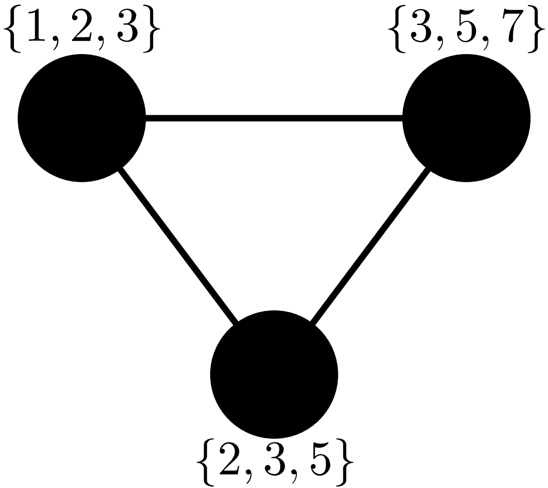
Choice number

The choice number $ch(G)$ of a graph G is the smallest k such that for every list assignment L of size k, the graph G is L-colorable.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$\{3, 5, 7\}$
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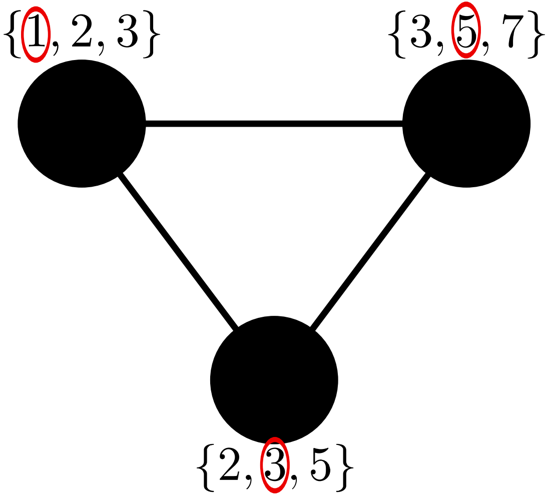
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Chromatic inequality

We have $\chi(G) \leq ch(G)$ for every graph G. There are graphs for which $\chi(G) \neq ch(G)$ (in fact, the gap can be arbitrarily large).

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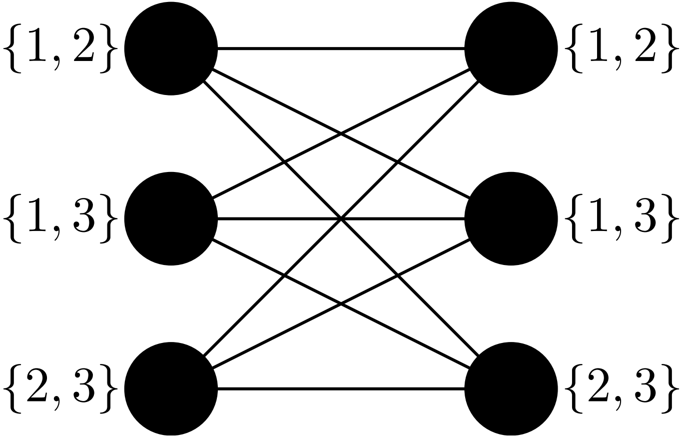

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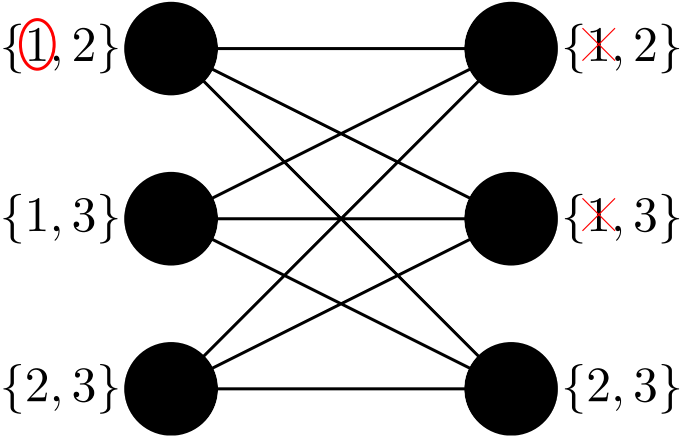

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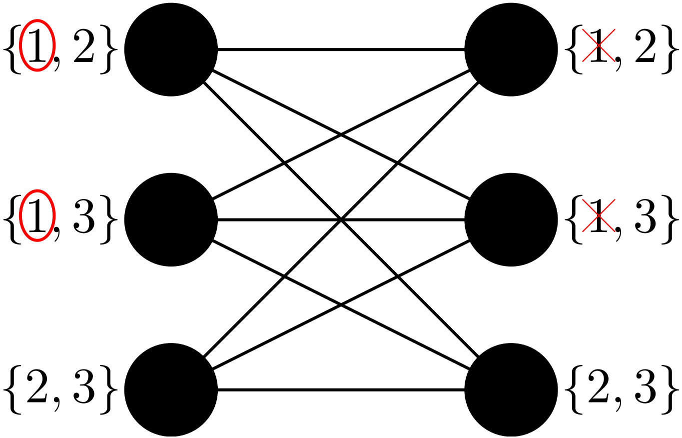

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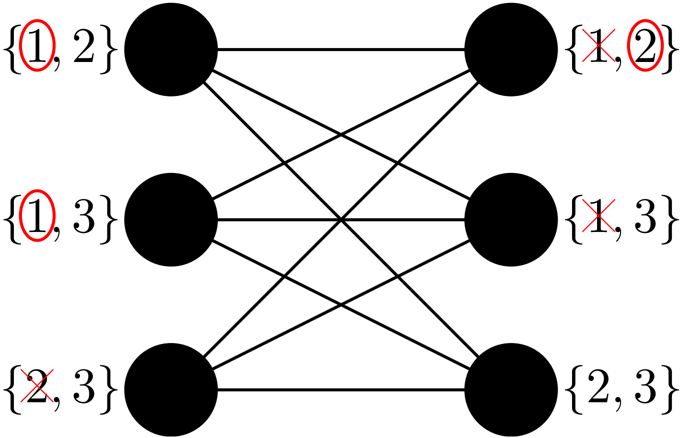

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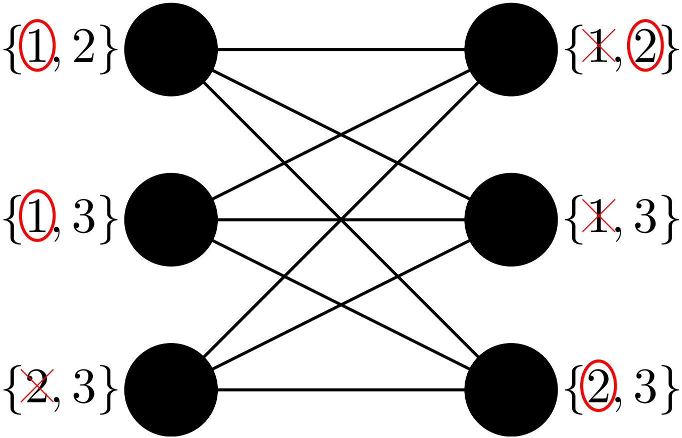

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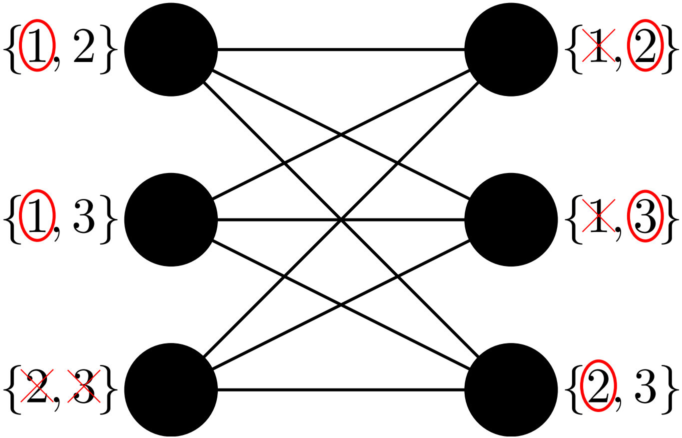

Vizing's conjecture

For every graph G, $\chi(\mathcal{L}(G)) = ch(\mathcal{L}(G))$. In other words, $\chi'(G) = ch'(G)$ with $ch'(G)$ the list chromatic index of G.

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Special case

We are interested in the case where G is perfect.

Theorem [Gravier, Maffray, P.]

Let G be a claw-free perfect graph with $\omega(G) \leq 4$. Then $\chi(G) = ch(G)$.

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A graph G is called perfect if for every induced subgraph H of G, $\chi(H) = \omega(H)$.

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Strong Perfect Graph Theorem

A graph G is perfect if and only if G does not contain an odd hole nor an odd antihole.

Claw-free graph

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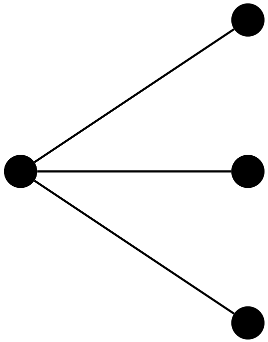
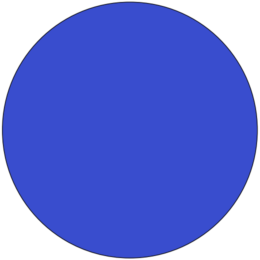
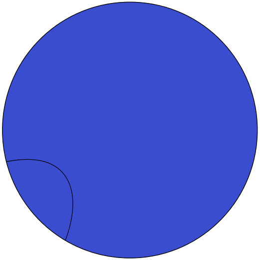
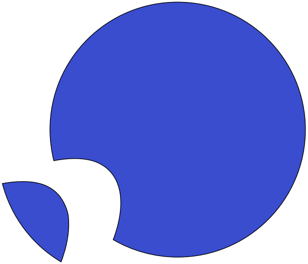


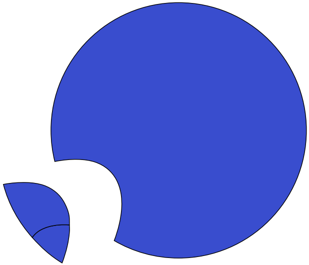
Theorem [Chvátal and Sbihi, 1988]

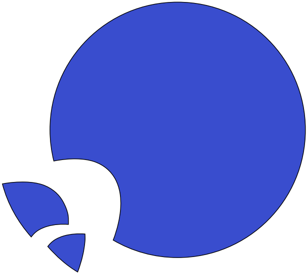
Every claw-free perfect graph either has a clique-cutset, or is a peculiar graph, or is an elementary graph.

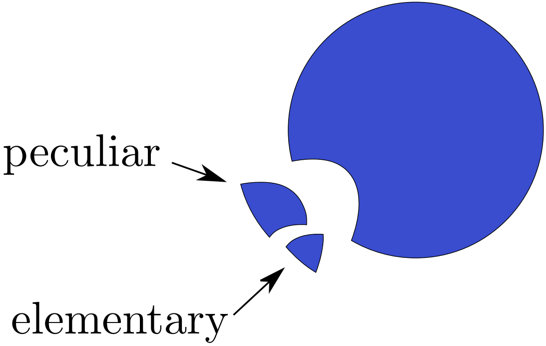


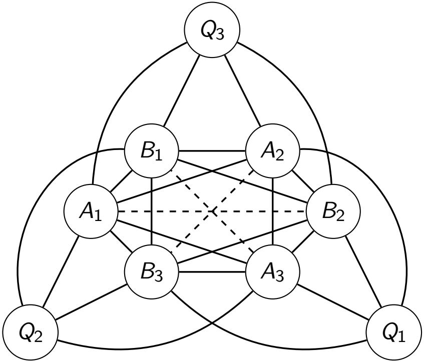









peculiar
elementary



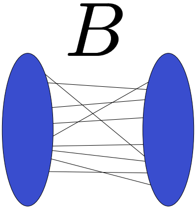
 clique

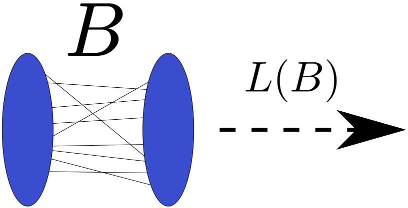
at least one non-edge

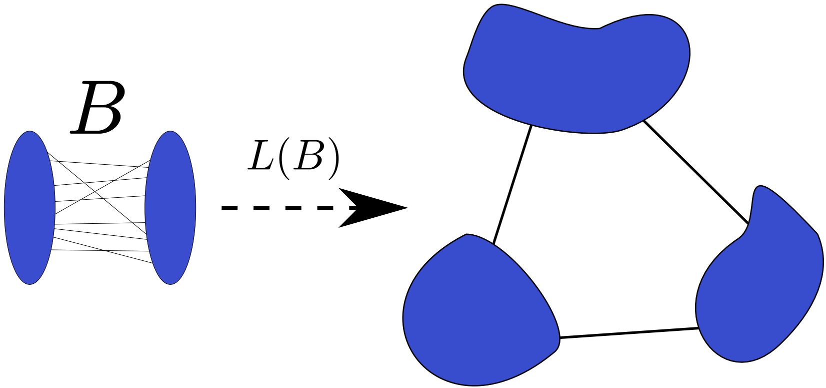
complete adjacency

Theorem [Maffray and Reed, 1999]

A graph G is elementary if and only if it is an augmentation of the line-graph H (called the **skeleton** of G) of a bipartite multigraph B (called the **root** graph of G).


B


B
$L(B)$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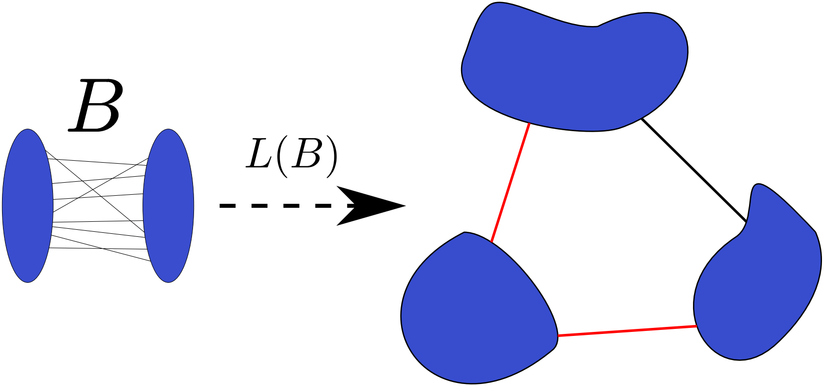
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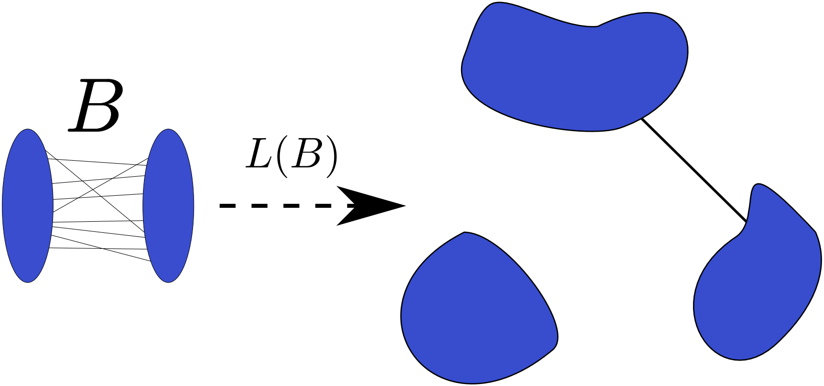
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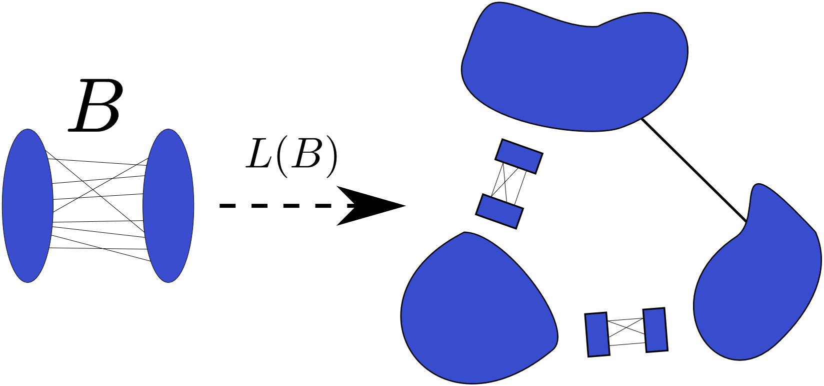
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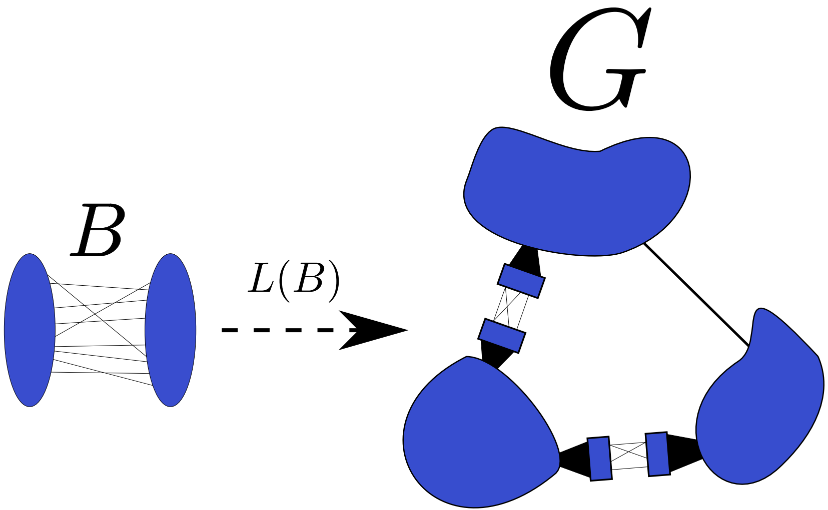
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G

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Lemma

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Let G be a peculiar graph with $\omega(G) \leq 4$ (unique in this case). Then G is 4-choosable.

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Let G be the line-graph of a bipartite multigraph. Then $\chi(G) = ch(G)$.

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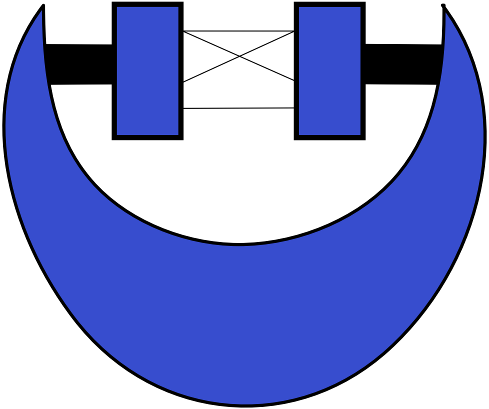
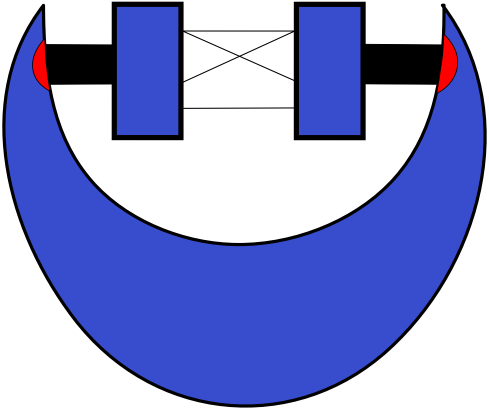
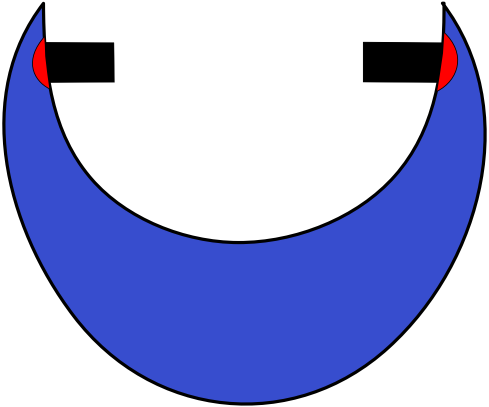
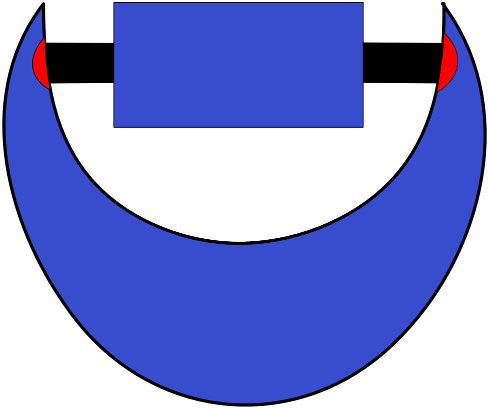
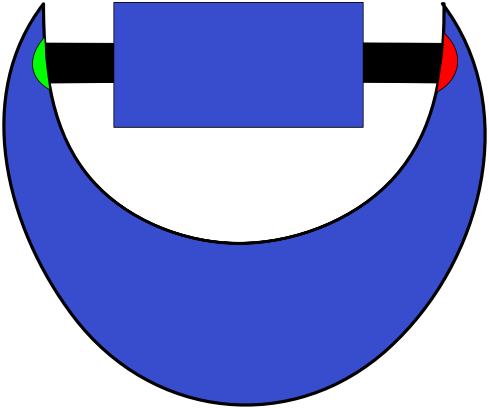
- We show that we can always extend the coloring to the last augmented flat edge

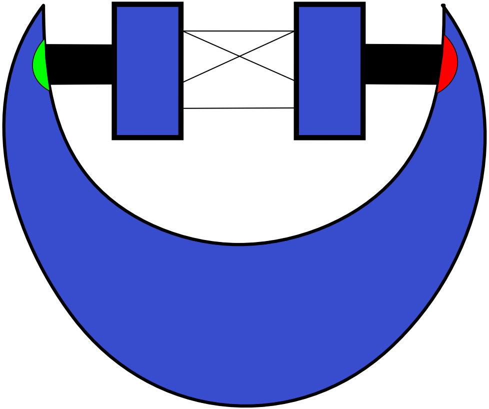












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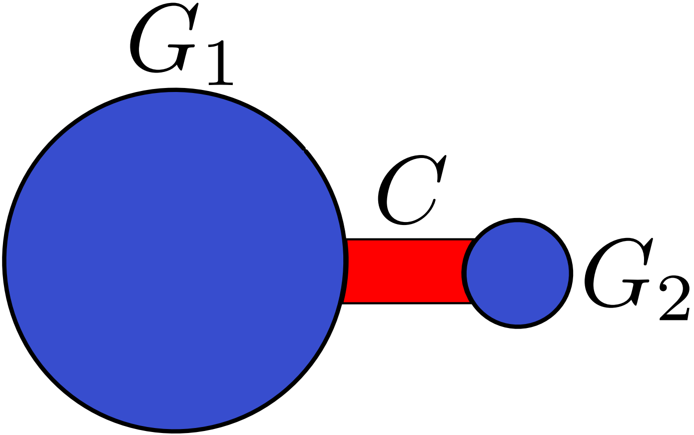
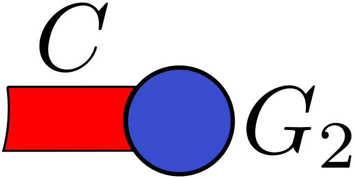
- Thanks to a Galvin's argument, we can show that G_2 is list-colorable with restriction of the list size of C


G_1
C
G_2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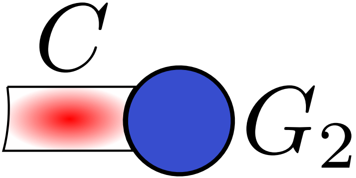
G_2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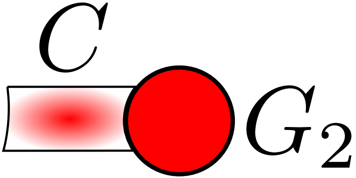
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- It seems to be hard to use this trick for the general case.
- We tried Galvin like arguments without any success.
- What about peculiar graphs?

Thank you for listening.