Is it hard to color a graph as a 4 years old child?

Édouard Bonnet[1], Florent Foucaud[2], Eunjung Kim[3], and Florian Sikora[3].

[1] Hungarian Academy of Sciences
[2]LIMOS – France
[3]LAMSADE, Université Paris Dauphine, CNRS – France

JGA 2015

Is it hard to color a graph as a 4 years old child?

Édouard Bonnet[1], Florent Foucaud[2], Eunjung Kim[3], and Florian Sikora[3].

[1] Hungarian Academy of Sciences
[2]LIMOS – France
[3]LAMSADE, Université Paris Dauphine, CNRS – France

JGA 2015

(Wrote most of these slides)

Is it hard to color a graph as a 4 years old child?

Édouard Bonnet[1], Florent Foucaud[2], Eunjung Kim[3], and Florian Sikora[3].

[1] Hungarian Academy of Sciences
[2]LIMOS – France
[3]LAMSADE, Université Paris Dauphine, CNRS – France

JGA 2015

Is it hard to color a graph as a 4 years old child?

Édouard Bonnet[1], Florent Foucaud[2], Eunjung Kim[3], and Florian Sikora[3].

[1] Hungarian Academy of Sciences
[2]LIMOS – France
[3]LAMSADE, Université Paris Dauphine, CNRS – France

JGA 2015

Outline

Outline

Warm Up

Exact algorithms

Weak Grundy Coloring

Grundy Colorings – Definition(s)

The worst way of reasonably coloring a graph.

Grundy Colorings – Definition(s)

The worst way of reasonably coloring a graph.

- Order the vertices $v_1, v_2 \ldots v_n$ to *maximize* the number of colors used by the greedy coloring: the Grundy Number (GN).
- That is, v_i is colored with $c(v_i)$ the first color that is not in its neighborhood (first-fit).

Grundy Colorings – Definition(s)

The worst way of reasonably coloring a graph.

- Order the vertices $v_1, v_2 \ldots v_n$ to *maximize* the number of colors used by the greedy coloring: the Grundy Number (GN).
- That is, v_i is colored with $c(v_i)$ the first color that is not in its neighborhood (first-fit).

Grundy Colorings – Definition(s)

The worst way of reasonably coloring a graph.

- Order the vertices $v_1, v_2 \ldots v_n$ to *maximize* the number of colors used by the greedy coloring: the Grundy Number (GN).
- That is, v_i is colored with $c(v_i)$ the first color that is not in its neighborhood (first-fit).
- **Connected version**: $\forall i, G[v_1 \cup \ldots \cup v_i]$ is connected.

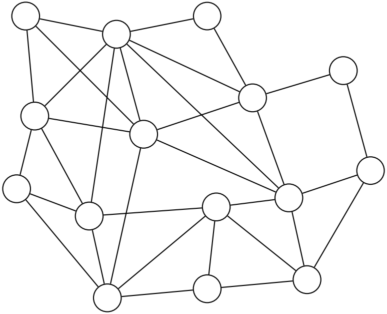
Grundy Colorings – Definition(s)

The worst way of reasonably coloring a graph.

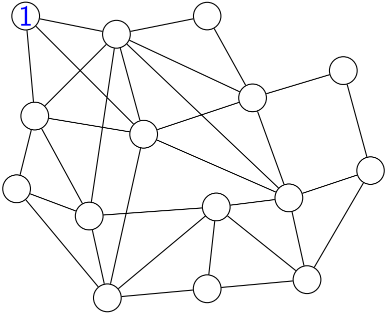
- Order the vertices $v_1, v_2 \dots v_n$ to *maximize* the number of colors used by the greedy coloring: the Grundy Number (GN).
- That is, v_i is colored with $c(v_i)$ the first color that is not in its neighborhood (first-fit).
- **Connected version**: $\forall i, G[v_1 \cup \dots \cup v_i]$ is connected.
- **Weak version**: v_i can be colored with any color in $\{1, \dots, c(v_i)\}$.

Algorithmic motivations

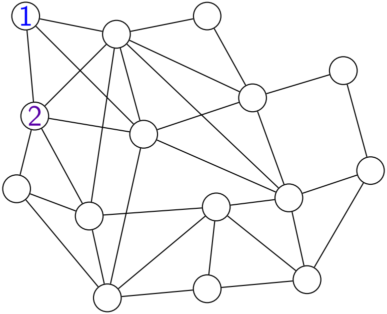
- $GN(G)$ upper bounds the number of colors used by any greedy heuristic for MIN COLORING.
- $GN(G) \leqslant C \cdot \chi(G)$ on some classes of graphs gives a C-approximation for MIN COLORING.
- See Sampaio's PhD thesis for further motivations.



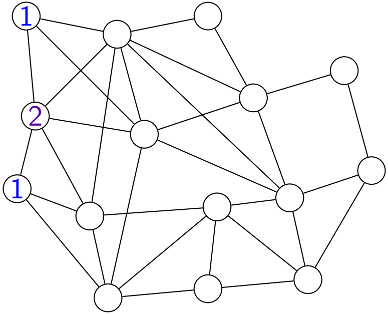
Can you achieve color 6?



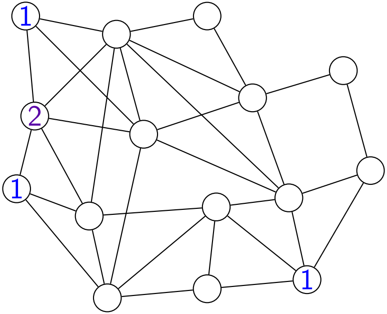
Can you achieve color 6?



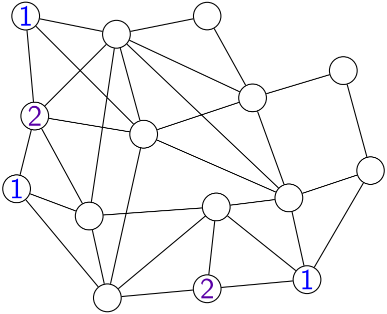
Can you achieve color 6?



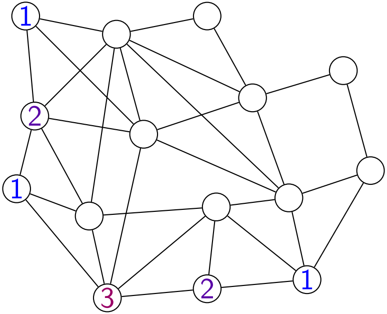
Can you achieve color 6?



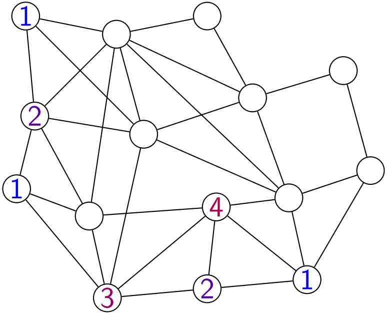
Can you achieve color 6?



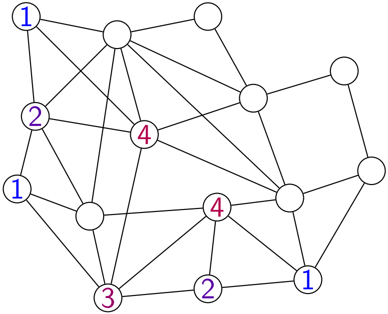
Can you achieve color 6?



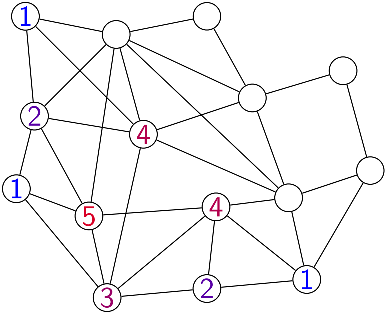
Can you achieve color 6?



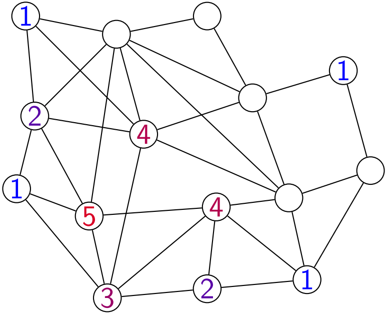
Can you achieve color 6?



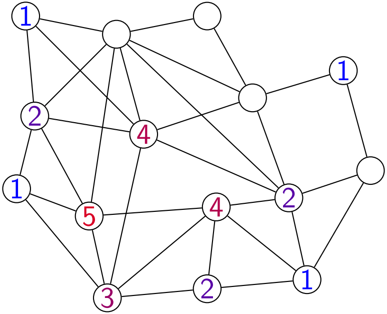
Can you achieve color 6?



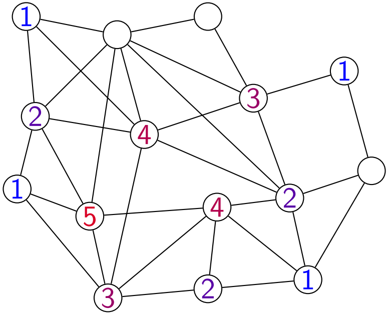
Can you achieve color 6?



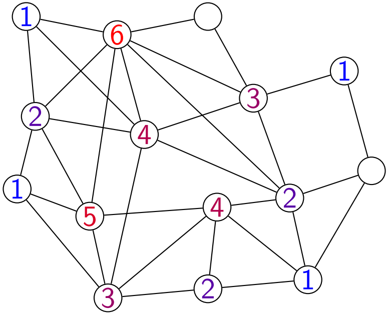
Can you achieve color 6?



Can you achieve color 6?

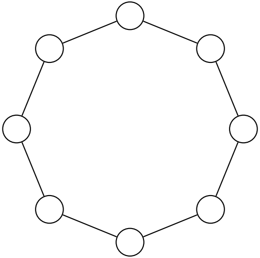


Can you achieve color 6?



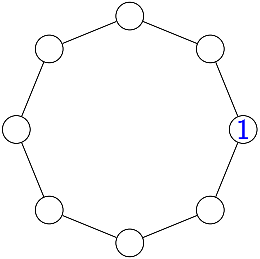
Can you achieve color 6?

Cycles



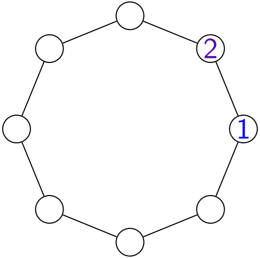
Grundy number =?

Cycles



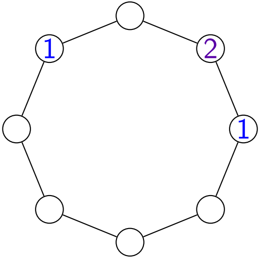
Grundy number =?

Cycles



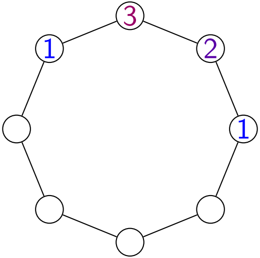
Grundy number =?

Cycles



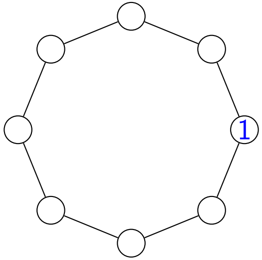
Grundy number =?

Cycles



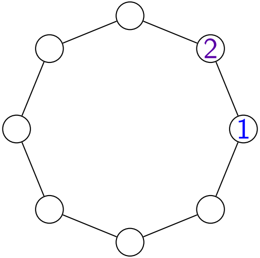
Grundy number = 3

(even) Cycles



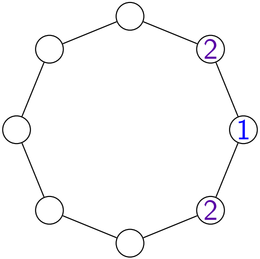
Connected Grundy number =?

(even) Cycles



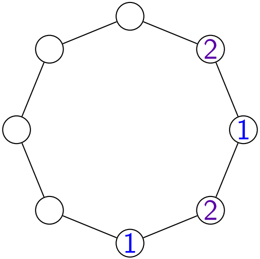
Connected Grundy number =?

(even) Cycles



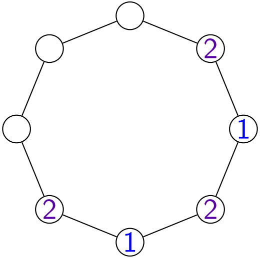
Connected Grundy number =?

(even) Cycles



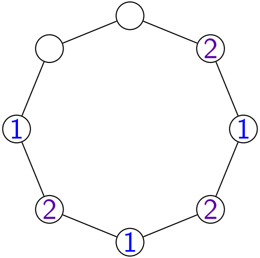
Connected Grundy number =?

(even) Cycles



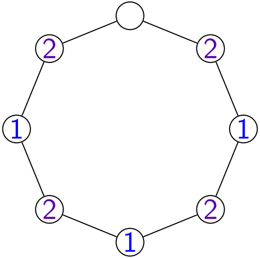
Connected Grundy number =?

(even) Cycles



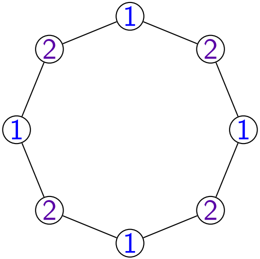
Connected Grundy number =?

(even) Cycles



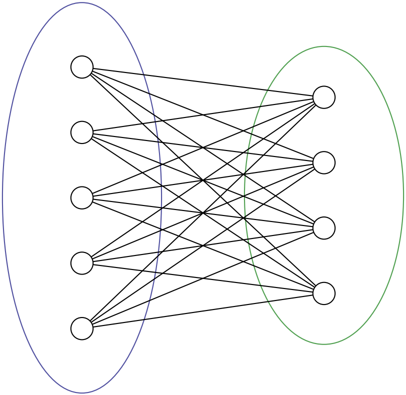
Connected Grundy number =?

(even) Cycles



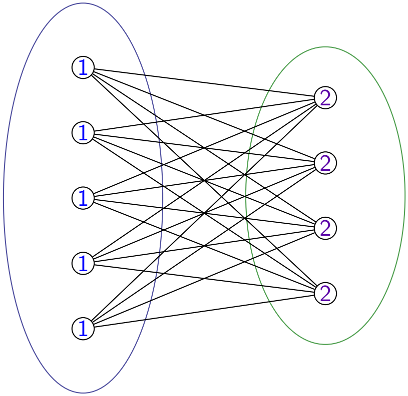
Connected Grundy number = 2

$K_{n,m}$



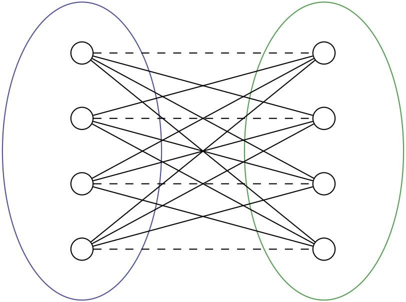
Grundy number =?

$K_{n,m}$



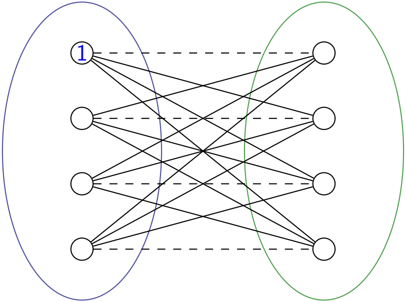
Grundy number = 2

$K_{p,p}$ minus a perfect matching



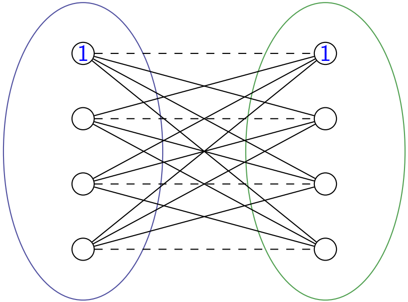
Grundy number =?

$K_{p,p}$ minus a perfect matching



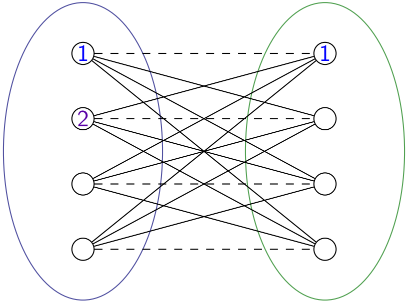
Grundy number =?

$K_{p,p}$ minus a perfect matching



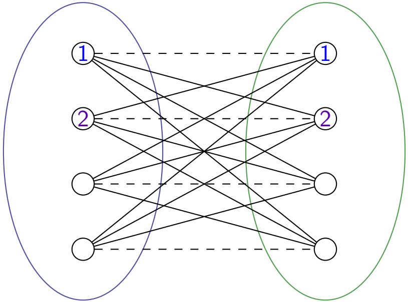
Grundy number =?

$K_{p,p}$ minus a perfect matching



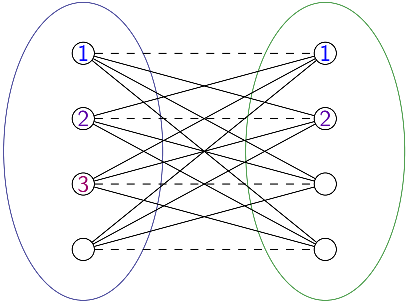
Grundy number =?

$K_{p,p}$ minus a perfect matching



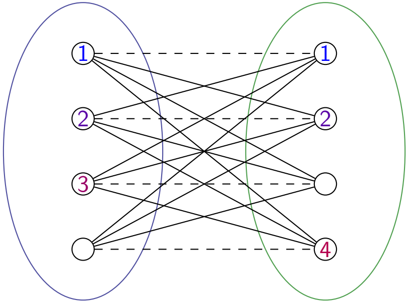
Grundy number =?

$K_{p,p}$ minus a perfect matching



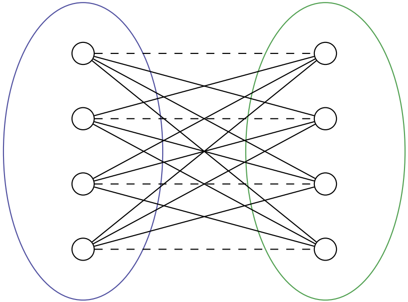
Grundy number =?

$K_{p,p}$ minus a perfect matching



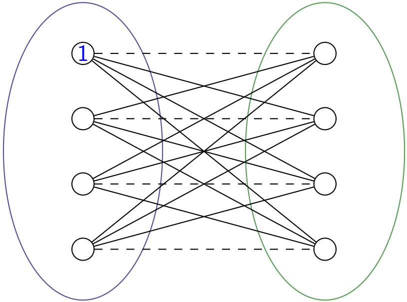
Grundy number $= p$

$K_{p,p}$ minus a perfect matching



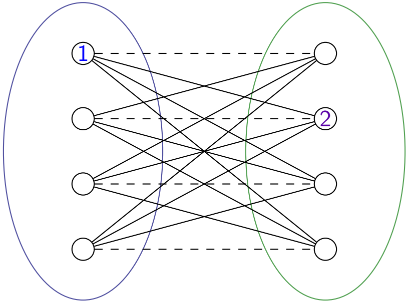
Connected Grundy number =?

$K_{p,p}$ minus a perfect matching



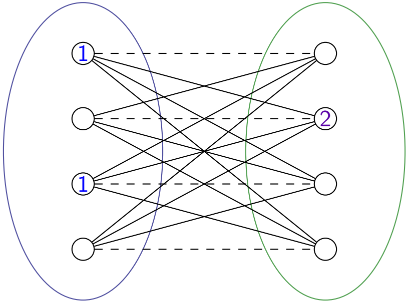
Connected Grundy number =?

$K_{p,p}$ minus a perfect matching



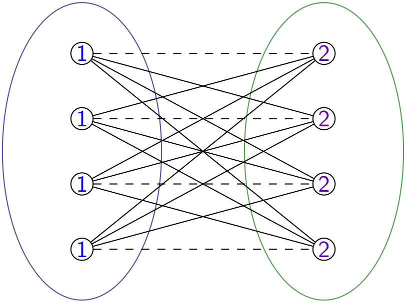
Connected Grundy number =?

$K_{p,p}$ minus a perfect matching



Connected Grundy number =?

$K_{p,p}$ minus a perfect matching



Connected Grundy number = 2

(minimal) Witnesses

How many vertices (at least) did we need to achieve color k?
(easy)

(minimal) Witnesses

How many vertices (at least) did we need to achieve color k?
(easy)

k (Clique of size k).

(minimal) Witnesses

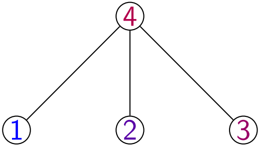
How many vertices (at most) did we need to achieve color k?

(minimal) Witnesses

How many vertices (at most) did we need to achieve color k?

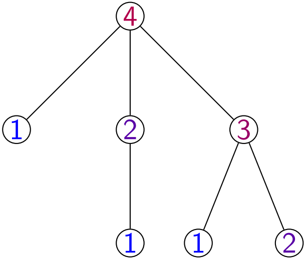
(minimal) Witnesses

How many vertices (at most) did we need to achieve color k?



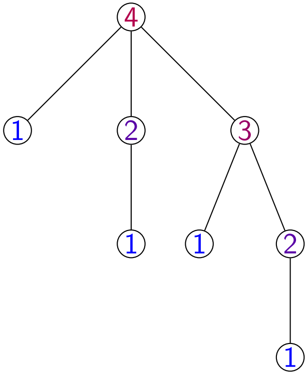
(minimal) Witnesses

How many vertices (at most) did we need to achieve color k?



(minimal) Witnesses

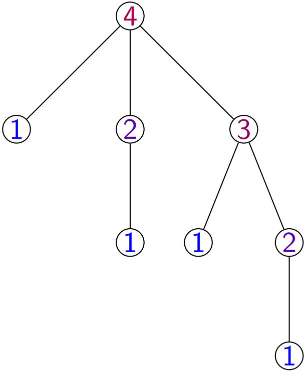
How many vertices (at most) did we need to achieve color k?



Binomial tree T_4.

(minimal) Witnesses

How many vertices (at most) did we need to achieve color k?



Binomial tree T_4.

- $|T_k| = \Sigma_{1 \leqslant i \leqslant k-1} |T_i|$, $|T_1| = 1$.
- So $|T_k| = 2^{k-1}$

(minimal) Witnesses – Consequences

- Algorithm:
  - For every subset of 2^{k-1} vertices, check if there is a witness.

Theorem (Zaker '05)

The Grundy number can be computed in $O(f(k)n^{2^{k-1}})$.

XP algorithm: $O(f(k)n^{g(k)})$: polynomial for fixed values of k.

(minimal) Witnesses – Consequences

- Algorithm:
  - For every subset of 2^{k-1} vertices, check if there is a witness.

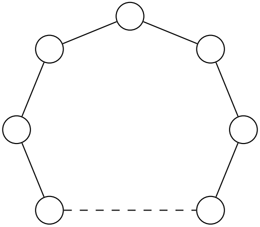
Theorem (Zaker '05)

The Grundy number can be computed in $O(f(k)n^{2^{k-1}})$.

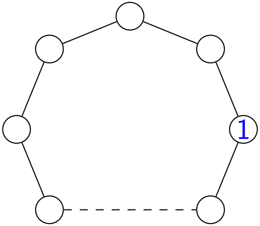
XP algorithm: $O(f(k)n^{g(k)})$: polynomial for fixed values of k.

Can we do the same for the connected Grundy number?

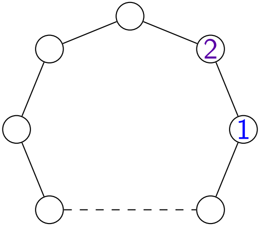
Witness for the connected version



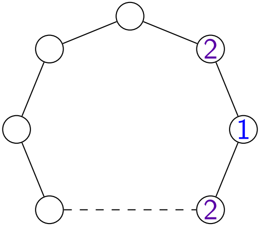
Witness for the connected version



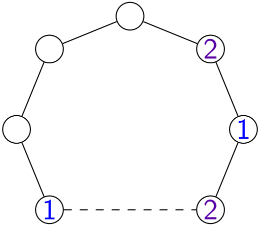
Witness for the connected version



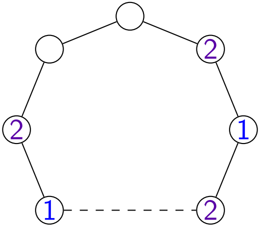
Witness for the connected version



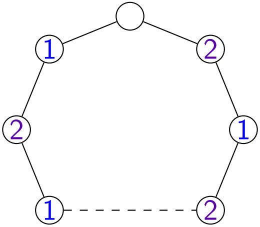
Witness for the connected version



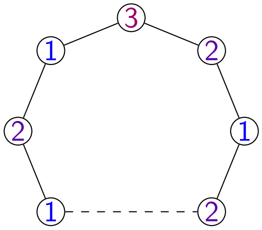
Witness for the connected version



Witness for the connected version

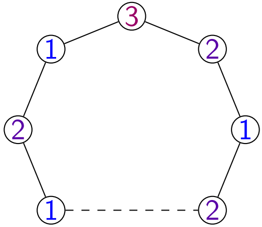


Witness for the connected version



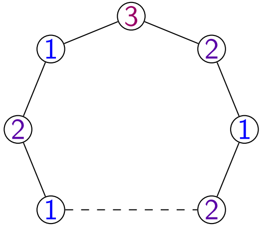
Connected Grundy number = 3 but unbounded witness.

Witness for the connected version



Connected Grundy number = 3 but unbounded witness.
Can't do the previous trick!

Witness for the connected version

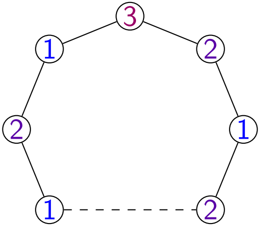


Connected Grundy number = 3 but unbounded witness.
Can't do the previous trick!

Theorem

CONNECTED GRUNDY COLORING *is* NP*-complete*

Witness for the connected version



Connected Grundy number = 3 but unbounded witness.
Can't do the previous trick!

Theorem

CONNECTED GRUNDY COLORING *is* NP*-complete even for* $k = 7$.

Outline

Warm Up

Exact algorithms

Weak Grundy Coloring

Solving Grundy Coloring

Try all possible ordering of the vertices and check if at least k colors are used by the greedy coloring: $\Theta(n!)$-time.

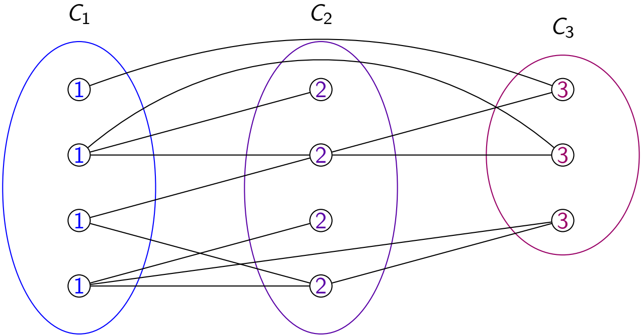
Solving Grundy Coloring

Try all possible ordering of the vertices and check if at least k colors are used by the greedy coloring: $\Theta(n!)$-time.

We can have a $O(c^n)$ algorithm.

Solving Grundy Coloring

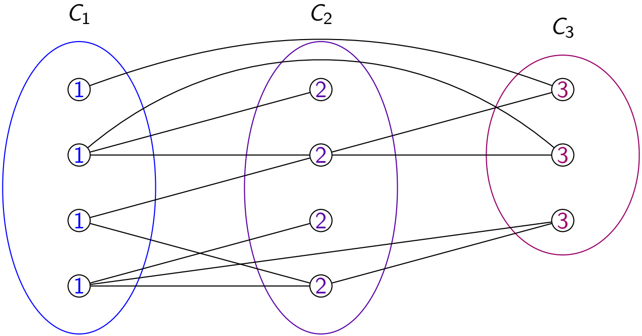
In a witness:



- Any color class is an independent dominating set in the graph induced by the next classes.

Solving Grundy Coloring

In a witness:



- Any color class is an independent dominating set in the graph induced by the next classes.
- $GN(S) = \max\{GN(S \setminus X), X \text{ ind. dom. set of } G[S]\} + 1$.

Solving Grundy Coloring

- A minimal independent dominating set is a maximal independent set.
- One can enumerate all maximal independent sets in $O(1.45^n)$ time.
  - Filling a cell in the table takes $O(1.45^i)$ time for a subset of size i.

Solving Grundy Coloring

- A minimal independent dominating set is a maximal independent set.
- One can enumerate all maximal independent sets in $O(1.45^n)$ time.
  - Filling a cell in the table takes $O(1.45^i)$ time for a subset of size i.

Theorem

One can solve grundy coloring in $\sum_{i=0}^{n} \binom{n}{i} 1.45^i = (1+1.45)^n$.

Solving Grundy Coloring

- A minimal independent dominating set is a maximal independent set.
- One can enumerate all maximal independent sets in $O(1.45^n)$ time.
  - Filling a cell in the table takes $O(1.45^i)$ time for a subset of size i.

Theorem
One can solve grundy coloring in $\sum_{i=0}^{n} \binom{n}{i} 1.45^i = (1+1.45)^n$.

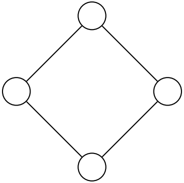
Cannot replace the n by the treewidth (even feedback vertex set).

Theorem
Under the ETH, Grundy Coloring cannot be solved in $O^*(c^{tw})$ *for any constant* c *(even* $O^*(2^{o(tw \log tw)})$*).*

Outline

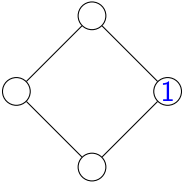
Are there graphs where weak Grundy exceeds Grundy number?

Are there graphs where weak Grundy exceeds Grundy number?



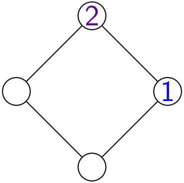
Weak Grundy number $= 3$, (connected) Grundy number $= 2$.

Are there graphs where weak Grundy exceeds Grundy number?



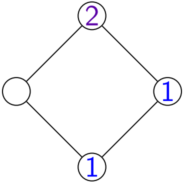
Weak Grundy number = 3, (connected) Grundy number = 2.

Are there graphs where weak Grundy exceeds Grundy number?



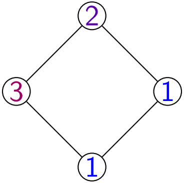
Weak Grundy number = 3, (connected) Grundy number = 2.

Are there graphs where weak Grundy exceeds Grundy number?



Weak Grundy number = 3, (connected) Grundy number = 2.

Are there graphs where weak Grundy exceeds Grundy number?



Weak Grundy number = 3, (connected) Grundy number = 2.

Color Coding

- Add colors between 1 and k uniformly at random to the instance.

Color Coding

- Add colors between 1 and k uniformly at random to the instance.
- The probability that a good solution was well colored is a function of k.

Color Coding

- Add colors between 1 and k uniformly at random to the instance.
- The probability that a good solution was well colored is a function of k.
- Solving the instance is easier with this extra information.

Color Coding

- Add colors between 1 and k uniformly at random to the instance.
- The probability that a good solution was well colored is a function of k.
- Solving the instance is easier with this extra information.
- Do a function of k tries to have a small probability of failure.

Theorem

WEAK GRUNDY COLORING *is in* FPT.

FPT algorithm: $O(f(k)n^c)$.

Theorem

WEAK GRUNDY COLORING *is in* FPT.

FPT algorithm: $O(f(k)n^c)$.

- Idea:
  - A witness is of size at most 2^{k-1}.
  - This witness is well colored with probability is $\frac{1}{k^{2^{k-1}}}$.

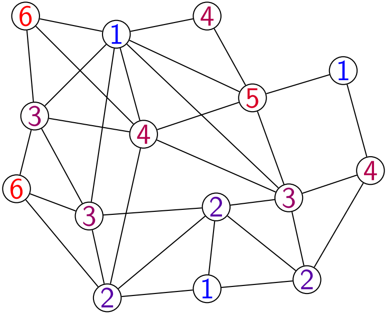
Theorem

WEAK GRUNDY COLORING *is in* FPT.

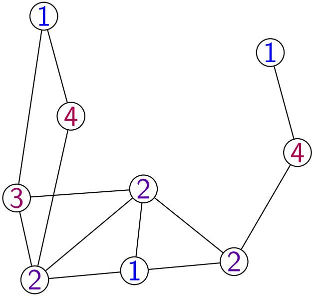
FPT algorithm: $O(f(k)n^c)$.

- Idea:
  - A witness is of size at most 2^{k-1}.
  - This witness is well colored with probability is $\frac{1}{k^{2^{k-1}}}$.
  - After $\log(\frac{1}{\varepsilon})k^{2^{k-1}}$ tries, the probability of success is at least $1-\varepsilon, \forall \varepsilon > 0$.

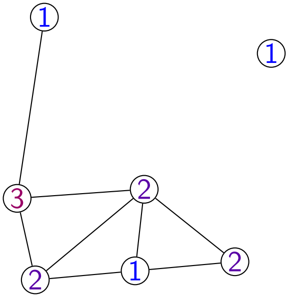
Guess #1



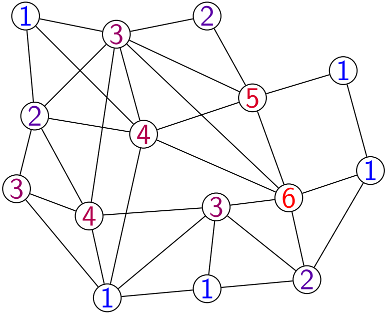
Guess #1



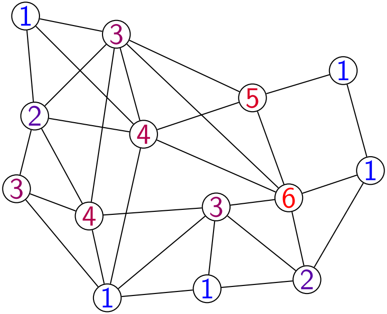
Guess #1



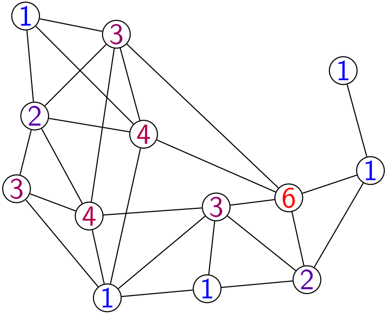
Guess #2



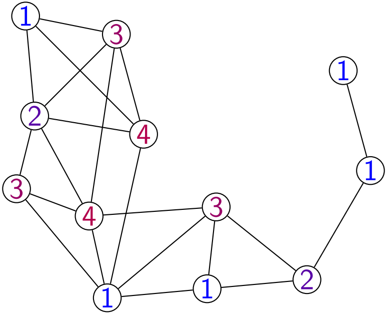
Guess #2



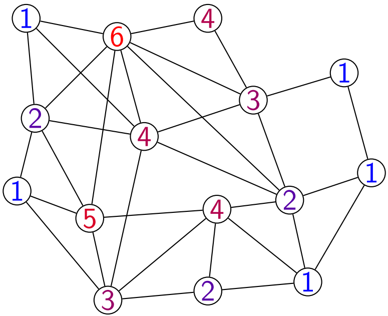
Guess #2



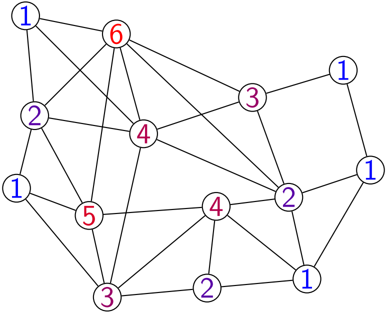
Guess #2



... $O(k^{2^k})$ unsuccessful guesses later ...



... $O(k^{2^k})$ unsuccessful guesses later ...



Open Problems

- Is GRUNDY COLORING solvable in $O(2^n)$?
- Is GRUNDY COLORING solvable in $O(f(tw)n^c)$?

Open Problems

- Is GRUNDY COLORING solvable in $O(2^n)$?
- Is GRUNDY COLORING solvable in $O(f(tw)n^c)$?
- Is GRUNDY COLORING solvable in $O(f(k)n^c)$? Even for bipartite graph?
  - True for chordal, claw-free and bounded degree graphs or graph excluding a fixed graph as a minor.
  - ETH fails if GRUNDY COLORING is solvable in $O^*(2^{2^{o(k)}}2^{o(n+m)})$.

Merci !