Linearity versus contiguity for encoding graphs

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ENS de Lyon

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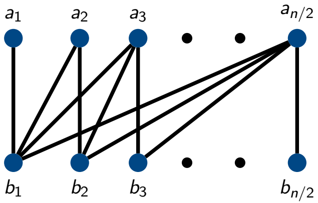
Encoding graphs



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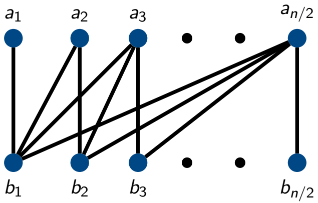

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Traditional scheme: **adjacency-lists**.

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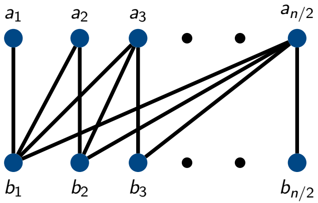

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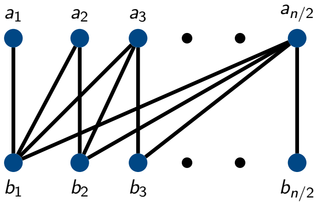

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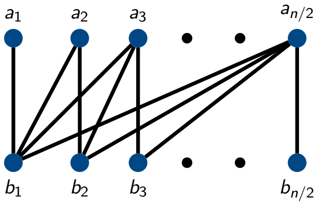

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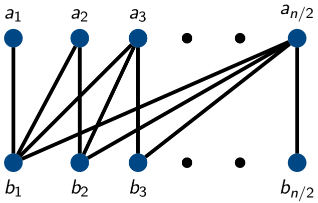

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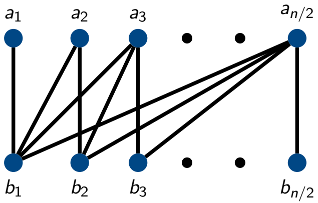

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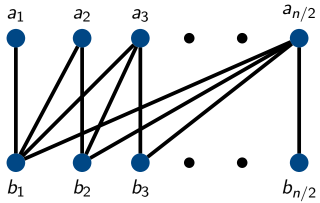

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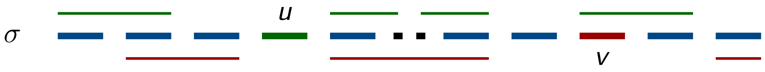

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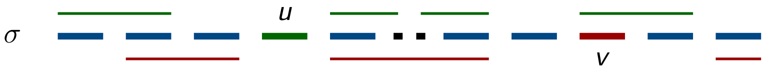

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- Complexity $\leq n + 2k_\sigma n$, where $k_\sigma = \max_u(\#\text{ intervals of } u \text{ in } \sigma)$.

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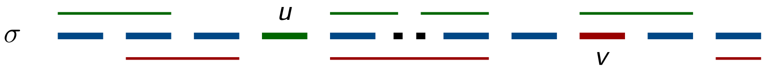

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- Complexity $\leq n + 2k_\sigma n$, where $k_\sigma = \max_u(\#\text{ intervals of } u \text{ in } \sigma)$.
- The smaller k_σ, the better encoding.

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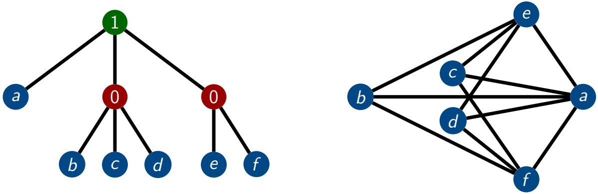
Question: Which graphs have small contiguity?

Contiguity of cographs

Theorem (Crespelle, Gambette' 2014)

- Contiguity of any **cograph** on n vertices is $O(\log n)$.



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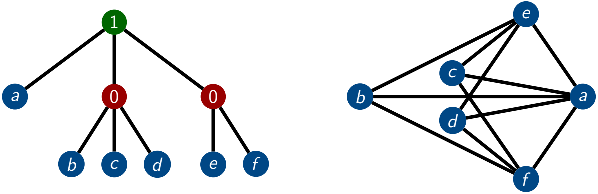
- Contiguity of any cograph corrseponding to some **complete binary cotree** is $\Theta(\log n)$.



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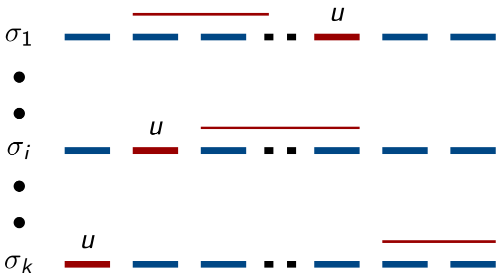
Alternative adjacency-intervals scheme

1. Store a **collection of permutations** $\Sigma = \{\sigma_1, \sigma_2, ..., \sigma_k\}$.
2. For each $u \in V$ and $\sigma_i \in \Sigma$, store **one** neighbor-interval of u per σ_i.



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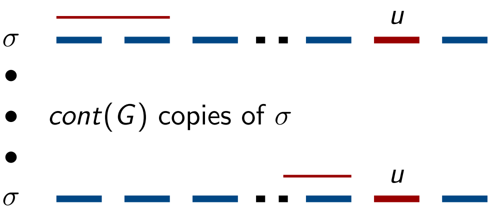

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Direct corollary

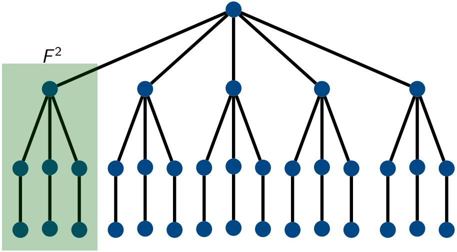
For any cograph G on n vertices corresponding to some complete binary cotree, $lin(G) = O\left(\frac{cont(G)}{\log \log n}\right) = o(cont(G))$.

Sketch of proof

Definition: Double factorial tree

The **double factorial tree** F^k is defined by induction:

- F^0 is a singleton.



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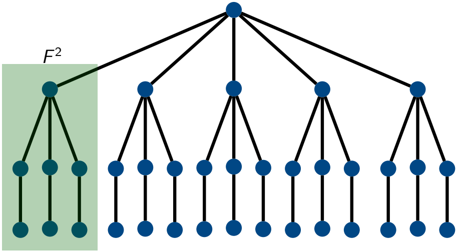

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Let T be a rooted tree.

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Definition: Critical node

A node u in T is **critical** if its rank is strictly greater than the rank of all its children.

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Combine with (i) in key lemma, $lin(G) = O\left(\frac{\log n}{\log\log n}\right)$. $\square$

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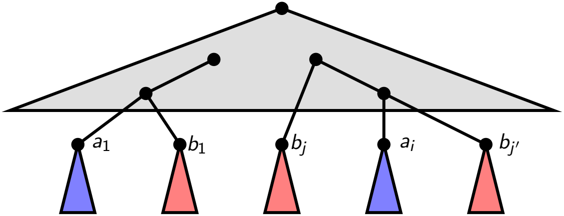

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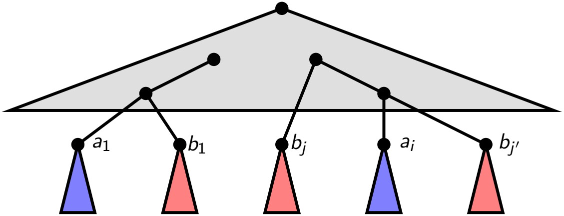

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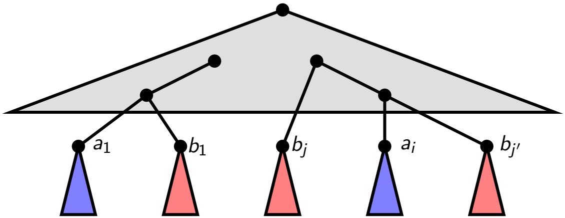

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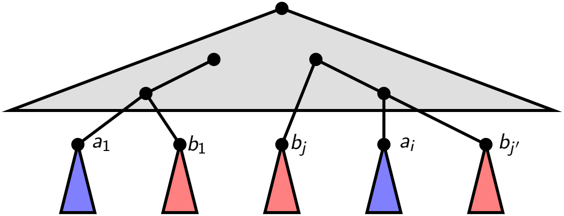
- $A = \{a_1, a_2, \ldots\}$: critical nodes of rank k (blue).
- $B = \{b_1, b_2, \ldots\}$: nodes of rank $k-1$, whose parent is non-critical of rank k (red).



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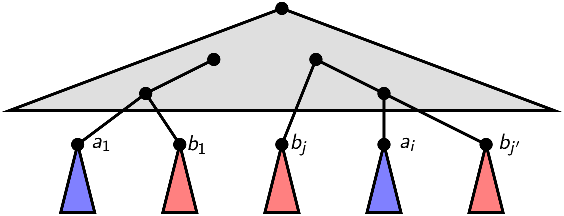

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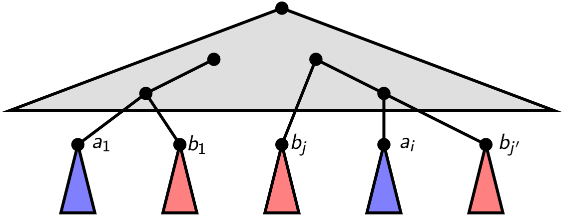

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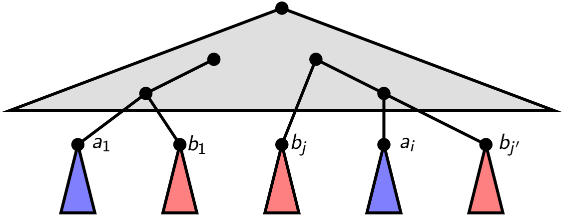

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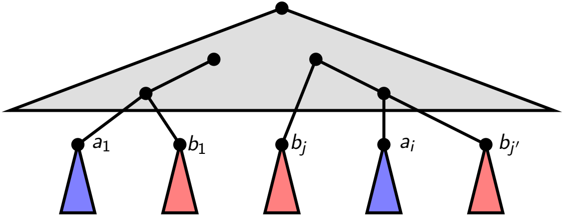

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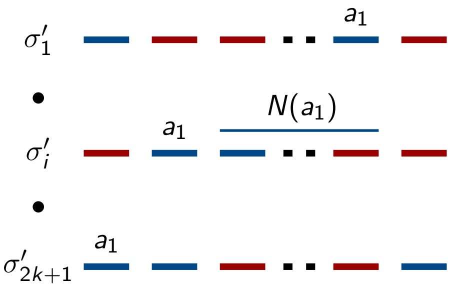
Claim

There exists an encoding of G' by $\Sigma' = \{\sigma'_1, \ldots, \sigma'_{2k+1}\}$ such that:

- Neighbor set of each a_i is encoded by only **one interval**.
- Neighbor set of each b_j is encoded by at most **two intervals** in two distinct permutations.



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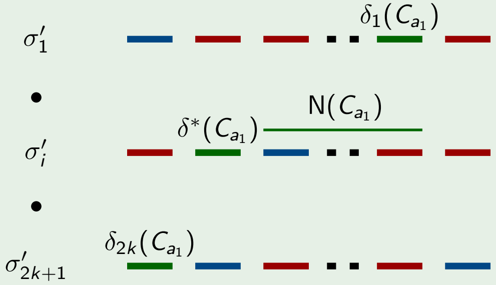
Replace a_1 by $\delta_1(C_{a_1}), \ldots, \delta_{2k}(C_{a_1})$:
(Note that all vertices in C_{a_1} have the same neighbors outside C_{a_1}).



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The lemma is proved. $\square$

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Encode G:

- Find a graph G' where $cont(G')$ is small, and $|V \Delta V'|, |E \Delta E'|$ are small.
- Encode G' by adjacency-intervals (long-term storage)

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Add/remove vertices/edges:

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- When temporary storage is full, re-encode G.

Perspective

Definition:

Let f be a function of n. A graph G is **nearly f-contiguous** if there exists a graph G' such that

- $|V \Delta V' \bigcup E \Delta E'| = O(fn)$,
- $cont(G') = O(f)$.

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Question: Which graphs are nearly $\log n$-contiguous?

The end

Thank you.