Random Tilings

Alexandra Ugolnikova

Laboratoire d'Informatique de Paris Nord

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1. Introduction

2. Tilings and Tiling Graphs

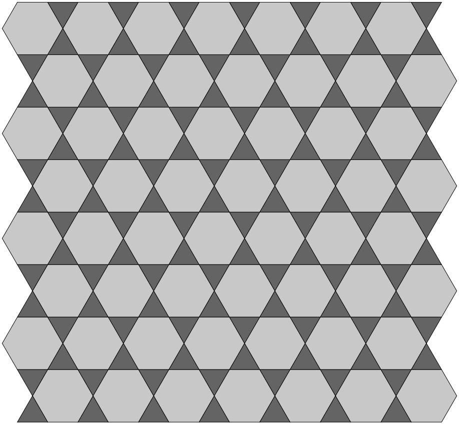
3. Mixing time

Introduction

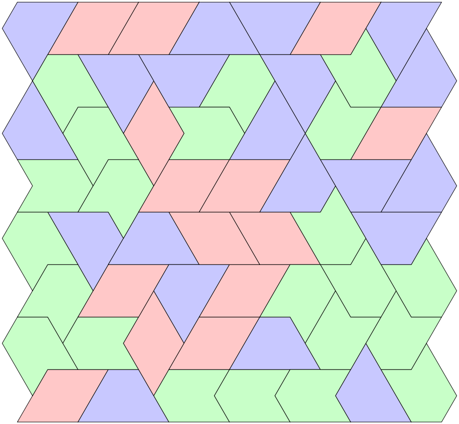
Families of tilings: dimers, balanced words, kagome tilings and others

- Random generation
- Tiling graph properties

Kagome tilings

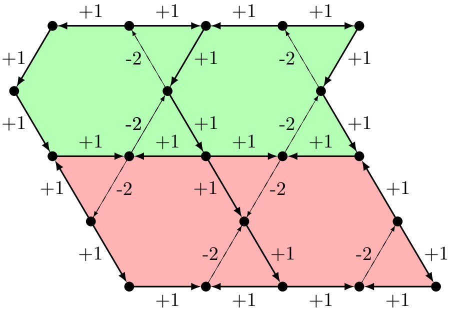


Kagome tilings



Height function, Flips

- *Flip*: local min ↔ local max.
- Tilable region $R \rightsquigarrow$ *tiling graph* G_R (distributive lattice).



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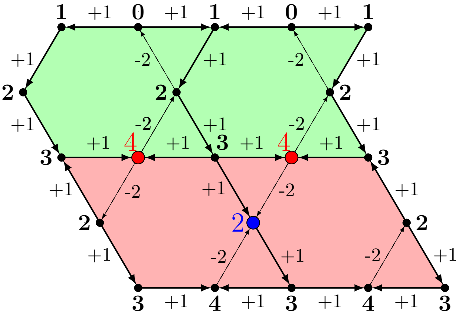

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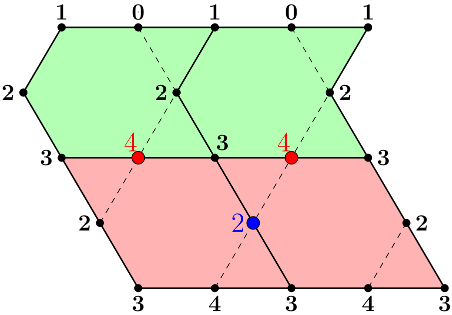

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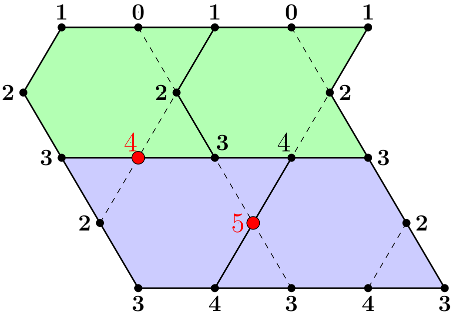

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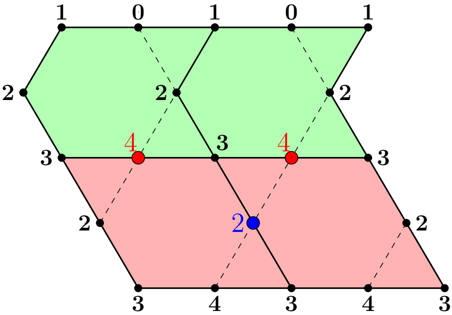

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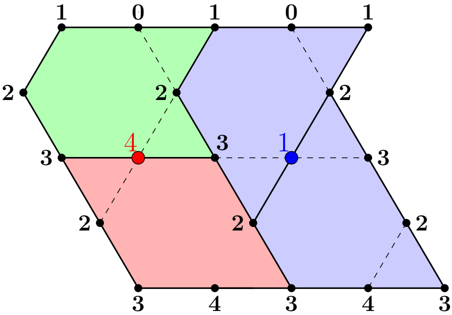

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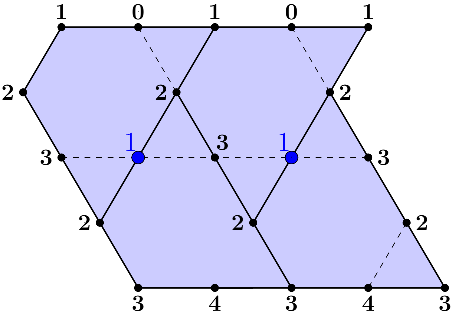
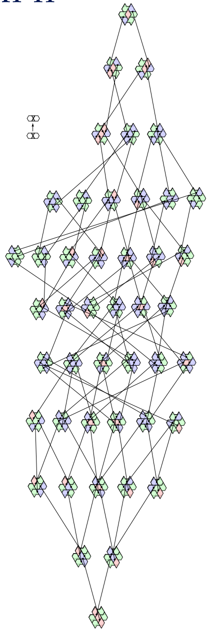


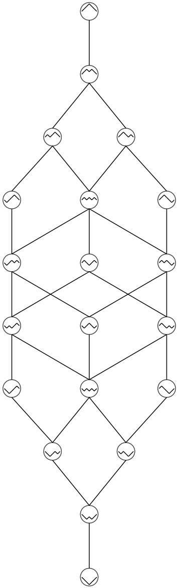
Kagome tiling graph

$D = O(n^{3/2})$



Balanced words on $A = \{a, b\}$

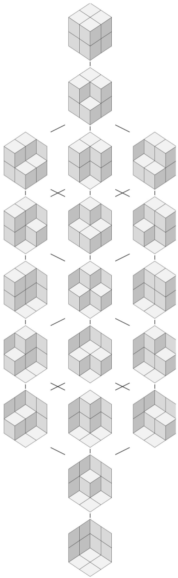
flip: ab $\longleftrightarrow$ ba
$D = O(n^2)$



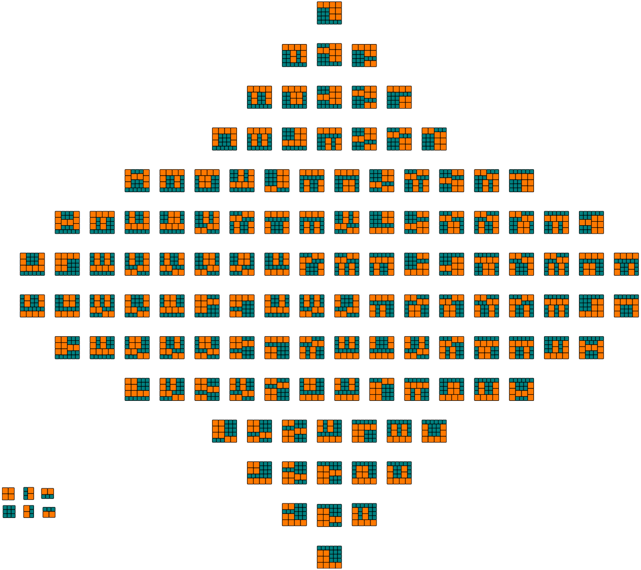
Lozange tilings

flip: add/remove a cube

$D = O(n^{3/2})$



Square tilings 2×2 & 3×3



Markov Chain on the Tiling Graph

State space $\Omega = \{$ tilings of the chosen region R $\} = \{$ vertices of G_R $\}$

Starting at t_0, repeat t times:

1. Pick inner vertex v of R
2. Pick a direction of a flip: "up" or "down"
3. Flip if possible, otherwise stay still.

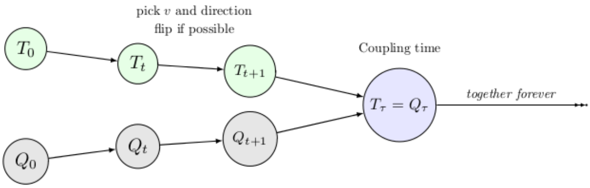
MC: aperiodic + irreducible = ergodic $\Rightarrow$ unique stationary distribution

Goal: bound *Mixing time* τ

(time it takes for MC to be at distance $\varepsilon < \frac{1}{2}$ from stationary distrib)

Coupling

- Coupling – MC on $\Omega \times \Omega$ defining $(T_t, Q_t)_{t=0}^{\infty}$:



- Theorem (Aldous): $\tau \leq c\tau_{coupling}$

Path coupling, modified chains (tower of flips)

- 2−letter words of size n: $\tau_{ab-words} = O(n^3)$
- k−colorings of a n−vertex graph: $\tau_{k-colorings} = O(n \ln(ne^{-1}))$
- Lozange tilings of the region of area n:
  $\tau_{lozange} = O(n^4)$ (Randall, Tetalli),
  $\tau_{lozange} = O(n^2 \ln n)$ (Wilson)
- Kagome tilings: $\tau_{kagome} = O(n^2) \leftarrow$ *Conjecture*

Mixing time via conductance

$$\Phi = \min_{S:\pi(S)\leq\frac{1}{2}} \frac{\sum_{e\in\delta S} Q(e)}{\pi(S)},$$

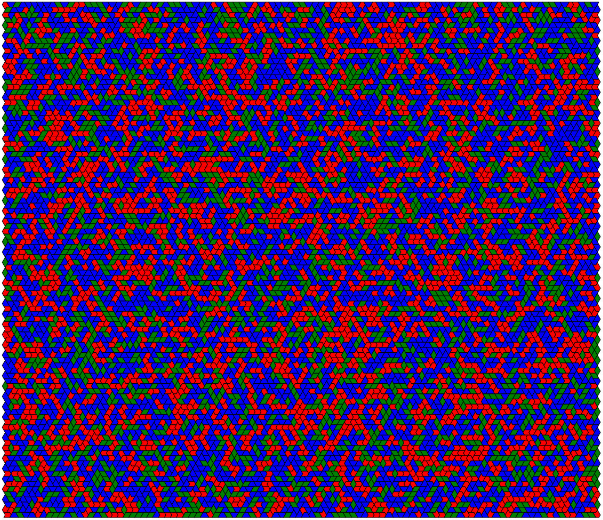
$$\frac{\Phi^2}{2} < Gap \leq 2\Phi,$$

$$\tau \leq \frac{\ln \pi^{-1}}{\Phi^2}.$$

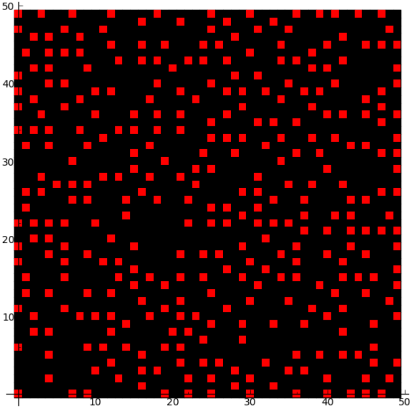
Can we prove that:

- $\Phi \geq \frac{1}{poly(n)}$,
- G has no bottleneck,
- G is a $(\frac{N}{2}, 1+\epsilon)$ vertex expander,
- there are $poly(n)$ canonical paths passing through each edge ?

Coupling from the Past for Kagome



Bounding chain for squares $1 \times 1 \& 2 \times 2$



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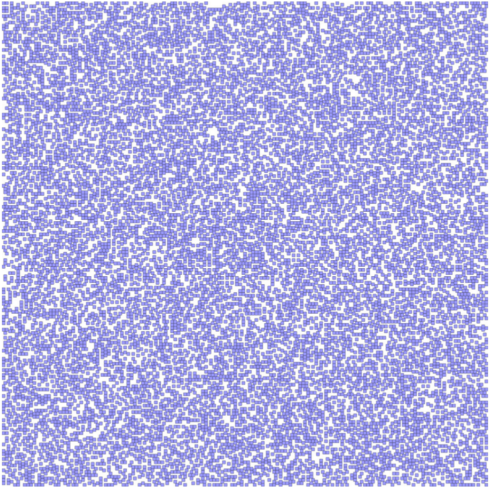

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