A
Small Minimal Aperiodic Reversible Turing machine

(hal-00975244)

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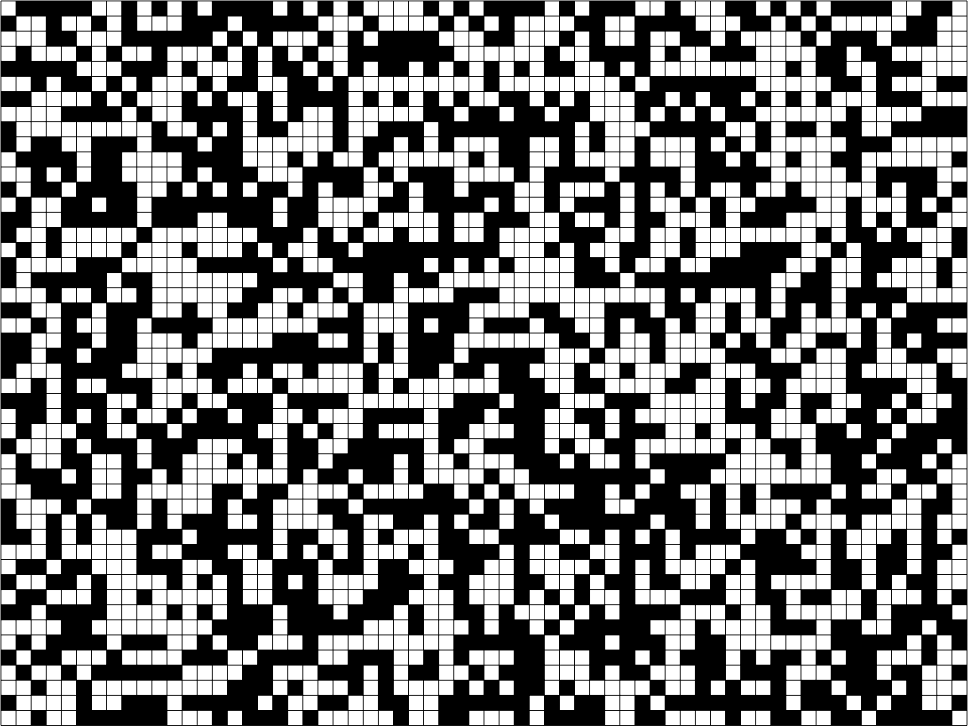
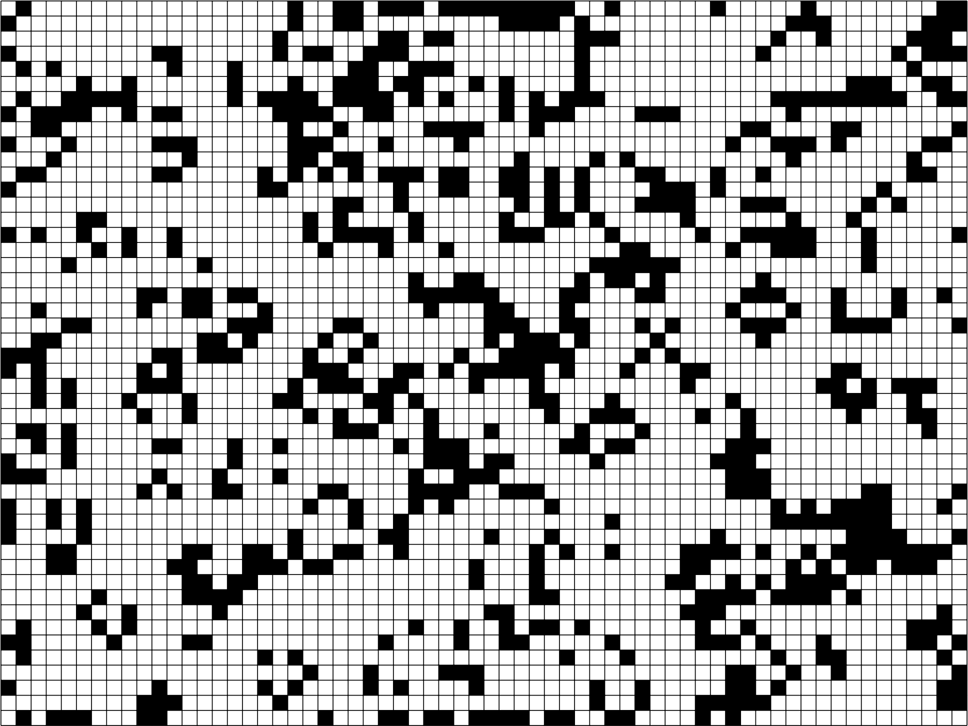
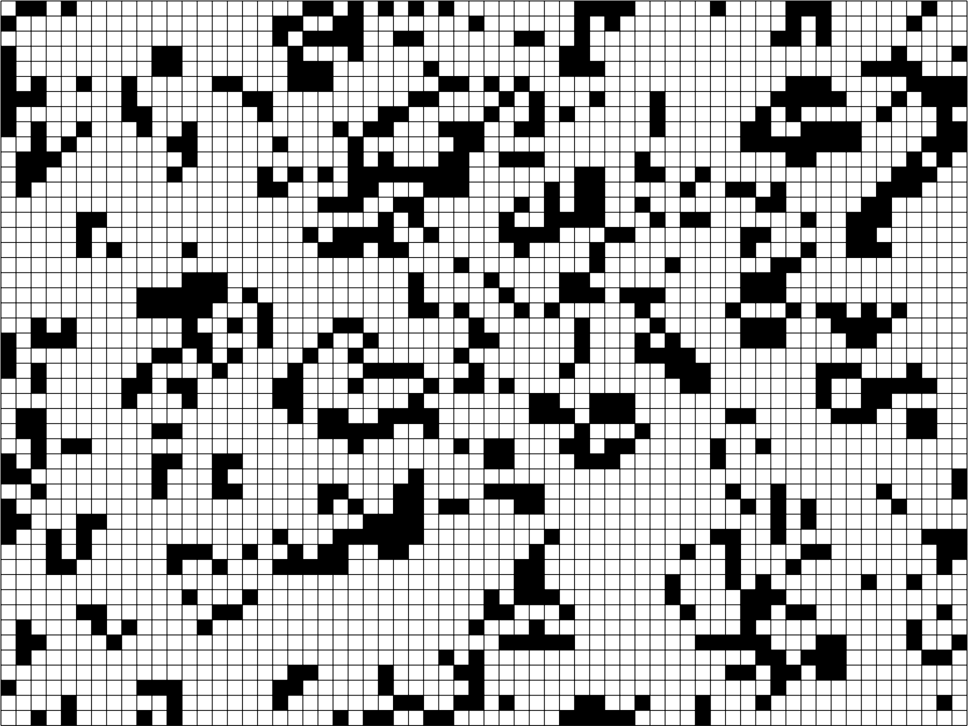
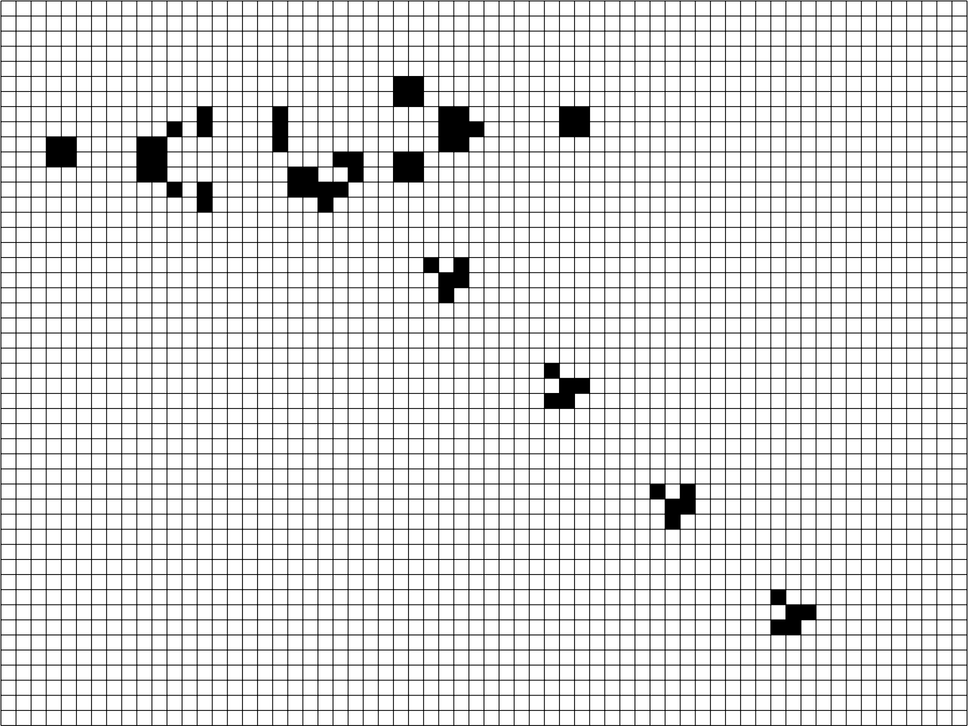
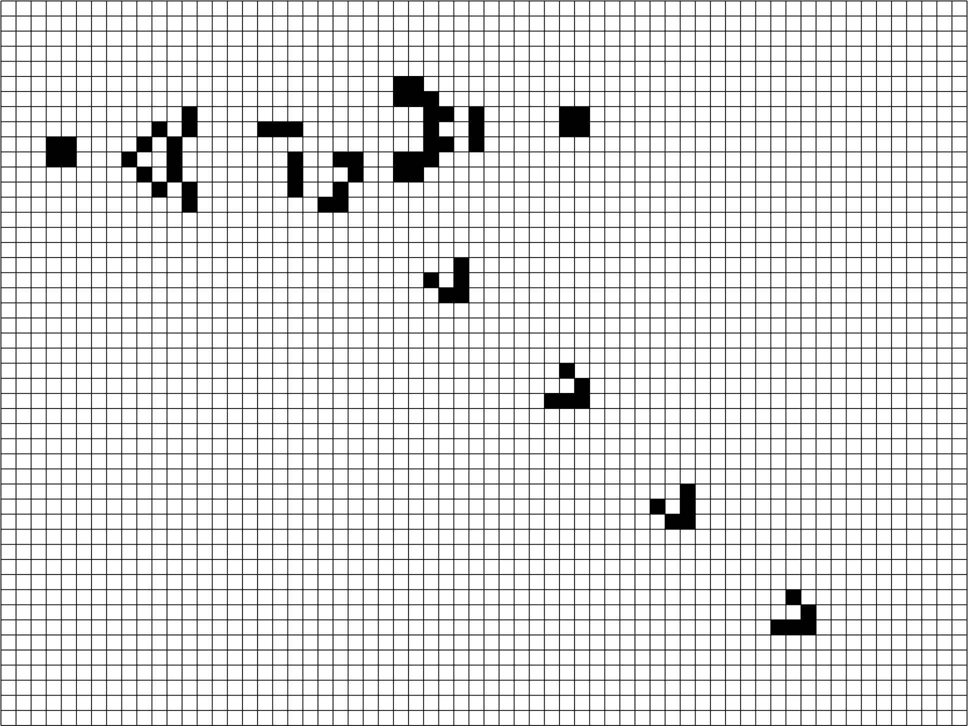
JIRC — April 15th, 2014

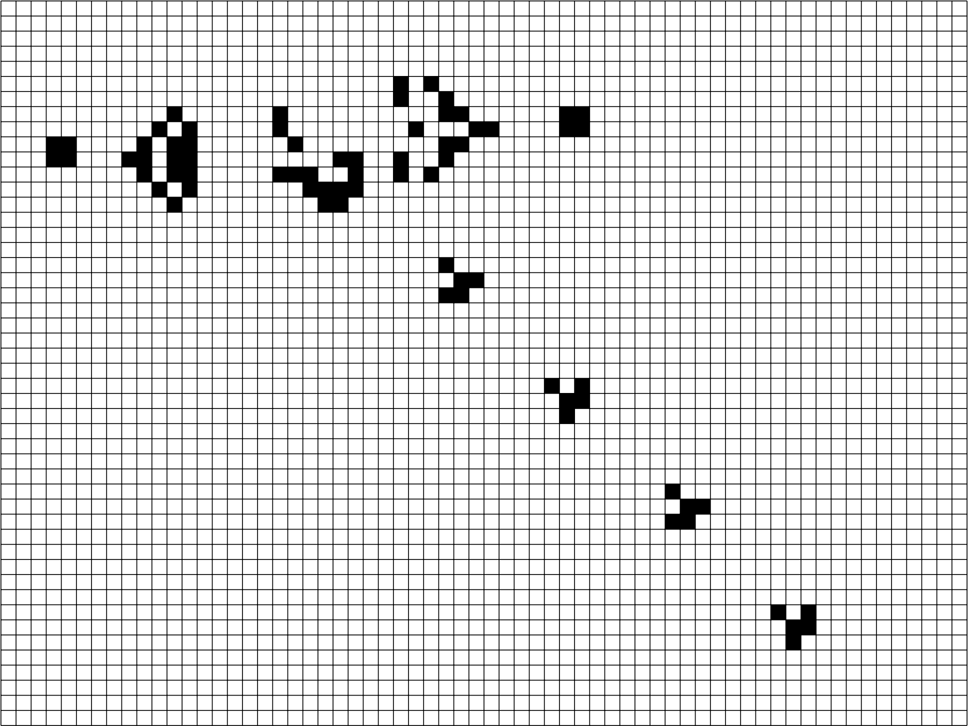


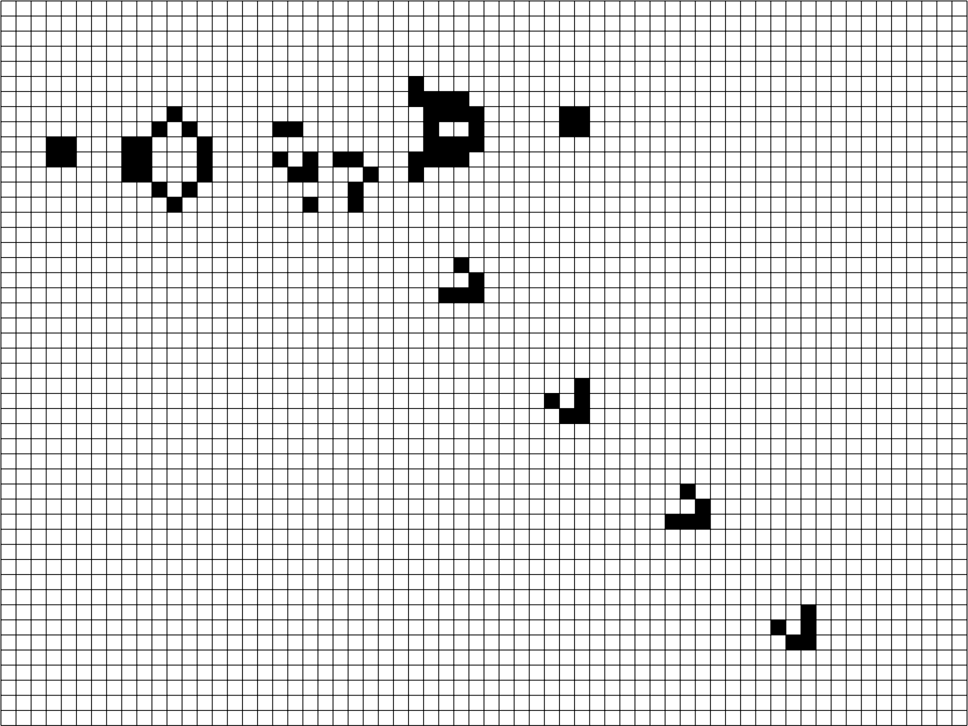


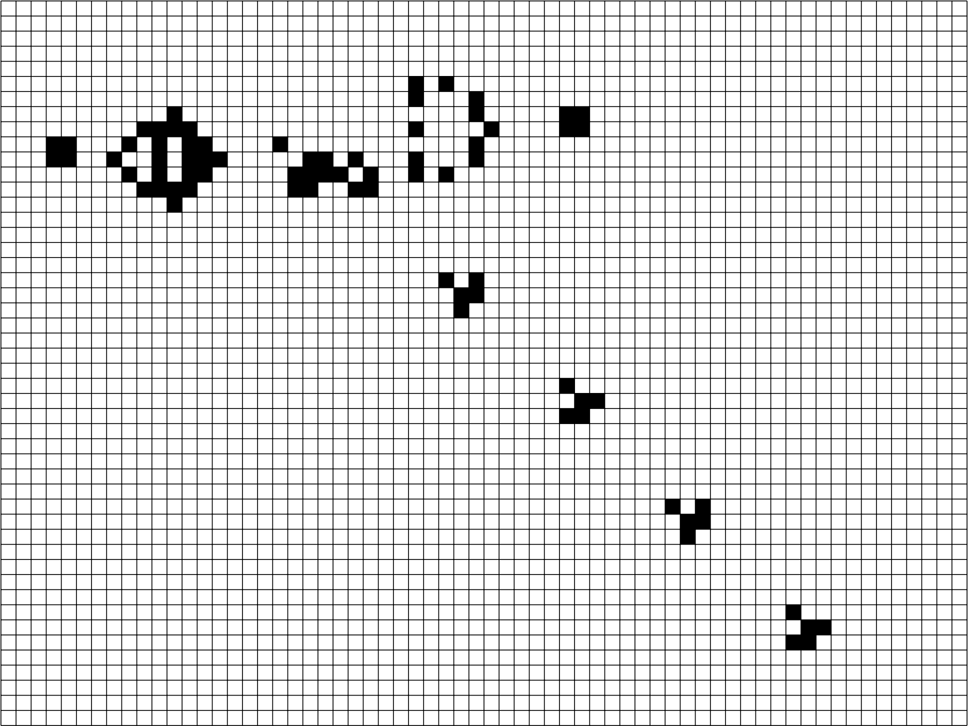


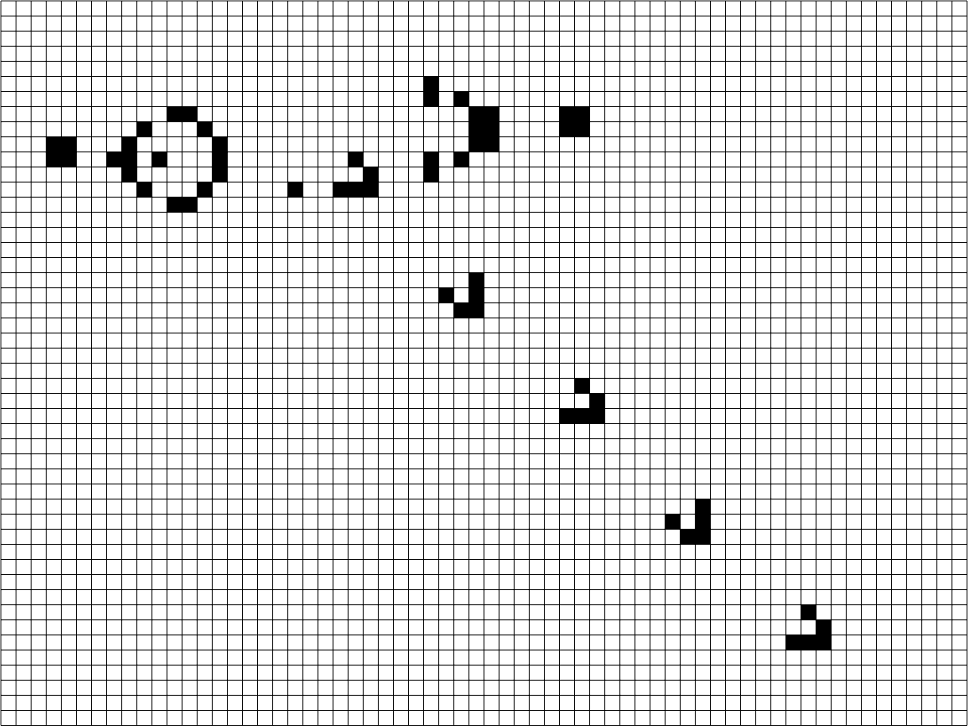


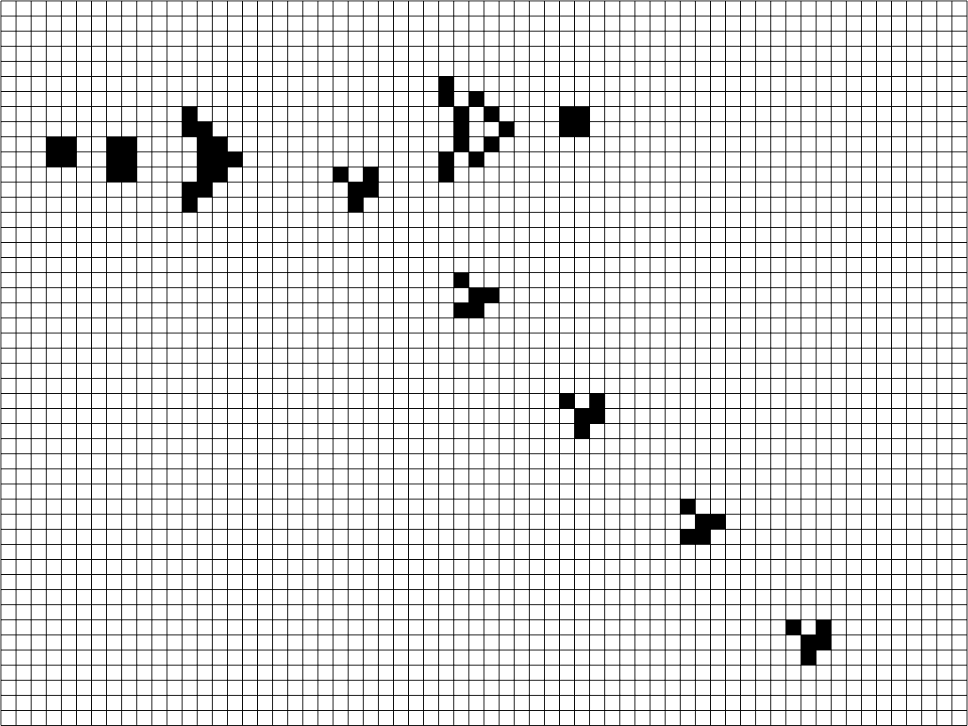


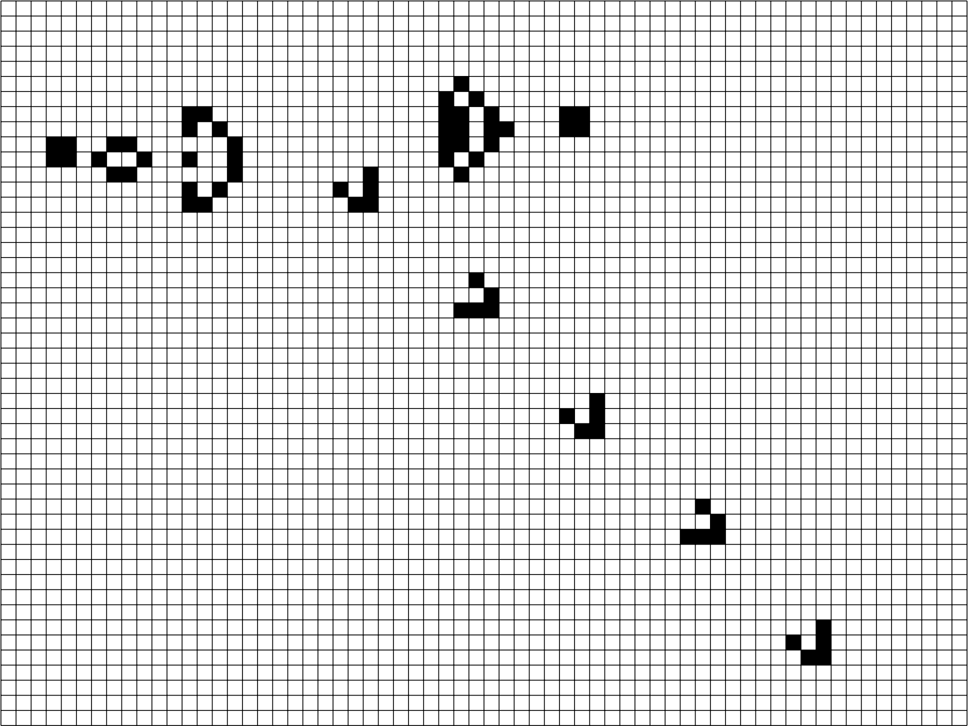


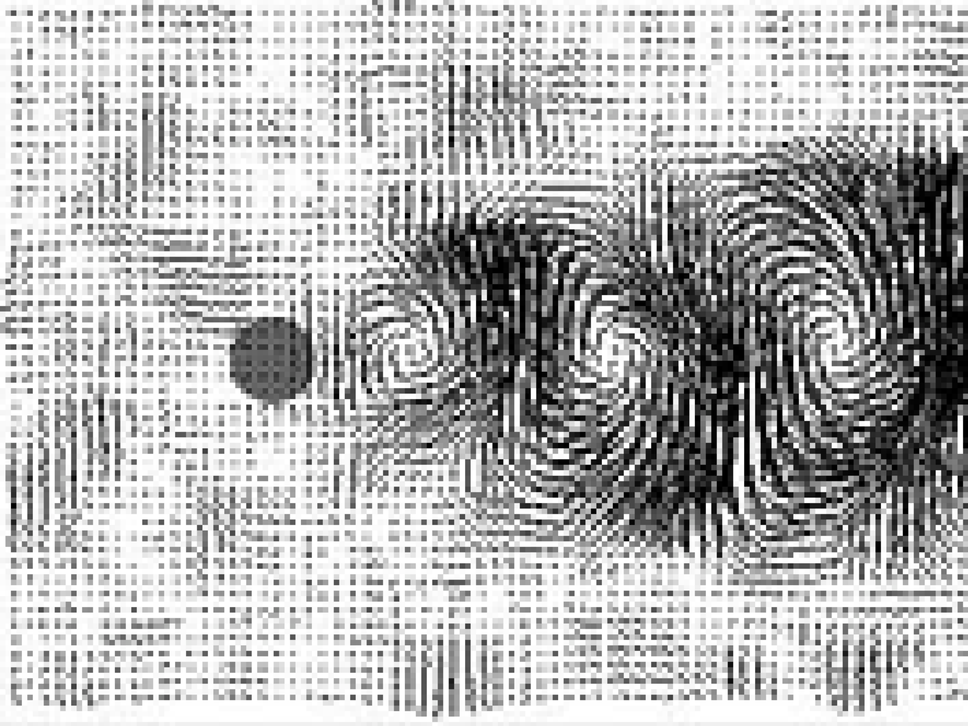






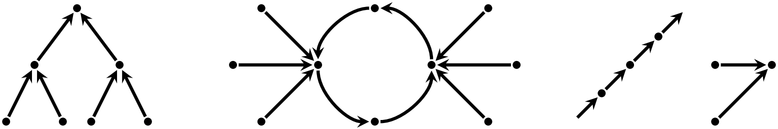






Discrete dynamical systems

Definition A **DDS** is a pair (X, F) where X is a topological space and $F : X \to X$ is a continuous map.



Definition The **orbit** of $x \in X$ is the sequence $(F^n(x))$ obtained by iterating F.

In this talk, $X = S^{\mathbb{Z}}$ or $Q \times S^{\mathbb{Z}}$ and is endowed with the **Cantor topology** (product of the discrete topology on S), and F is a continuous map **invariant by translation**.

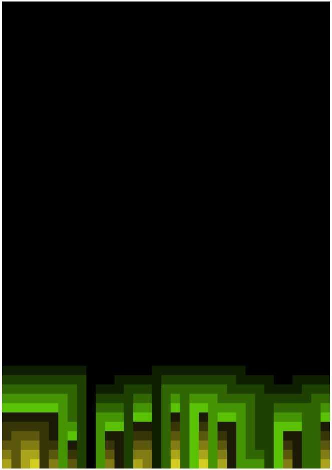
The nilpotency problem (Nil)

Definition A DDS is **nilpotent** if
$\exists z \in X, \forall x \in X, \exists n \in \mathbb{N}, F^n(x) = z$.

Given a recursive encoding of the DDS, can we **decide** nilpotency?

A DDS is **uniformly nilpotent** if
$\exists z \in X, \exists n \in \mathbb{N}, \forall x \in X, F^n(x) = z$.

Given a recursive encoding of the DDS, can we **bound recursively** n?



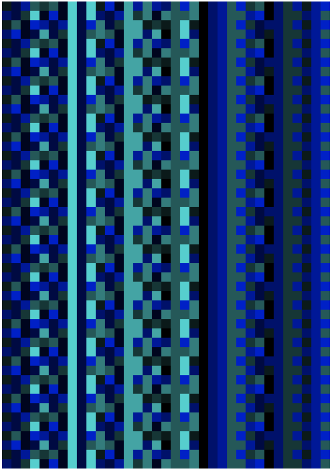
The periodicity problem (Per)

Definition A DDS is **periodic** if
$\forall x \in X, \exists n \in \mathbb{N}, F^n(x) = x$.

Given a recursive encoding of the DDS, can we **decide** periodicity?

A DDS is **uniformly periodic** if
$\exists n \in \mathbb{N}, \forall x \in X, F^n(x) = x$.

Given a recursive encoding of the DDS, can we **bound recursively** n?



Cellular automata

Definition A **CA** is a triple (S, r, f) where S is a **finite set of states**, $r \in \mathbb{N}$ is the **radius** and $f : S^{2r+1} \to S$ is the **local rule** of the cellular automaton.

A **configuration** $c \in S^{\mathbb{Z}}$ is a coloring of $\mathbb{Z}$ by S.

The **global map** $F : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ applies f uniformly and locally:

$$\forall c \in S^{\mathbb{Z}}, \forall z \in \mathbb{Z}, \qquad F(c)(z) = f(c(z-r), \ldots, c(z+r)).$$

Space-time diagram

$S = \{0, 1, 2\}$, $r = 1$, $f(x, y, z) = \lfloor 6450288690466/3^{9x+3y+z} \rfloor \pmod 3$

Undecidability results

Theorem Both **Nil** and **Per** are **recursively undecidable**.

The proofs inject **computation** into **dynamics**.

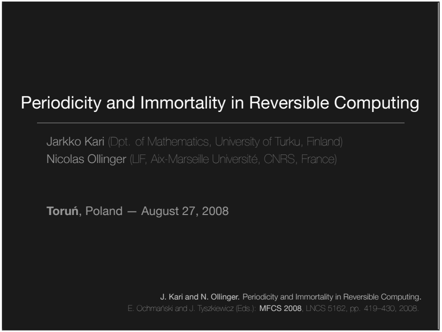
Undecidability is not necessarily a negative result:
it is a **hint of complexity**.

Remark Due to **universe configurations** both nilpotency and periodicity are uniform.

The bounds grow **faster than any recursive function**: there exists simple nilpotent or periodic CA with huge bounds.

Motivation for today

Solve a conjecture that we had with J. Kari a few years ago:



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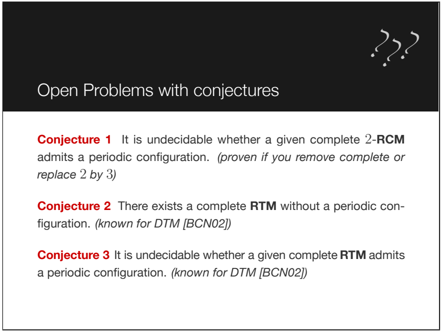
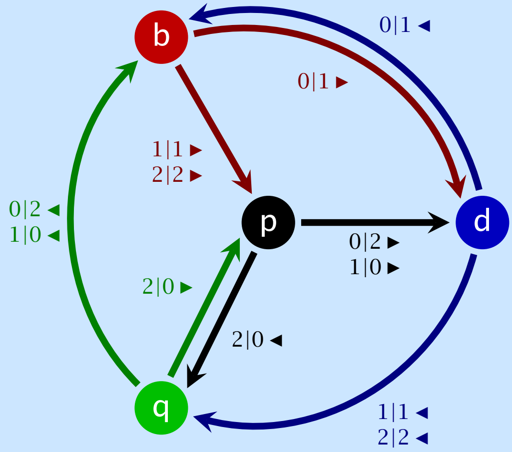


Theorem To find if a given **complete reversible Turing machine** admits a **periodic orbit** is Σ_1-complete.



1. Dynamics of Turing machines

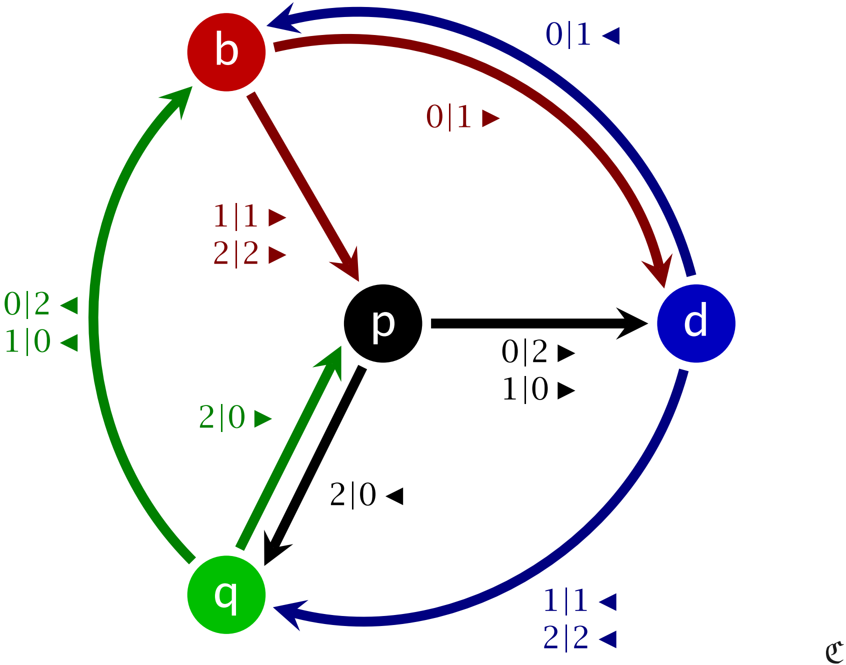
Turing machines

Definition A **Turing machine** is a triple (Q, Σ, δ) where Q is the finite set of states, Σ is the finite set of tape symbols and $\delta : Q \times \Sigma \to Q \times \Sigma \times \{◀, ▶\}$ is the transition function.

Transition $\delta(s, a) = (t, b, d)$ means:

"in state s, when reading the symbol a on the tape,
replace it by b move the head in direction d and enter state t."

Remark We do not care about blank symbol or initial and final states, we see Turing machines as dynamical systems.



b
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0|1 ▶
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1|0 ◀
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1|0 ▶
2|0 ▶
2|0 ◀
1|1 ◀
2|2 ◀

Moving head dynamics

$$X_H = Q \times \mathbb{Z} \times \Sigma^{\mathbb{Z}} \cup \Sigma^{\mathbb{Z}}$$
$$T_H : X_H \to X_H$$

$\cdots$000000**b**000000000$\cdots$
$\cdots$0000001**d**00000000$\cdots$
$\cdots$000000**b**110000000$\cdots$
$\cdots$0000001**p**10000000$\cdots$
$\cdots$00000010**d**0000000$\cdots$
$\cdots$0000001**b**01000000$\cdots$
$\cdots$00000011**d**1000000$\cdots$
$\cdots$0000001**q**11000000$\cdots$
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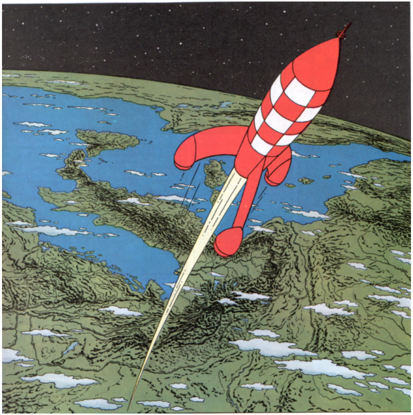
$\vdots$

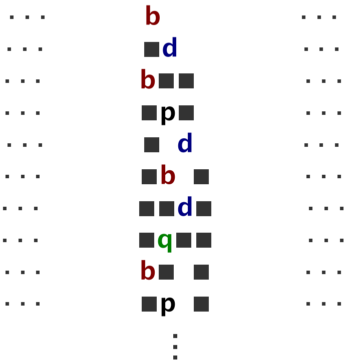
Hergé. *On a marché sur la lune.* Casterman, 1954.



Long shot

Moving head dynamics

$$X_H = Q \times \mathbb{Z} \times \Sigma^{\mathbb{Z}} \cup \Sigma^{\mathbb{Z}}$$
$$T_H : X_H \to X_H$$



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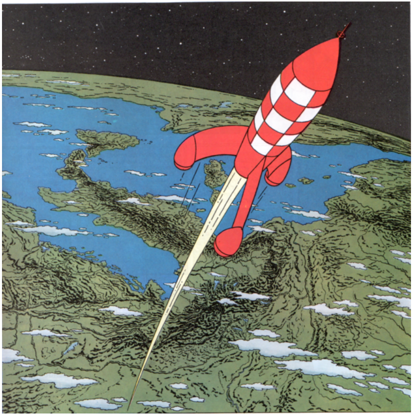

Long shot

Moving tape dynamics

$$\begin{aligned} X_T &= Q \times \Sigma^{\mathbb{Z}} \\ T_T &: X_T \to X_T \end{aligned}$$

$\cdots 0000000\mathbf{b}00000000 \cdots$
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$\cdots 0000000\mathbf{b}11000000 \cdots$
$\cdots 0000001\mathbf{p}10000000 \cdots$
$\cdots 0000010\mathbf{d}00000000 \cdots$
$\cdots 0000001\mathbf{b}01000000 \cdots$
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$\cdots 0000001\mathbf{q}11000000 \cdots$
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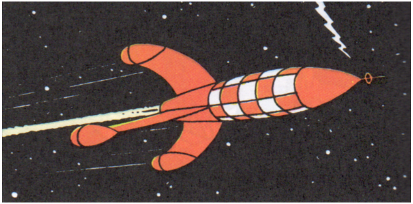
$\vdots$

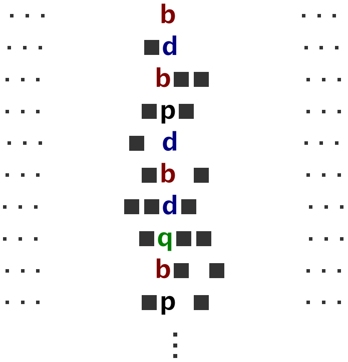
Hergé. *On a marché sur la lune*. Casterman, 1954.



Tracking shot

Moving tape dynamics

$$X_T = Q \times \Sigma^{\mathbb{Z}}$$
$$T_T : X_T \to X_T$$



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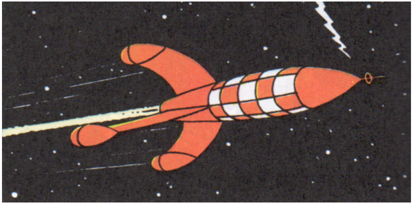


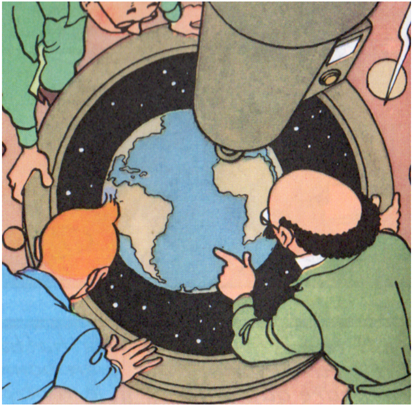
Tracking shot

Trace subshift

$$S_T \subseteq (Q \times \Sigma)^\omega$$

0 0 1 1 0 0 1 1 1 0
b d b p d b d q b p . . .

Hergé. *On a marché sur la lune*. Casterman, 1954.



Point of view shot

Simple dynamical properties

Definition A point $x \in X$ is **periodic** if it admits a **period** $p > 0$ such that $T^p(x) = x$.

Definition A machine is **periodic** if every point is periodic.

Remark Periodicity implies uniform periodicity: $T^p = \mathrm{Id}$.

Theorem[KO08] The **periodicity problem** is Σ_1-complete.

Definition A machine is **aperiodic** if it has no periodic point.

Partial vs complete machines

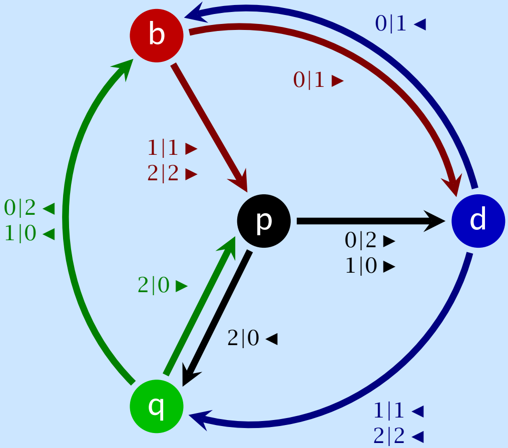
Definition A TM is **complete** if δ is completely defined, otherwise **undefined transitions** of a partial δ correspond to **halting configurations**.

Definition A point is **mortal** if it eventually **halts**.

Thm[Hooper66] The **immortality problem** is Π_1-complete.

Rk **Mortality** is different from **totality** which is Π_2-complete.

Thm[KO08] The result is the same for **reversible TM**.



2. Reversible Turing machines

Reversible Turing machines

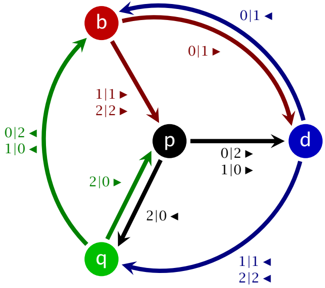
Intuitively, a TM is **reversible** if there exists another TM to compute backwards: "$T_2 = T_1^{-1}$". **Forget technical details**...

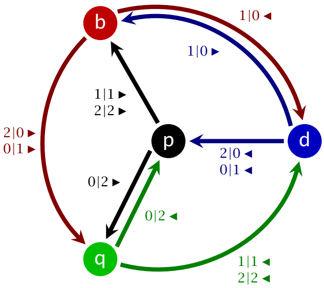
Definition A TM is **reversible** if δ can be decomposed as:

$$\begin{aligned} \delta(s,a) &= (t,b,\rho(t)) \qquad \text{where } (t,b) = \sigma(s,a) \\ \rho &: Q \to \{\blacktriangleleft, \blacktriangleright\} \\ \sigma &\in \mathfrak{S}_{Q\times\Sigma} \end{aligned}$$

Remark $\delta^{-1}(t,b) = (s,a,\blacklozenge(\rho(s)))$

A complete RTM



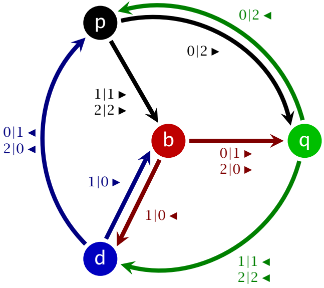


It is **time-symmetric**:

its own inverse up to

state/symbol permutation.

1	⇔	2
b	⇔	p
d	⇔	q



Searching for a reduction

We want to prove the following:

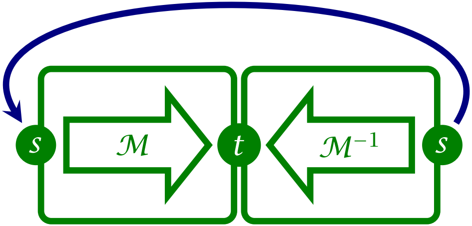
Theorem To find if a given **complete reversible Turing machine** admits a **periodic orbit** is Σ_1-complete.

In the partial case we use the following tool:

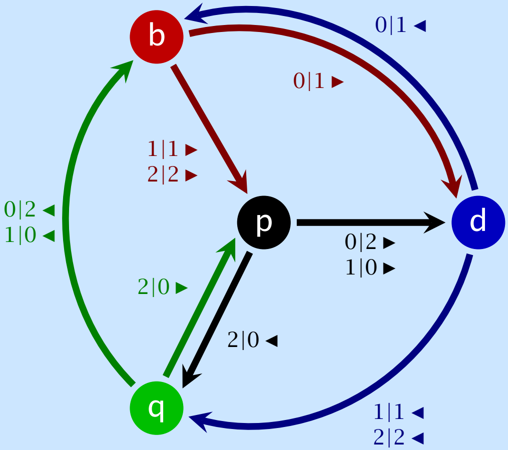
Prop[KO08] To find if a given **(aperiodic) RTM** can reach a given state t from a given state s is Σ_1-complete.

The partial case

Principle of the reduction Associate to an (aperiodic) RTM $\mathcal{M}$ with given s and t a new machine with a periodic orbit if and only if t is reachable from s.



We need to find a way to **complete** the constructed machine.



3. a SMART machine

The SMART machine $\mathfrak{C}$

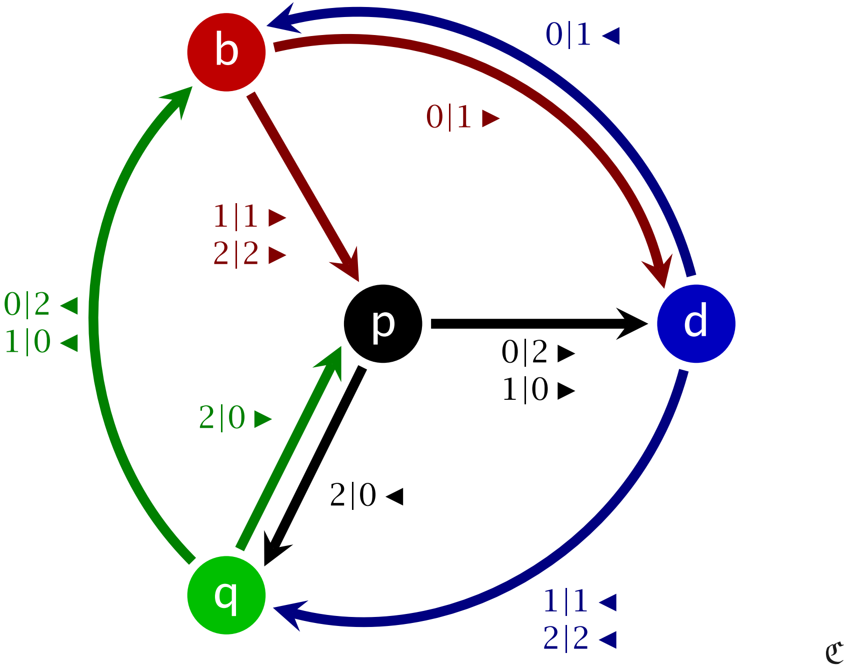
Conj[Kůrka97] Every **complete** TM has a **periodic** point.

Thm[BCN02] No, here is an **aperiodic** complete TM.

Rk It relies on the **bounded search** technique [Hooper66].

In 2008, I asked **J. Cassaigne** if he had a reversible version of the BCN construction. . .

. . . he answered with a small machine $\mathfrak{C}$ which is a reversible and (drastic) simplification of the BCN machine.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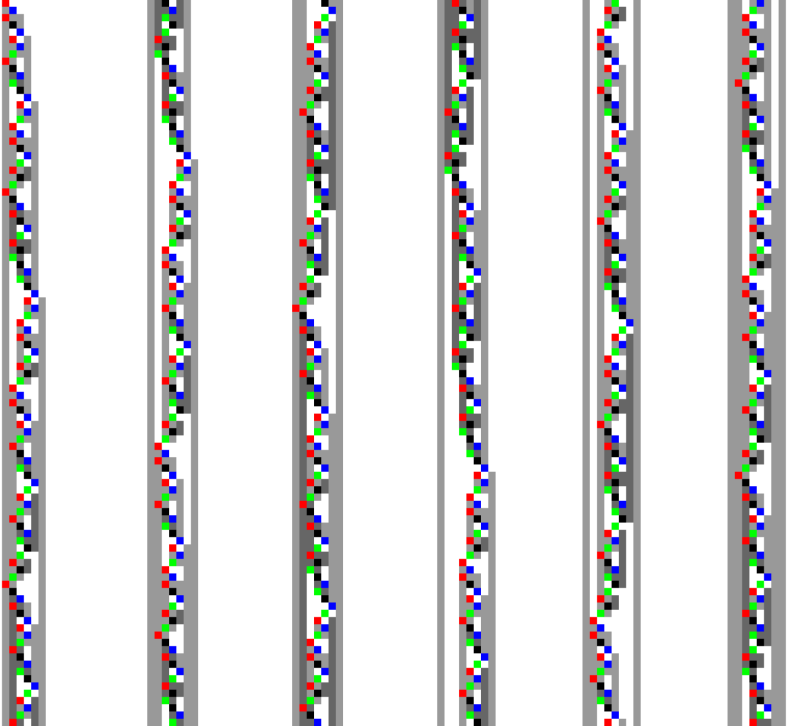
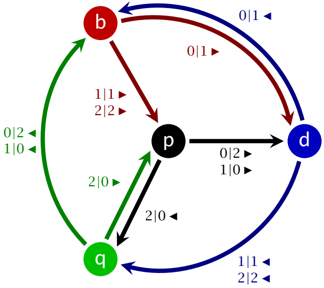
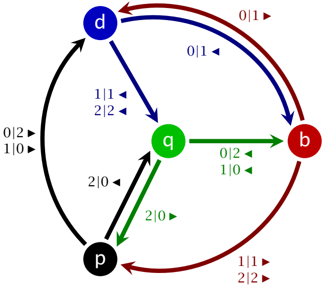


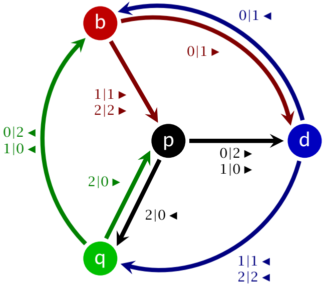
Symmetry





It is both **space-** and **time-symmetric**.

◀	⇔	▶
b	⇔	d
p	⇔	q



Aperiodicity

Proposition The machine $\mathfrak{C}$ is **aperiodic**.

Idea of the proof

1. The behavior starting from a tape of 0 is aperiodic;
2. Every block of 0 eventually grows;
3. Thus $\mathfrak{C}$ is aperiodic.

Minimality

The behavior of $\mathfrak{C}$ can be precisely described.

Proposition The behavior starting from a tape of 0 is **dense**.

Proposition The trace subshift of $\mathfrak{C}$ is **minimal**.

Proposition The trace subshift of $\mathfrak{C}$ is **substitutive**.

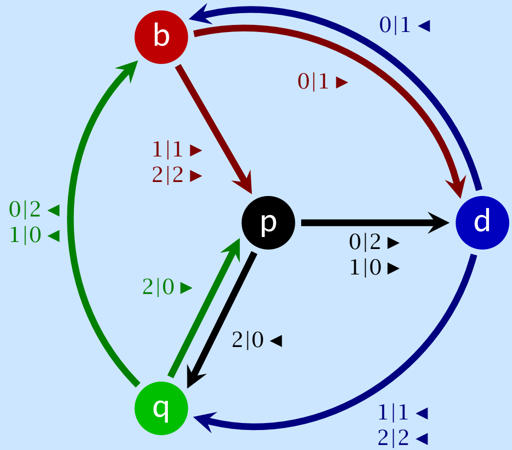
Substitutive trace subshift

$$\varphi\begin{pmatrix}0\\ \mathbf{b}\end{pmatrix} = \begin{matrix}0\ 0\ 1\ 1\\ \mathbf{b}\ \mathbf{d}\ \mathbf{b}\ \mathbf{p}\end{matrix} \qquad \varphi\begin{pmatrix}0\\ \mathbf{d}\end{pmatrix} = \begin{matrix}0\ 0\ 1\ 1\\ \mathbf{d}\ \mathbf{b}\ \mathbf{d}\ \mathbf{q}\end{matrix}$$

$$\varphi\begin{pmatrix}x\\ \mathbf{b}\end{pmatrix} = \begin{matrix}x\\ \mathbf{b}\end{matrix} \qquad \varphi\begin{pmatrix}x\\ \mathbf{d}\end{pmatrix} = \begin{matrix}x\\ \mathbf{d}\end{matrix}$$

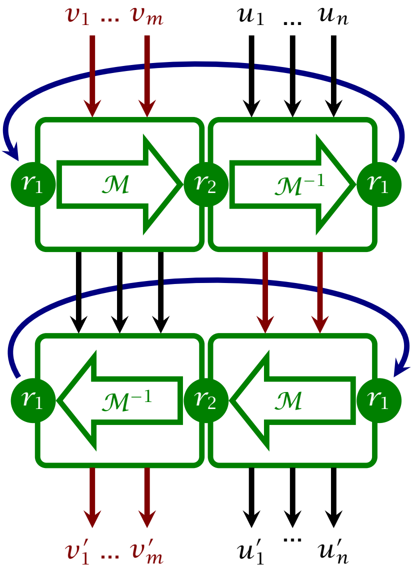
$$\varphi\begin{pmatrix}0\\ \mathbf{p}\end{pmatrix} = \begin{matrix}0\ 0\ 2\ 1\\ \mathbf{p}\ \mathbf{d}\ \mathbf{b}\ \mathbf{p}\end{matrix} \qquad \varphi\begin{pmatrix}0\\ \mathbf{q}\end{pmatrix} = \begin{matrix}0\ 0\ 2\ 1\\ \mathbf{q}\ \mathbf{b}\ \mathbf{d}\ \mathbf{q}\end{matrix}$$

$$\varphi\begin{pmatrix}x\\ \mathbf{p}\end{pmatrix} = \begin{matrix}0\ x\ 2\ x\\ \mathbf{p}\ \mathbf{d}\ \mathbf{q}\ \mathbf{p}\end{matrix} \qquad \varphi\begin{pmatrix}x\\ \mathbf{q}\end{pmatrix} = \begin{matrix}0\ x\ 2\ x\\ \mathbf{q}\ \mathbf{b}\ \mathbf{p}\ \mathbf{q}\end{matrix}$$

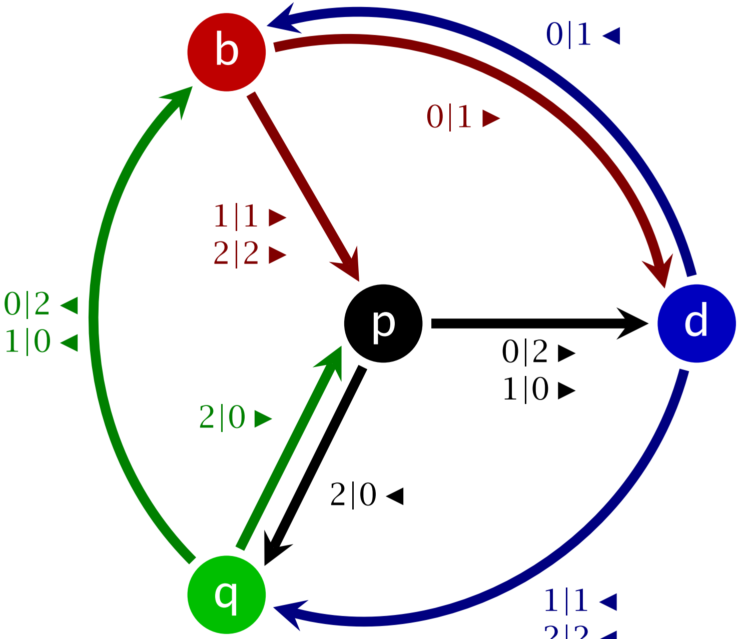


4. Embedding the machine

Embedding trick



Use the transitions of the **Cassaigne machine** to connect the u'_i to the u_i and the v'_i to the v_i.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