Gamburgers and Graphs with Small Game Domination Numbers

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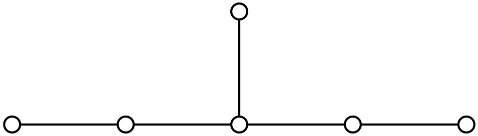
Domination Game: rules set.

- Two players Dominator and Staller.
- Alternately choose a vertex to build a dominating set.
- A least one new vertex is dominated !
- The game ends when the whole graph is dominated.
- Dominator wants to shorten the game.
- Staller wants to prolong it.

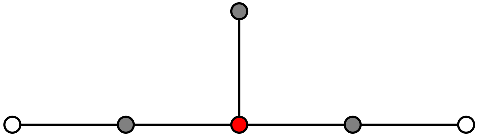
D-game

D-game: if Dominator starts.



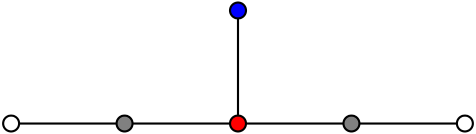
D-game

D-game: if Dominator starts.



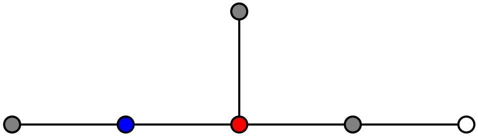
D-game

D-game: if Dominator starts. Forbidden move !



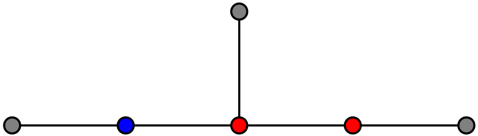
D-game

D-game: if Dominator starts.



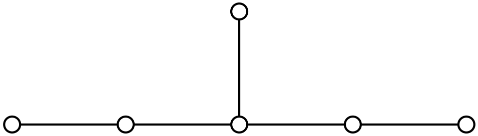
D-game

D-game: if Dominator starts. Dominating set of size 3



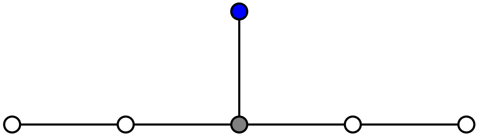
S-game

S-game: if Staller starts.



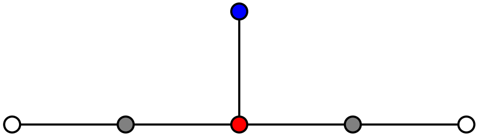
S-game

S-game: if Staller starts.



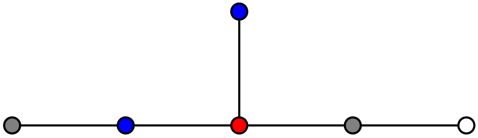
S-game

S-game: if Staller starts.



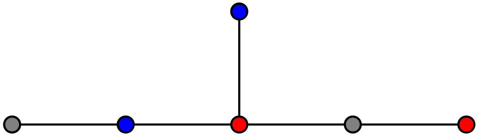
S-game

S-game: if Staller starts.



S-game

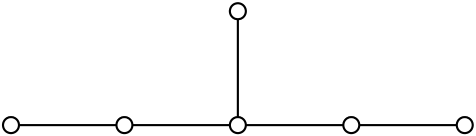
S-game: if Staller starts. Dominating set of size 4



γ_g and γ_g'

Assuming both players **play optimally**:

- $\gamma_g(G)$ size of dominating set for D-game on G
- $\gamma_g'(G)$ size of dominating set for S-game on G



$\gamma_g = 3$

$\gamma_g' = 4$

γ_g and γ_g'

Assuming both players **play optimally**:

- $\gamma_g(G)$ size of dominating set for D-game on G
- $\gamma_g'(G)$ size of dominating set for S-game on G

Theorem Brešar et al. (2010), Kinnersley et al. (2013)

$$|\gamma_g(G) - \gamma_g'(G)| \leq 1$$

Complexity

Theorem Brešar et al. (2014)

Problem: D-Game

Input: A graph G, an integer k

Output: whether $\gamma_g(G) \leq k$

The problem D-game is PSPACE-Complete.

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Remark:

If k is not part of the input: D-game is solvable in time $\mathcal{O}(\Delta.|V(G)|^k)$.

Graphs with $\gamma_g = 1$ or $\gamma'_g = 1$

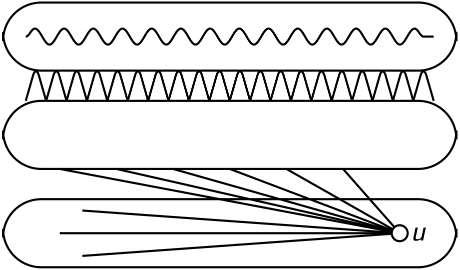
$$\gamma_g(G) = 1 \iff \Delta(G) = n - 1$$

$$\gamma'_g(G) = 1 \iff G \text{ is a clique.}$$

Graphs with $\gamma_g = 2$

$\gamma_g(G) = 2$

$\Longleftrightarrow$



Graphs with $\gamma_g' = 2$

$$\gamma_g'(G) = 2$$

$$\iff$$

Any vertex belongs to a dominating set of size 2 and G is not a clique.

Isometric paths

Proposition

If P is an isometric path of G: $\gamma_g(P) \leq \gamma_g(G)$ and $\gamma'_g(P) \leq \gamma'_g(G)$

Proposition If $n \geq 1$, then

(i) $\gamma_g(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil - 1; & n \equiv 3 \pmod 4, \\ \lceil \frac{n}{2} \rceil; & \text{otherwise.} \end{cases}$

(ii) $\gamma'_g(P_n) = \lceil \frac{n}{2} \rceil$.

Diameter bounds

Proposition

If G is a graph, then

(i) $\mathrm{diam}(G) \leq \begin{cases} 2\gamma_g(G); & \gamma_g(G) \text{ odd}, \\ 2\gamma_g(G) - 1; & \text{otherwise.} \end{cases}$

(ii) $\mathrm{diam}(G) \leq 2\gamma_g'(G) - 1$.

Extremal graphs:

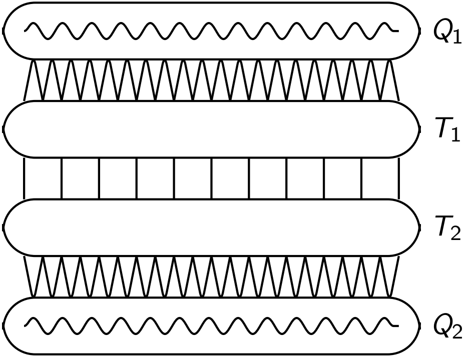
$\gamma_g = 2$	$\mathrm{diam} = 3$	$\gamma_g = 4$	$\mathrm{diam} = 7$
$\gamma_g' = 2$	$\mathrm{diam} = 3$	$\gamma_g' = 4$	$\mathrm{diam} = 7$
$\gamma_g = 3$	$\mathrm{diam} = 6$	...	...
$\gamma_g' = 3$	$\mathrm{diam} = 5$	...	...

Gamburger

Burger condition

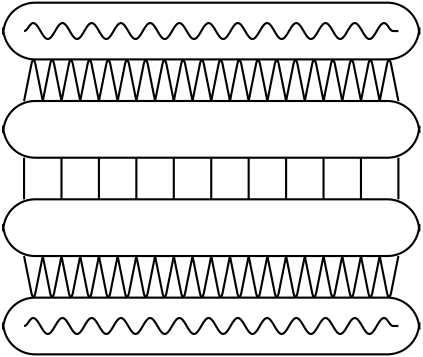
$$\forall u \in T_i,\ \exists v \in T_{3-i} \cup Q_{3-i}$$

$$T_1 \cup T_2 \subset N[u] \cup N[v]$$



Gamburger

A burger has $\mathrm{diam} = 3$

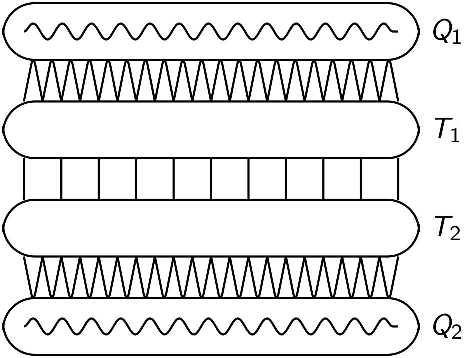


Graphs with $\mathrm{diam} = 3$ and $\gamma'_g = 2$

Theorem *Klavžar, Košmrlj, S. (2015)*

$\mathrm{diam}(G) = 3$
and
$\gamma'_g(G) = 2$

$\iff$



Graphs with $\mathrm{diam} = 3$ and $\gamma_g' = \gamma_g = 2$

Theorem *Klavžar, Košmrlj, S. (2015)*

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and

$\gamma_g'(G) = 2$

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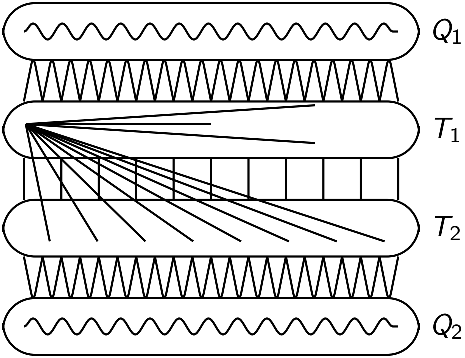
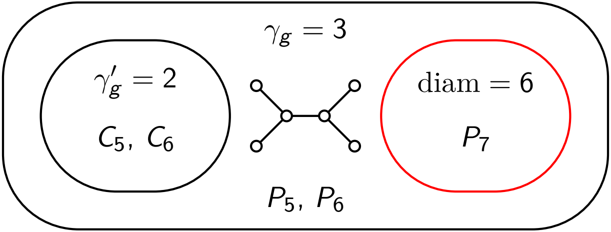


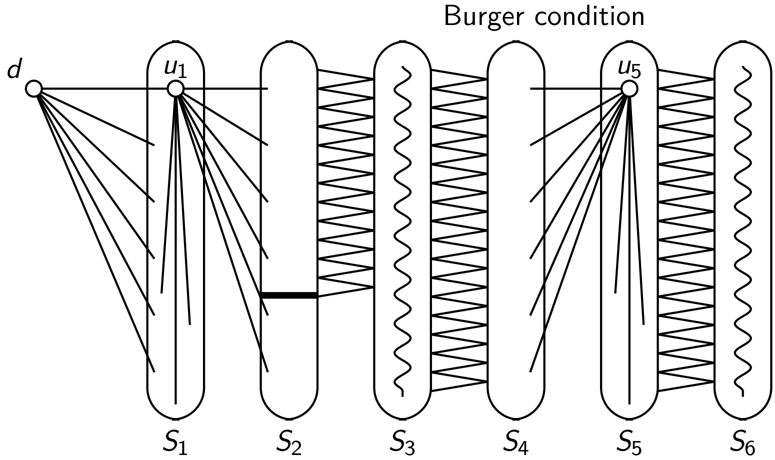
Graphs with $\gamma_g = 3$

- Graphs with $\gamma_g = 3$ and $\gamma_g' = 2$ caracterized by Brešar et al. (2015)
- What about extremal graphs (with respect to diam) ?



Tasty vertex

A vertex d is tasty if:



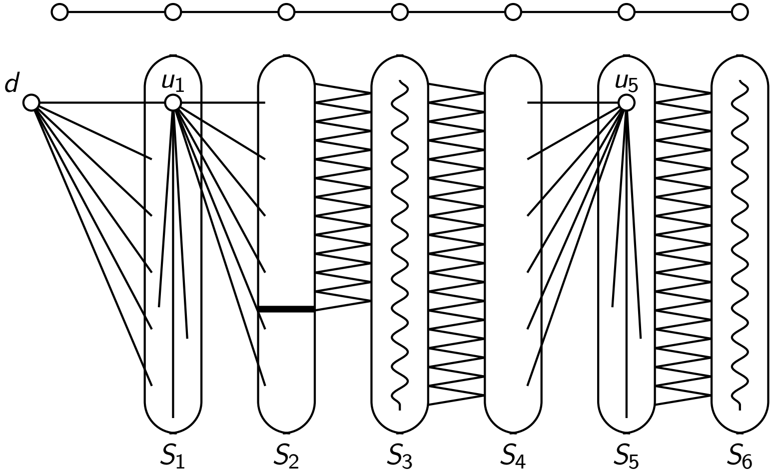
Graphs with $\gamma_g = 3$ and $\mathrm{diam} = 6$

Theorem *Klavžar, Košmrlj, S. (2015)*

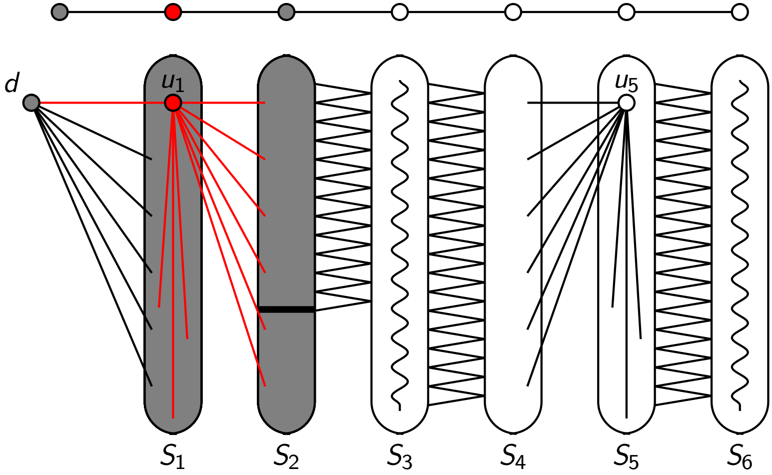
The followings statements are equivalent:

- $\gamma_g(G) = 3$ and $\mathrm{diam}(G) = 6$
- Any diametrical pair of vertices contains at least one **tasty** vertex
- There exists one **tasty** diametrical vertex

$\gamma_g = 3$ & $\mathrm{diam} = 6 \Longrightarrow$ any diam. pair has a **tasty** vertex



$\gamma_g = 3$ & $\mathrm{diam} = 6 \implies$ any diam. pair has a **tasty** vertex



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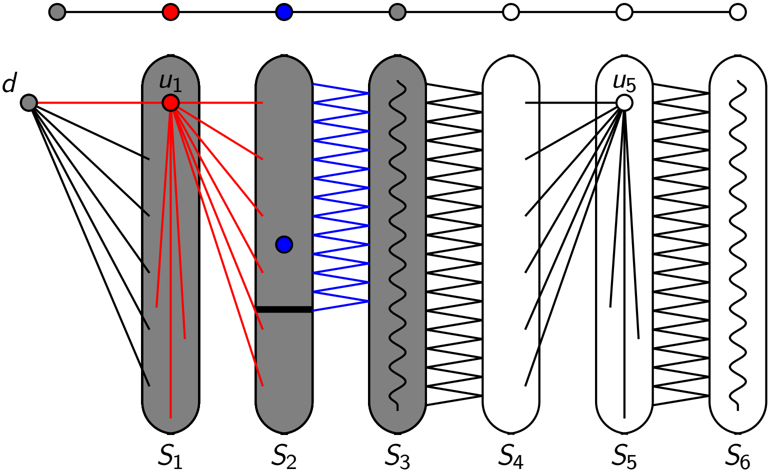

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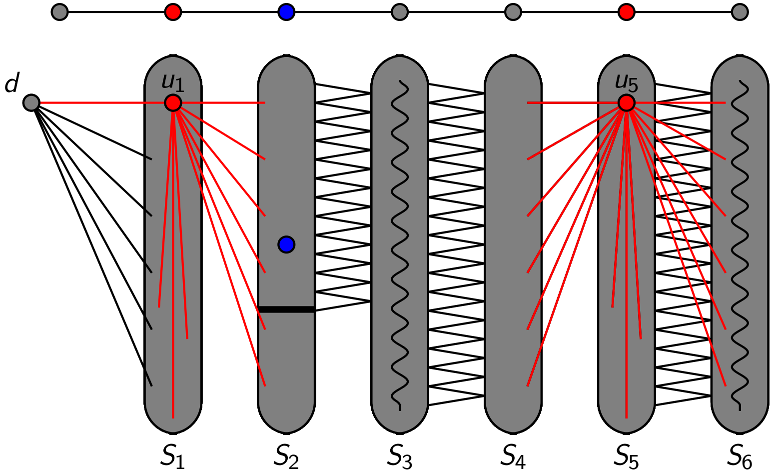
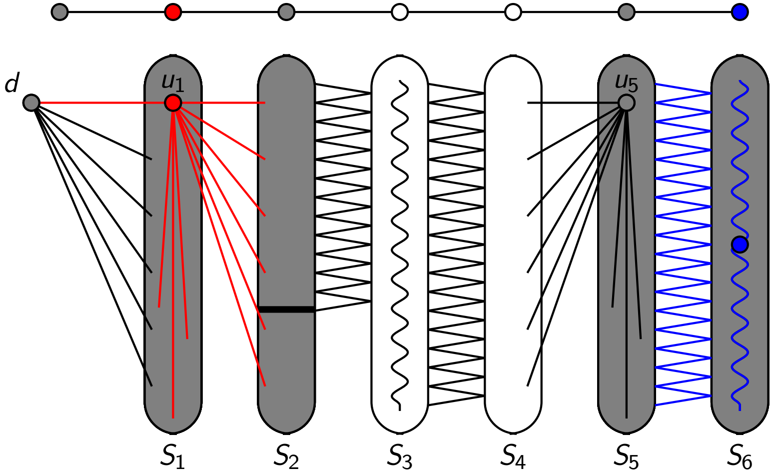


$\gamma_g = 3$ & $\text{diam} = 6 \implies$ any diam. pair has a **tasty** vertex



Recognition algorithm

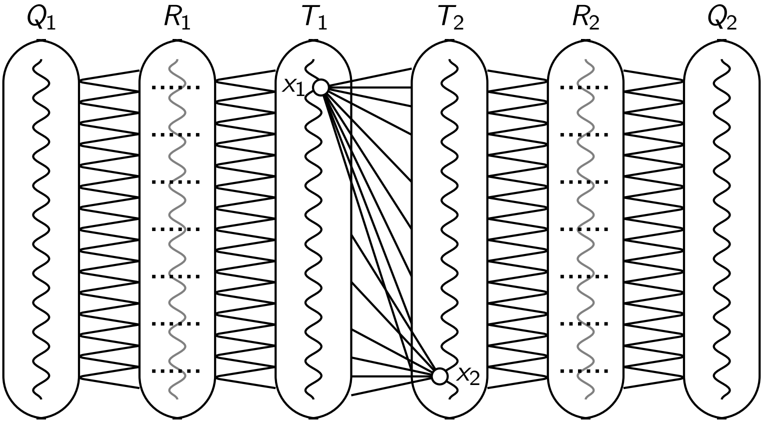
Corollary *Klavžar, Košmrlj, S. (2015)*

Graphs with $\gamma_g = 3$ and $\mathrm{diam} = 6$ can be recognized in time $\mathcal{O}(|E(G)||V(G)| + \Delta^3)$.

(instead of $\mathcal{O}(\Delta|V(G)|^3)$)

Double-gamburger

Matching condition: If M_i perfect, then $M_{3-i} = \emptyset$ and join between T_1, T_2



Graphs with $\gamma_g' = 3$ and $\mathrm{diam} = 5$

Theorem *Klavžar, Košmrlj, S. (2015)*

The followings statements are equivalent:

- $\gamma_g'(G) = 3$ and $\mathrm{diam}(G) = 5$

- The “distance layers” of any diam. vertex induce a double-gamburger

- There exists one diam. vertex whose “distance layers” induce a double-gamburger

Corollaries

- Graphs with $\gamma'_g = 3$ and $\mathrm{diam} = 5$ can be recognized in time $\mathcal{O}(|E(G)||V(G|)$.

- If $\gamma'_g(G) = 3$ and $\mathrm{diam}(G) = 5$, then $\gamma_g(G) = 3$

- There is no $\mathrm{diam} = 5$ graph with $\gamma_g = 4$ and $\gamma'_g = 3$

Further work

Extend the results to **Gambanica** !