

Cellular automata in hyperbolic spaces, universality and beyond the Turing barrier

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this talk:

1. a very short introduction to hyperbolic geometry
2. the splitting method
3. implementing cellular automata in the hyperbolic plane
4. classical universality in the hyperbolic plane
5. beyond the Turing barrier

1. hyperbolic geometry

absolute geometry

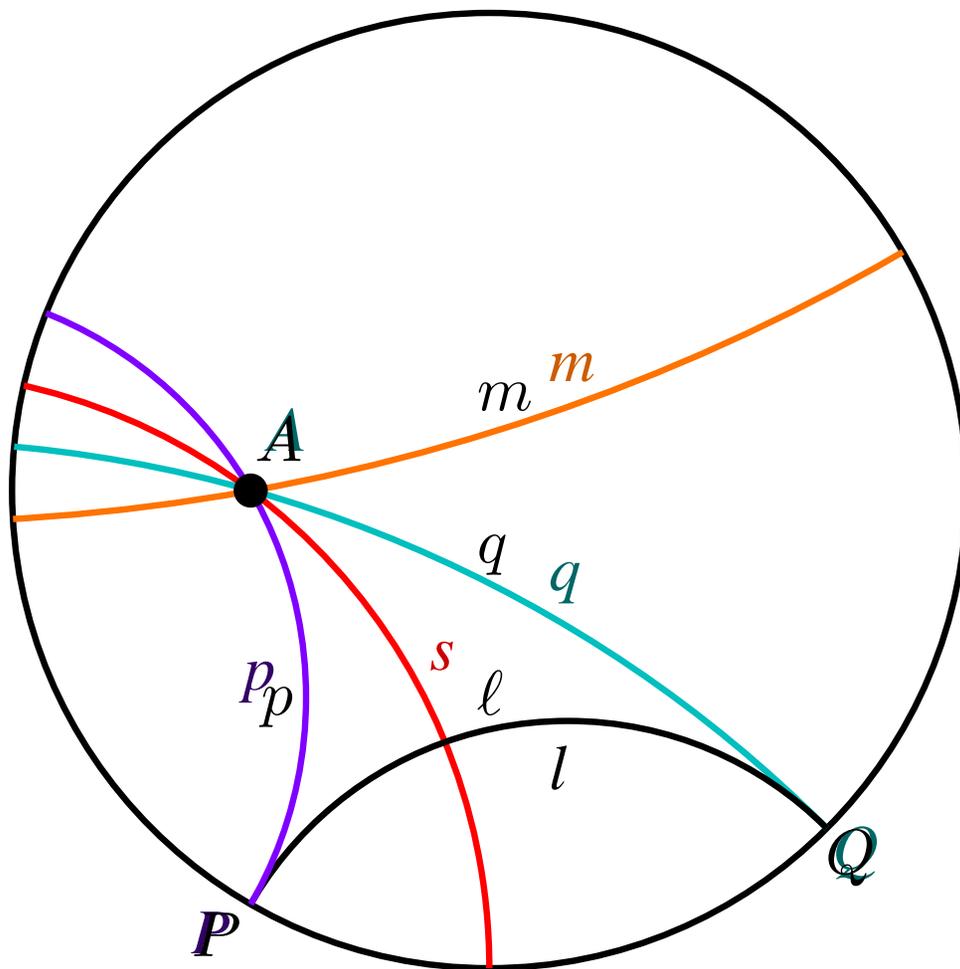
+ new axiom (Lobachevsky-Bolyai):
from a point A out of a line L ,
at least two parallel lines to L

extension to any dimension

a lot of models:

Beltrami, Klein, Poincaré,...

Poincaré's disc model



usefull properties

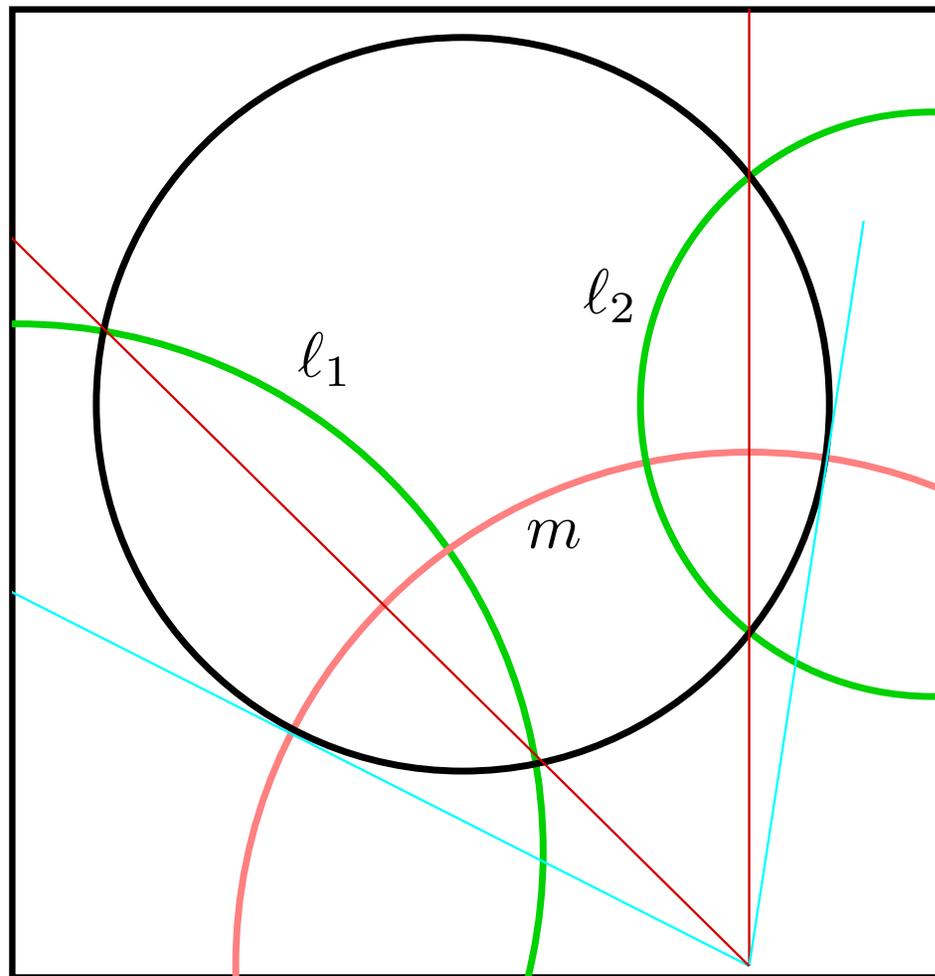
sum of the angles in a triangle:

always less than π

non secant lines:

always a unique common perpendicular

common perpendicular of non-secant lines



displacements in the hyperbolic plane

definition:

finite product of reflections

classification theorem:

any displacement is a product of at most three reflections; the parity of the number of factors is an invariant

positive displacements:

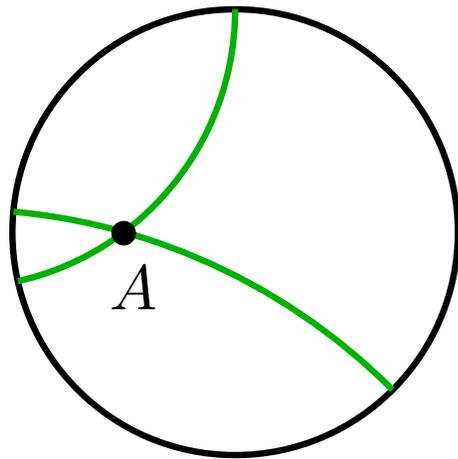
do not change orientation:
products of 2 reflections

classification of positive displacements

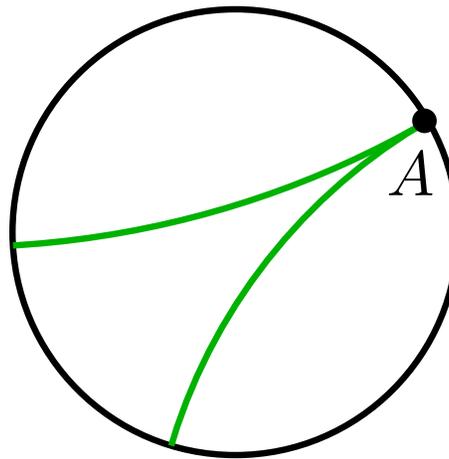
definition:

finite product of reflections

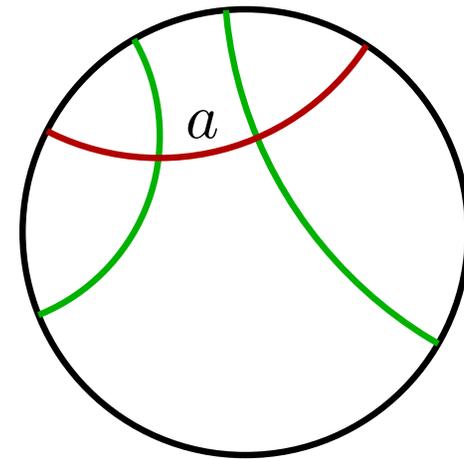
three cases, depending on intersection:



rotation



ideal rotation



shift along
a line

extension to higher dimension

generalisation of the Poincaré's disc:

the space: the open unit hyper-ball

points at infinity: the unit hyper-sphere ∂U

hyper-planes: hyper-spheres H , $H \perp \partial U$
and hyper-diameters

planes = hyper-plane in $3D$

= in $4D$:

intersection of 2 secant hyper-planes

lines: intersection of 2 secant planes

locally: the same as the euclidean analog

how to get some intuition in the
hyperbolic plane ?

a good site: Matthew Cook's site

[http://paradise.caltech.edu/
~cook/Workshop/Math/HyperbolicLand.html](http://paradise.caltech.edu/~cook/Workshop/Math/HyperbolicLand.html)

imagine the following:

no housing problem: just a new street around the city

very easy to get lost without Ariadna thread

while moving: completely changing landscape in each direction

you can see the straightforward continuation of your path either turning to the left or turning to the right

the location problem

key problem to implement cellular automata

very easy in the Euclidean plane:

the group of positive displacements possesses a non-trivial subgroup: the shifts

by contrast, in the hyperbolic plane:

the group of positive displacements is simple

which is very bad !

some features of the books:

volume 1:

the splitting method

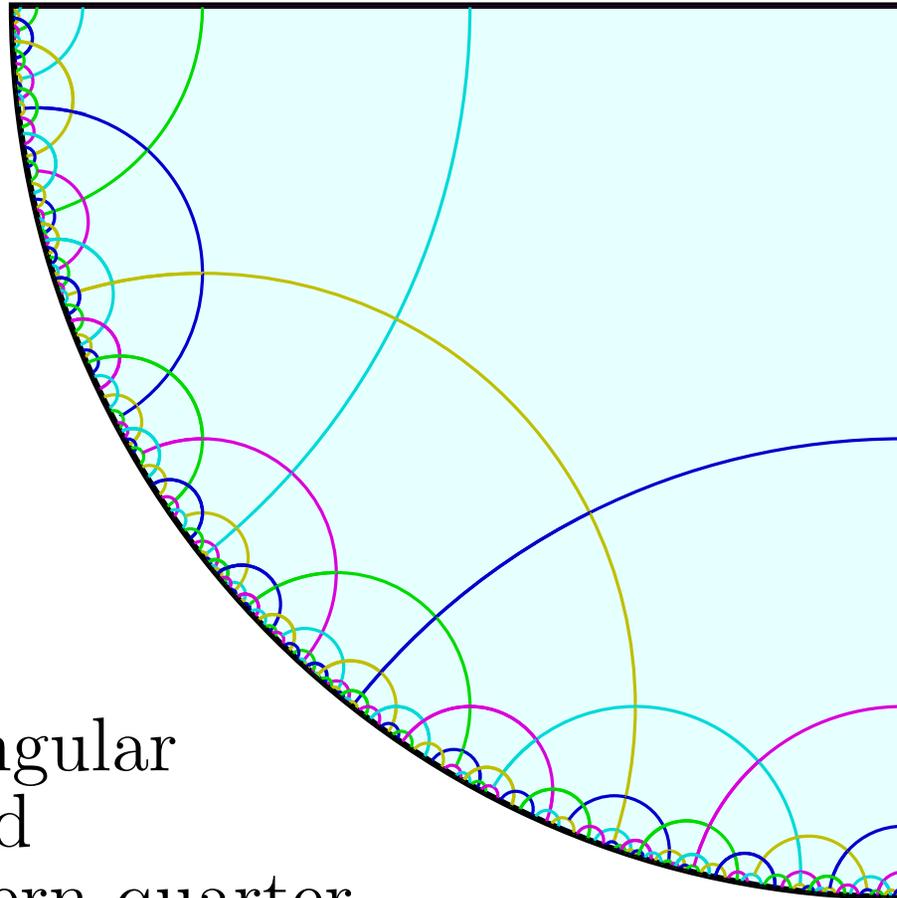
application to combinatoric tilings

volume 2:

then, implementation of cellular automata

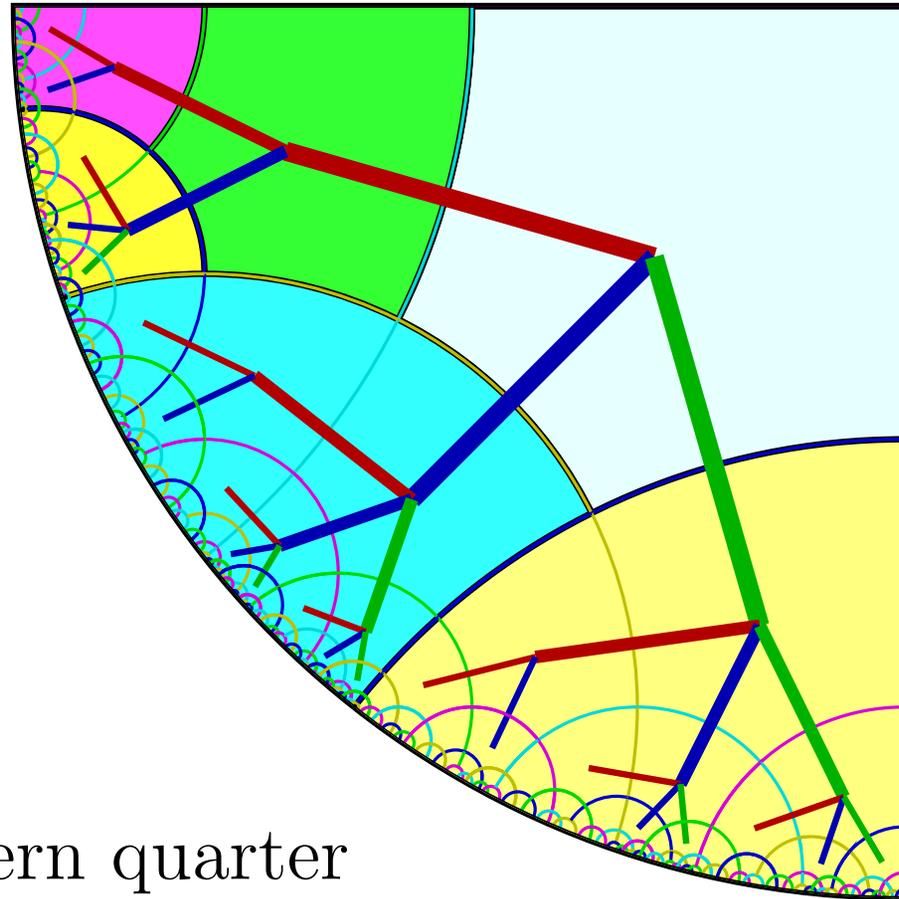
application to universality and super Turing computations

the classic case of the pentagrid



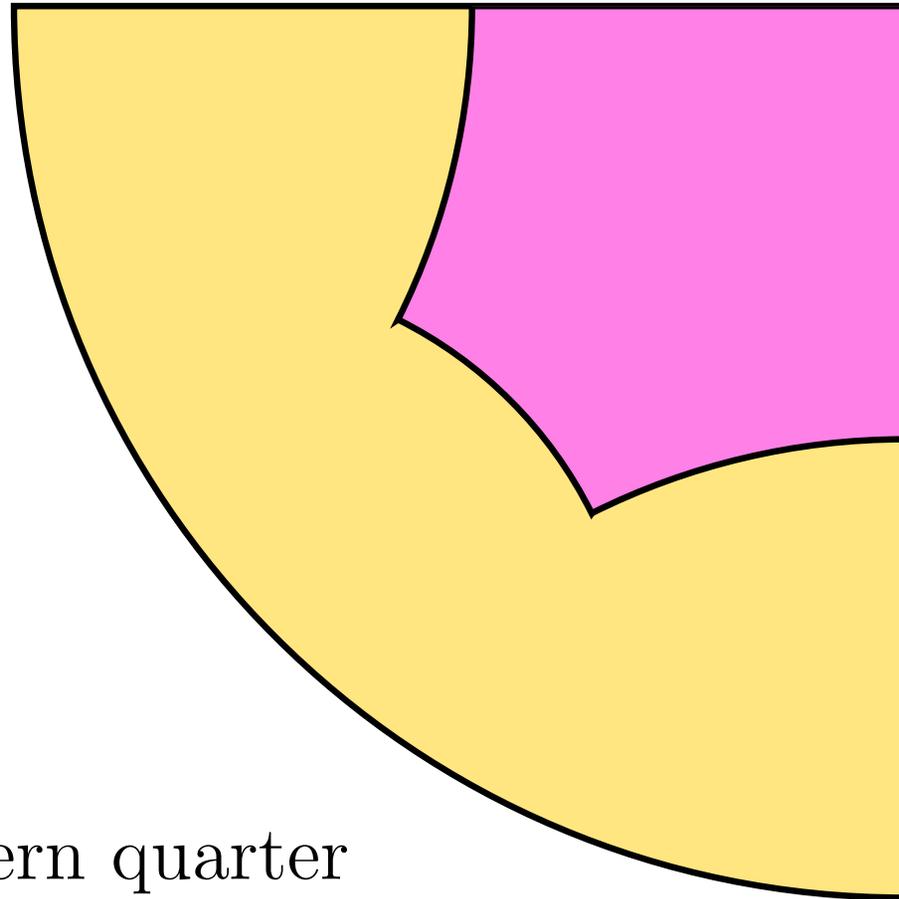
the simplest rectangular
grid, here restricted
to the South-Western quarter

splitting the pentagrid:



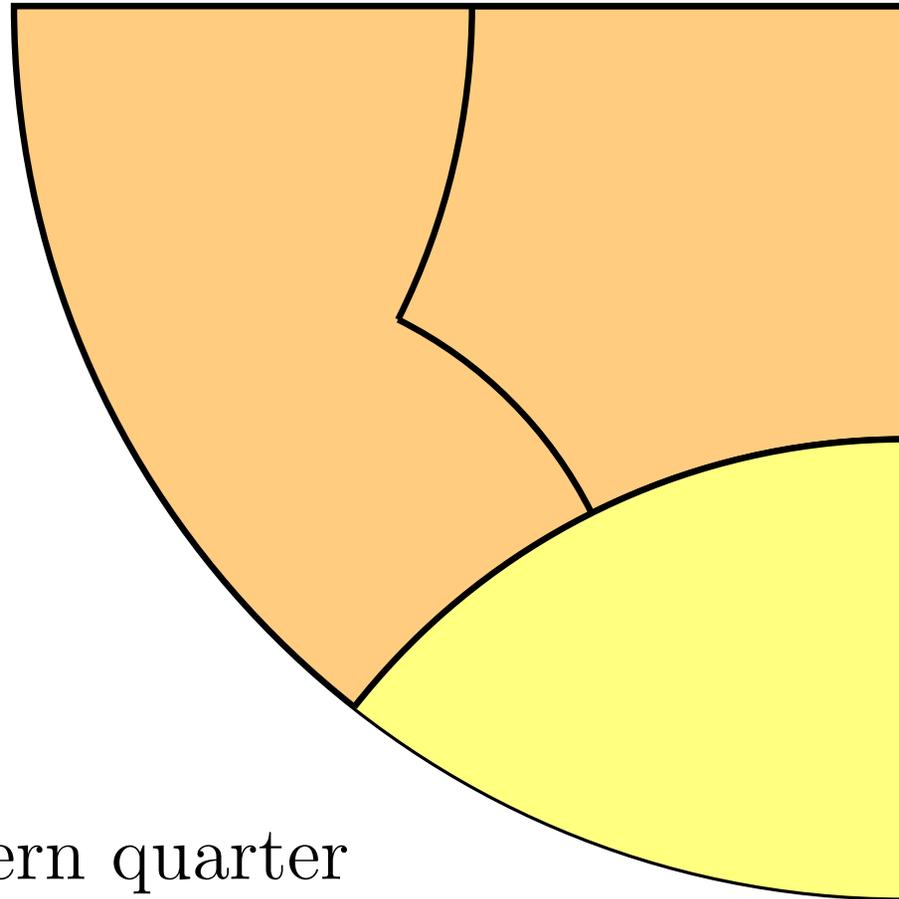
standard splitting
of the South-Western quarter

splitting the pentagrid:



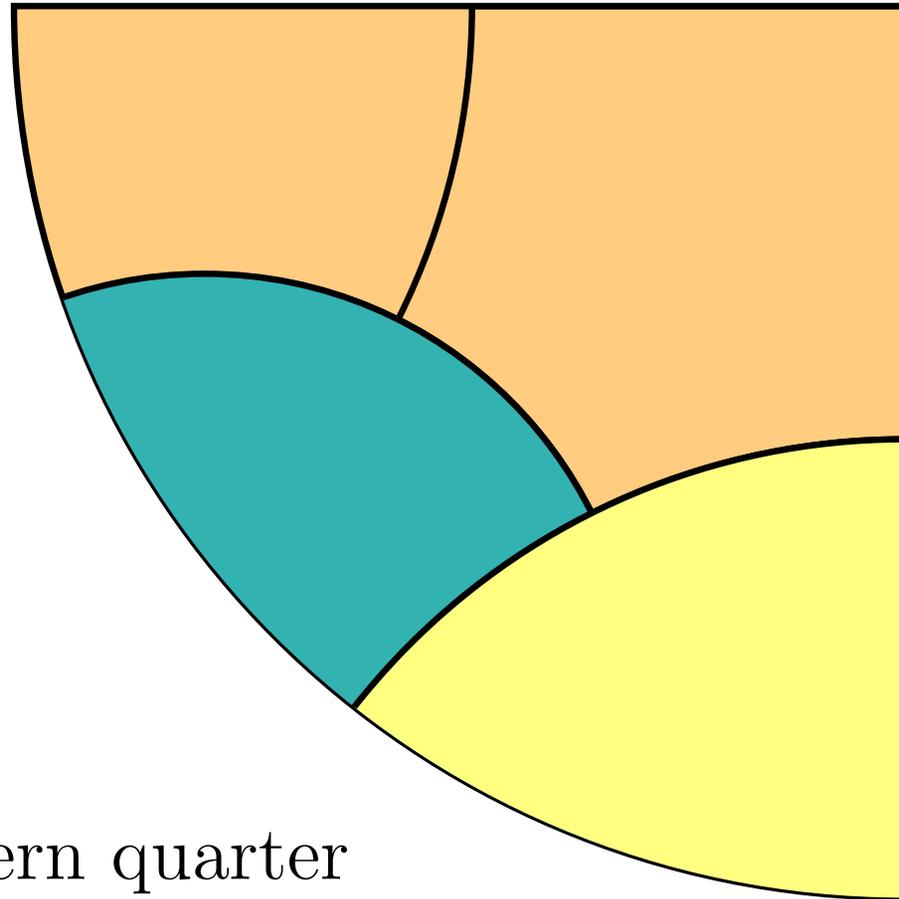
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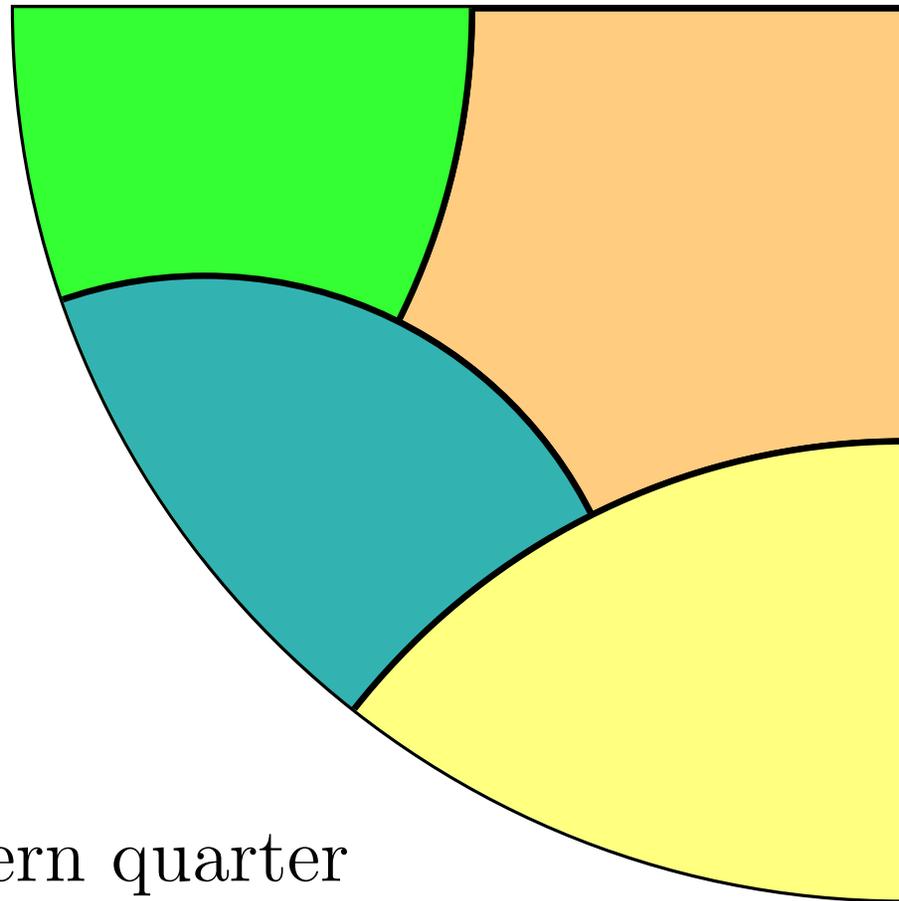
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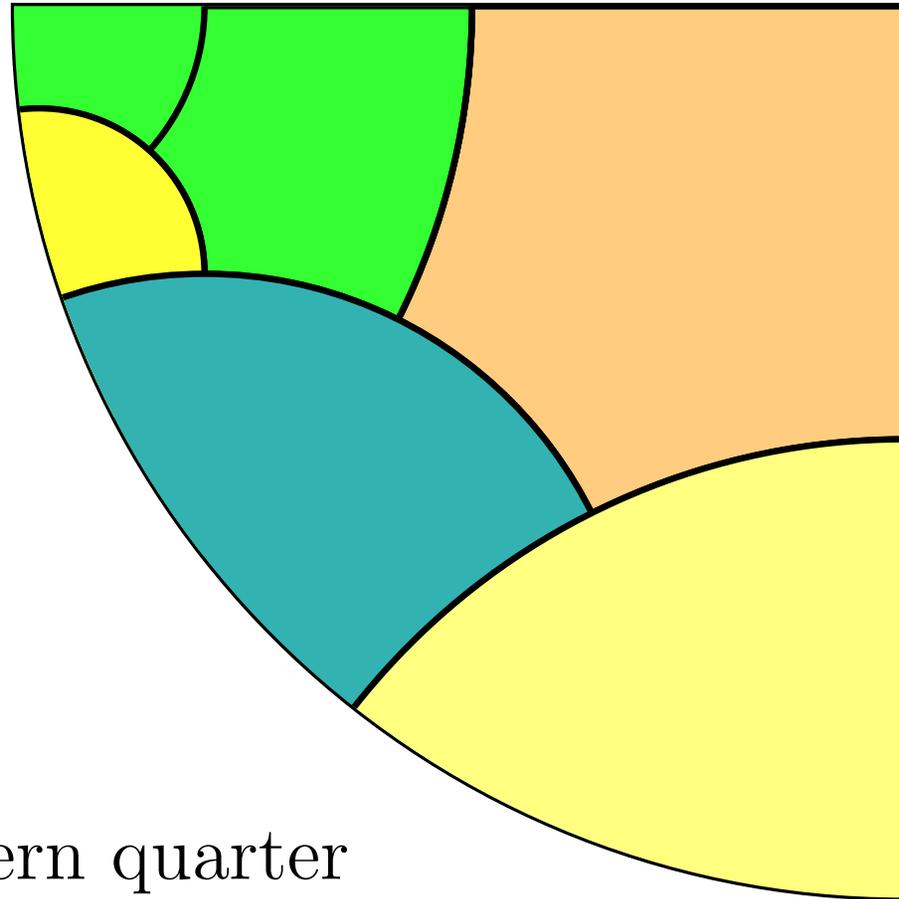
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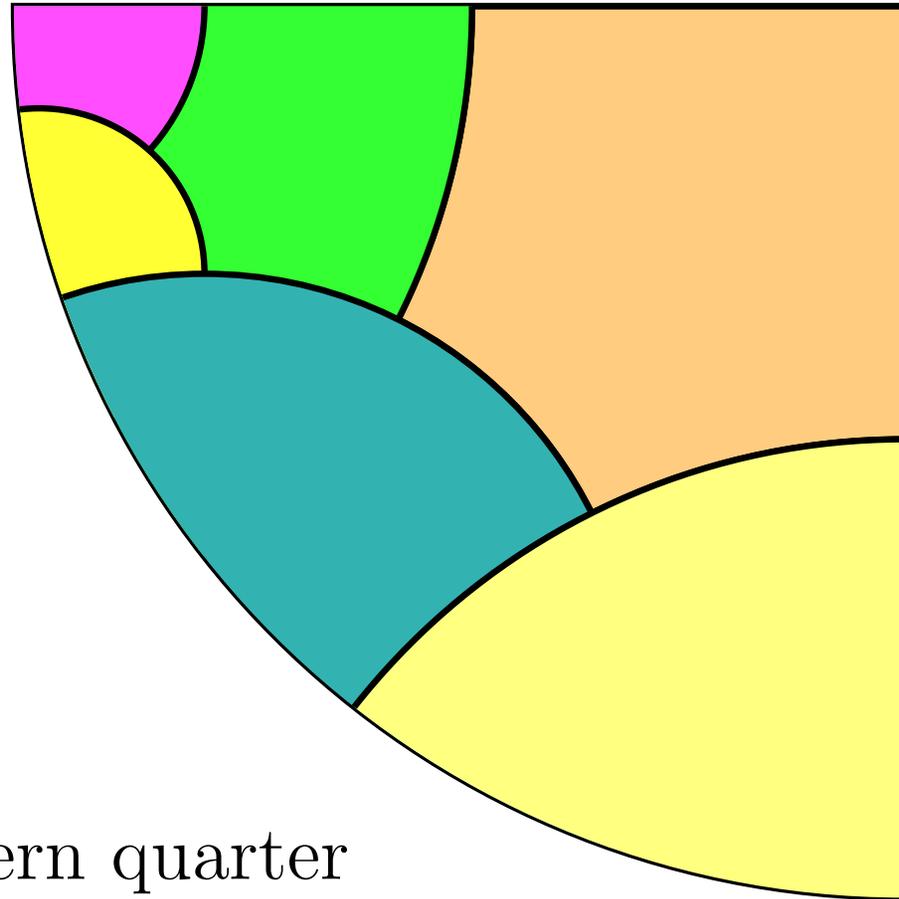
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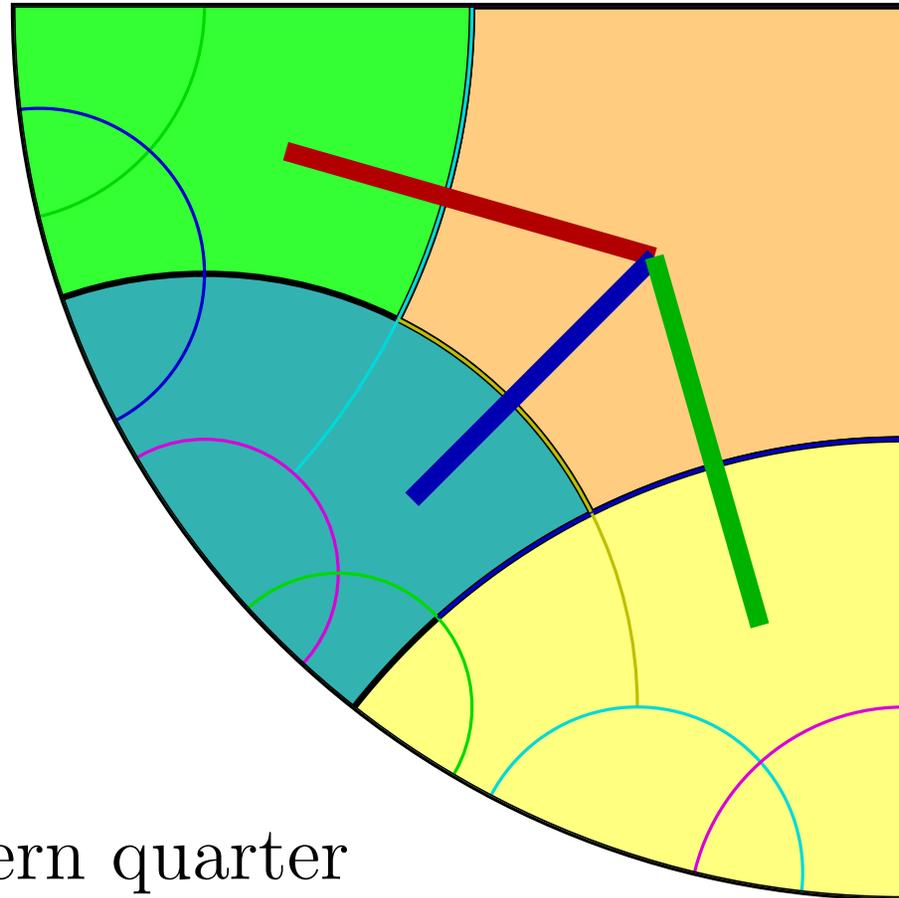
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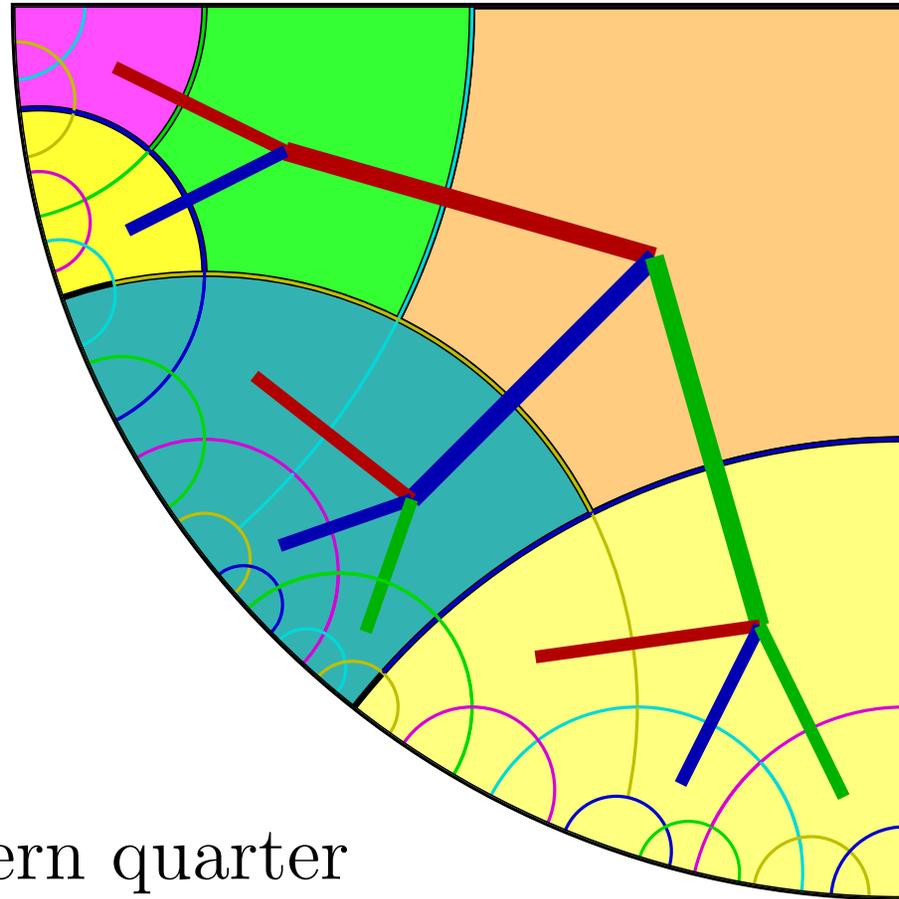
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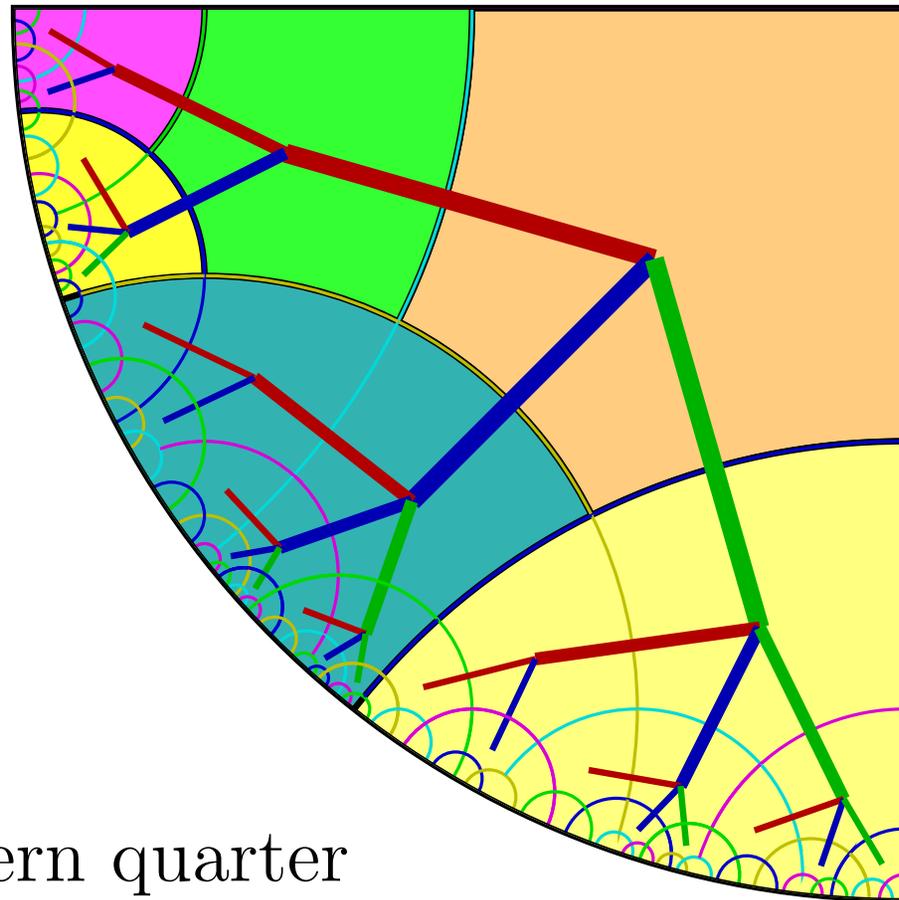
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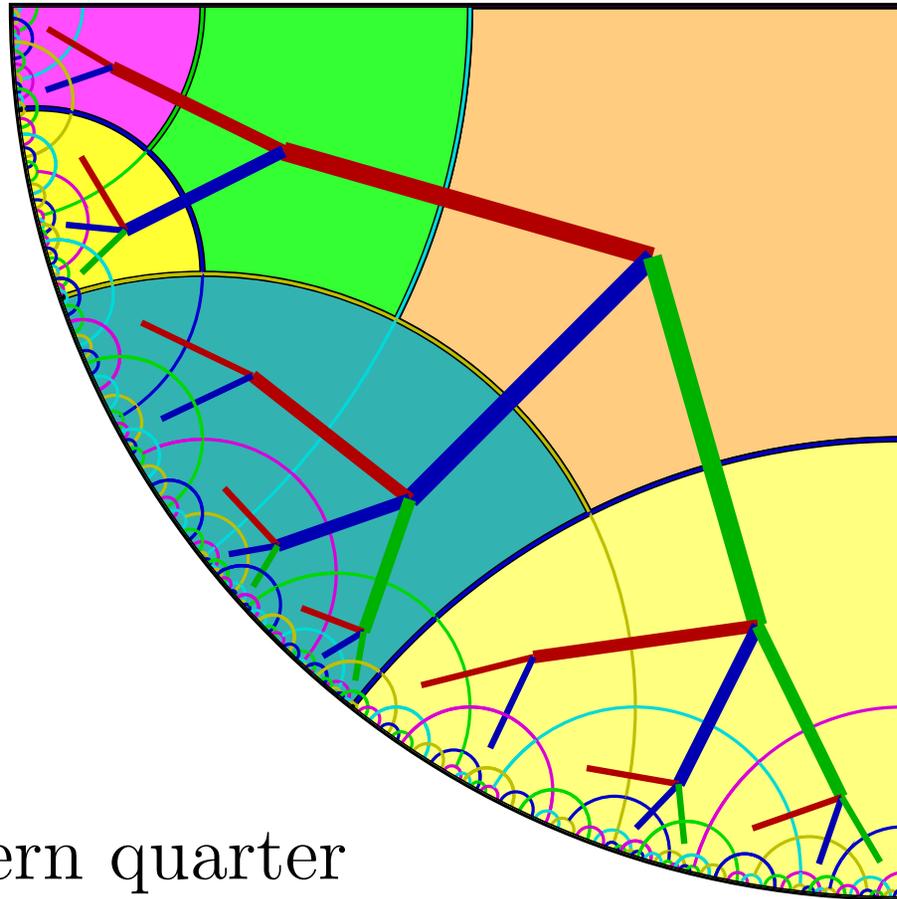
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splitting the pentagrid:



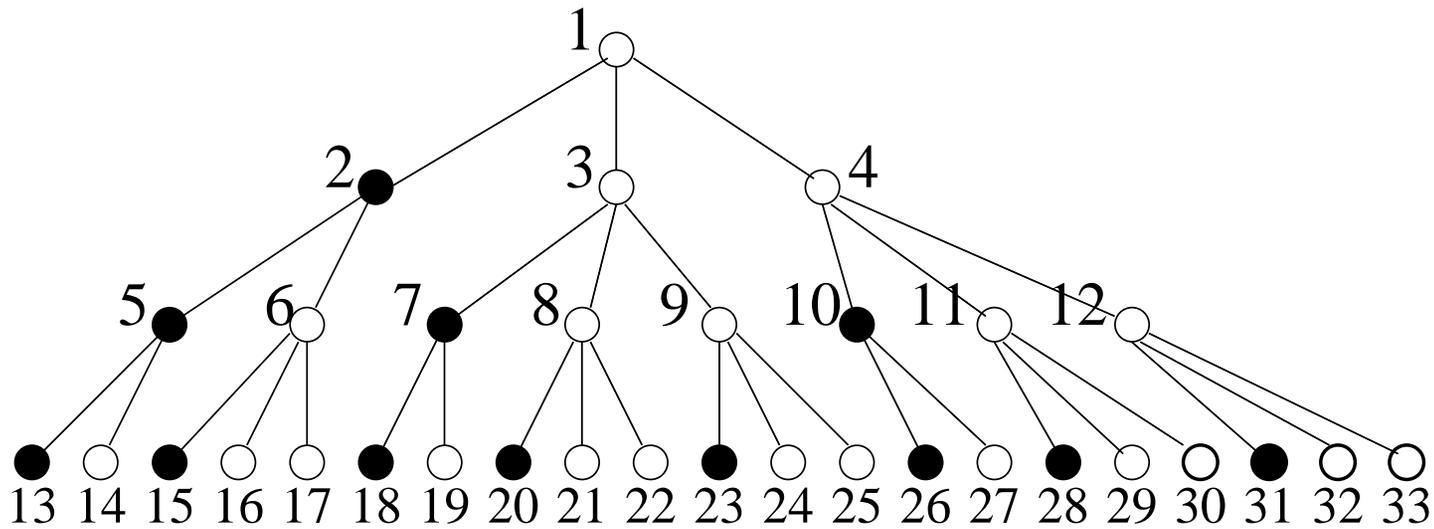
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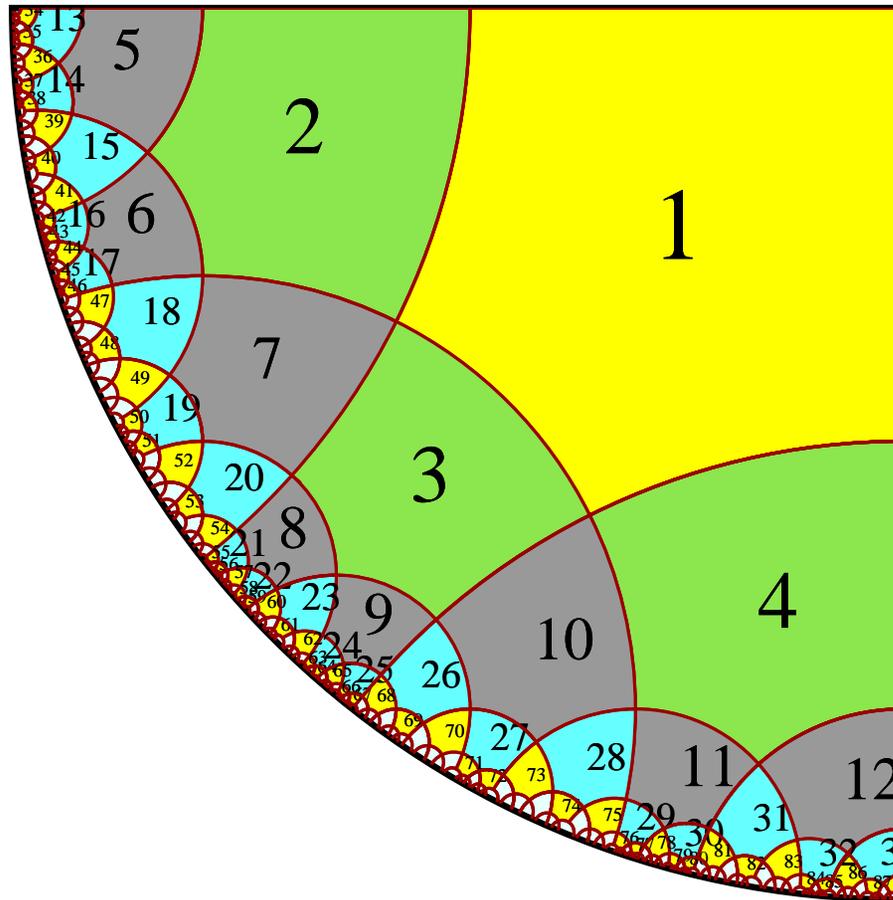
standard splitting
of the South-Western quarter

the tree, associated to the pentagrid



with its numbering

look at the numbering in the pentagrid



Fibonacci technology:

recall: for any number n : $n = \sum_{i=0}^k a_i f_i$,

where f_i : Fibonacci sequence

the representation is not unique;

uniqueness, obtained by a rule: 11 forbidden

this representation is called **coordinate** of n

the language of the coordinates is regular

the property of the preferred son:

let ν with $\alpha_k \dots \alpha_0$ as a coordinate
coordinates of the sons:

if ν is a 2node: $\boxed{\alpha_k \dots \alpha_0 00}$, $\boxed{\alpha_k \dots \alpha_0 01}$

if ν is a 3-node: $\boxed{\beta_h \dots \beta_0 10}$, $\boxed{\alpha_k \dots \alpha_0 00}$,
 $\boxed{\alpha_k \dots \alpha_0 01}$,

where $\beta_h \dots \beta_0$ represents $\nu-1$

the node with $\boxed{\alpha_k \dots \alpha_0 00}$ as a coordinate
is called the **preferred** son of ν

each node has one preferred node exactly

2. the splitting method:

(Margenstern, 2002):

basis of splitting:

finitely many sets S_0, \dots, S_k , unbounded simply connected subsets of \mathbb{H}^n and finitely many P_0, \dots, P_h , simply connected bounded subsets of \mathbb{H}^n , $h \leq k$, such that:

- (i) \mathbb{H}^n split into finitely many copies of S_0
(copy = isometric image)
- (ii) any S_i splits into a copy of some P_ℓ and finitely many copies of S_j 's

P_ℓ being detached: the **leading tile** of S_i

the spanning tree of the splitting

the root: the leading tile of S_0

assume level n being defined

level $n+1 := \emptyset$

then, for each node ν of level n :

let $C_j =$ region whose leading tile is ν

let $S =$ leading tiles of the copies of S_k 's
in the splitting of C_j

level $n+1 :=$ level $n+1 \cup S$

by induction: an infinite tree

combinatoric tilings

tiling \mathcal{T} is **combinatoric** if there is a basis of splitting such that the associated spanning tree is in bijection with the restriction of the tiling to S_0 , all the tiles of \mathcal{T} being copies of the P_ℓ 's

later on, a single generating tile $P = P_0$

matrix and polynomial of the splitting
if combinatoric tiling

then the spanning tree of the splitting:
 $k+1$ types of nodes: type i means S_i
moreover:

let $M_{i,j}$ = number of S_j 's in splitting S_i ;
 \Rightarrow then number of nodes of level n
when root of type i :
= sum of row $i+1$ in M^n

M is the **matrix of the splitting**
polynomial of the splitting:

characteristic polynomial P of M

the language of the splitting

let $u_n = \#\{\text{nodes on level } n \text{ of } \quad \}$

where \quad is the spanning tree

$\{u\}_n$ satisfies a recurrent equation induced by P

number the nodes of \quad from 1, starting from the root and then level by level

coordinate of node ν = maximal greedy representation of ν

language of the splitting = language of the coordinates

greedy representation in a basis

let $\{u_n\}_{n \in \mathbb{N}}$ be positive numbers with $u_0 = 1$,
 $u_n < u_{n+1}$ and $\limsup \frac{u_n}{u_{n+1}} < \infty$

let $b = \lfloor \limsup \frac{u_n}{u_{n+1}} \rfloor$

then $n = \sum_{i=0}^k \alpha_i u_i$ with $\alpha_i \in [0..b]$

maximal greedy representation :

when k maximal, it is unique

results: the following tilings are **combinatoric**:

- \mathbb{H}^2 : $\{5, 4\}$: pentagrid (MM-KM, MM)
 $\{s, 4\}$: s -gons with right angles (MM-GS)
 $\{p, q\}$: p -gons with angle $\frac{2\pi}{q}$ (MM-GS),
in progress
most of cases of Poincaré's theorem
(MM 2002, see later)
 $\{\infty, q\}$: * (MM) (a bit special)
- \mathbb{H}^3 : $\{5, 3, 4\}$: dodecahedra with right angles
(MM-GS, 2002)
- \mathbb{H}^4 : $\{5, 3, 3, 4\}$: the 120-grid (MM, 2003)

3. 3D hyperbolic space \mathbb{H}^3

the space: the open unit ball ;

unit sphere Σ == points at infinity

planes: trace of spheres S with $S \perp \Sigma$
and limit case of diametral planes

\Rightarrow a *line* of points at infinity

when non empty, intersection of 2 planes are
trace of a circle C in a diametral plane Π
with $C \perp \Sigma \cap \Pi$

hence two common points at infinity
defining the common line

limit case, parallel planes:

one common point, at infinity

lines: intersection of secant planes

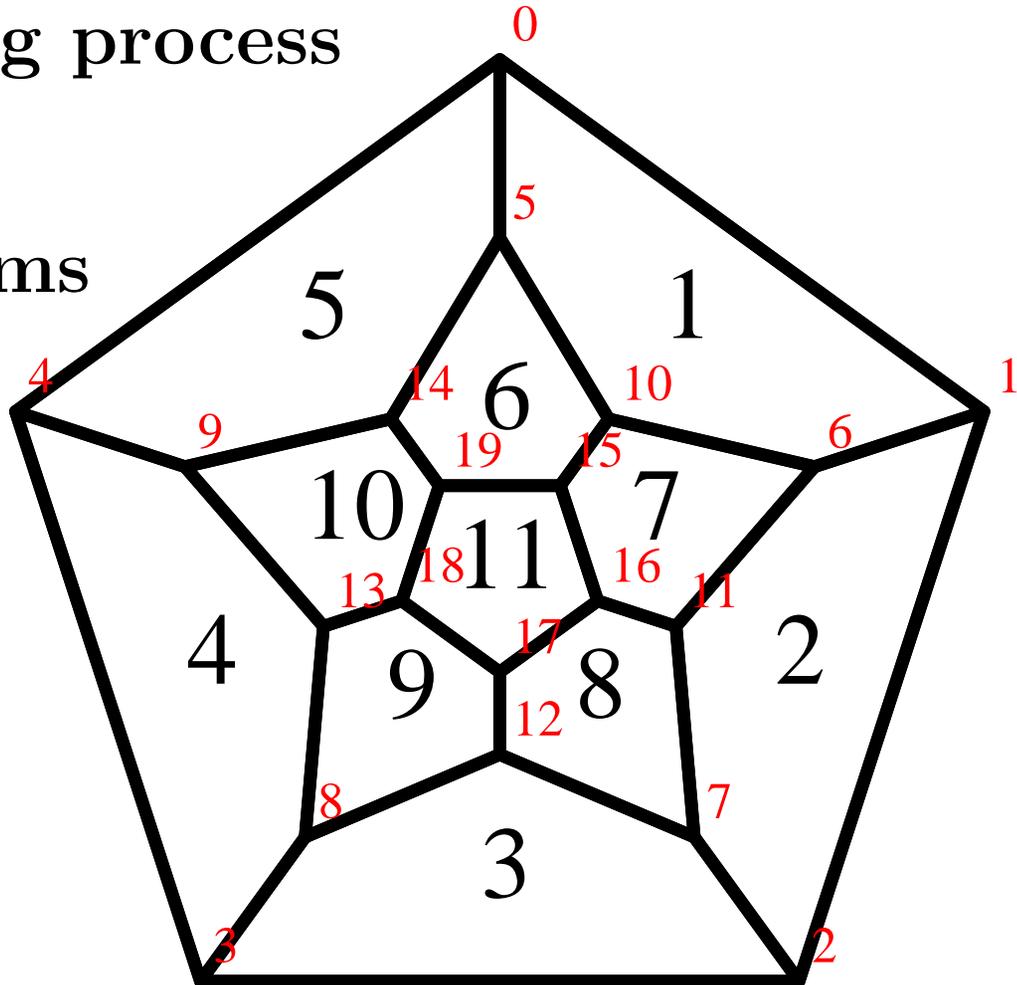
parallel planes: the lines at infinity meet
on the common point which is on Σ

when Π diametral plane, $\mathbb{H}^3 \cap \Pi = \mathbb{H}^2$,
otherwise a copy of \mathbb{H}^2

\mathbb{H}^3 , rectangular dodecagrid:
the splitting process

using an old tool:
Schlegel diagrams

projection
on the bottom
(0)
from below
seen from above

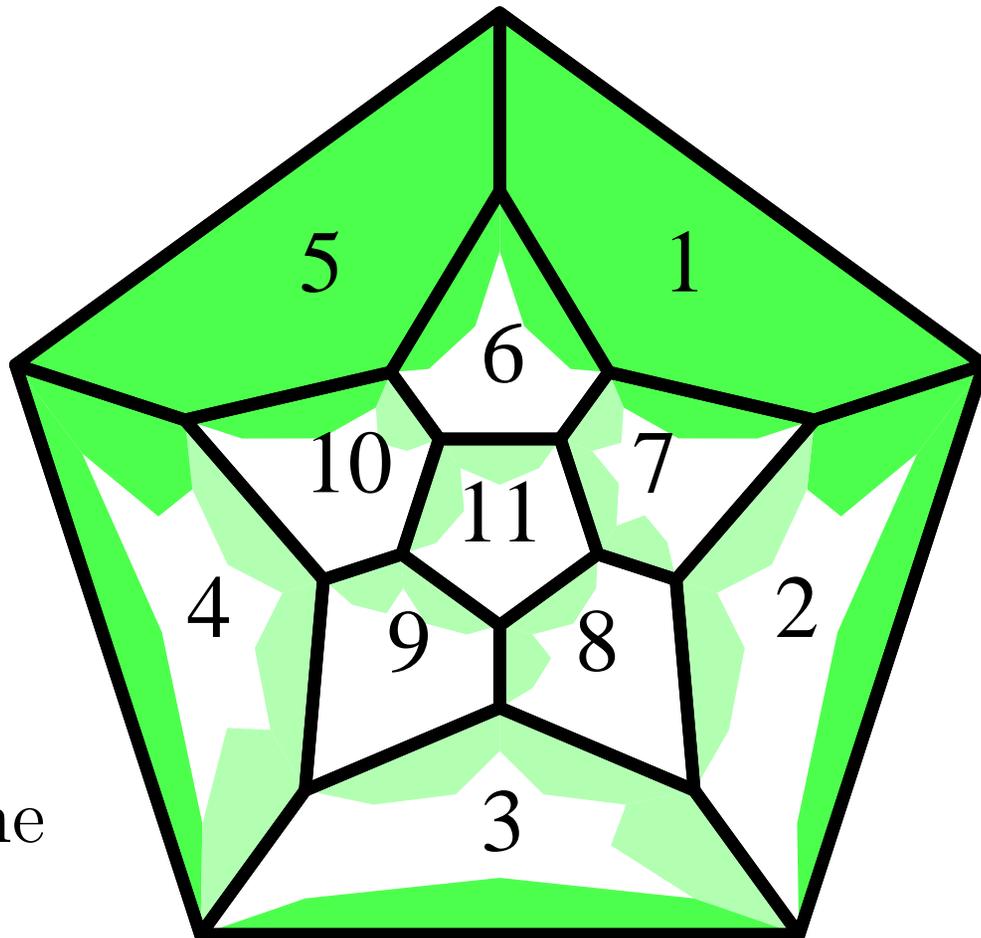


Schlegel diagram
represents both:

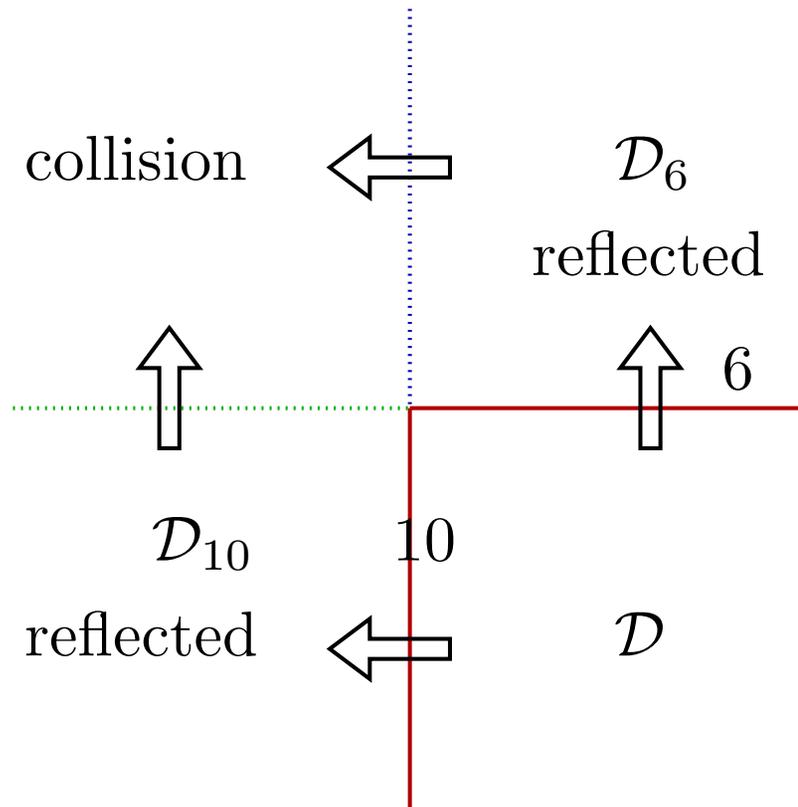
- a dodecahedron
- a corner of \mathbb{H}^3

green and blue
colours represent
a **wall**:

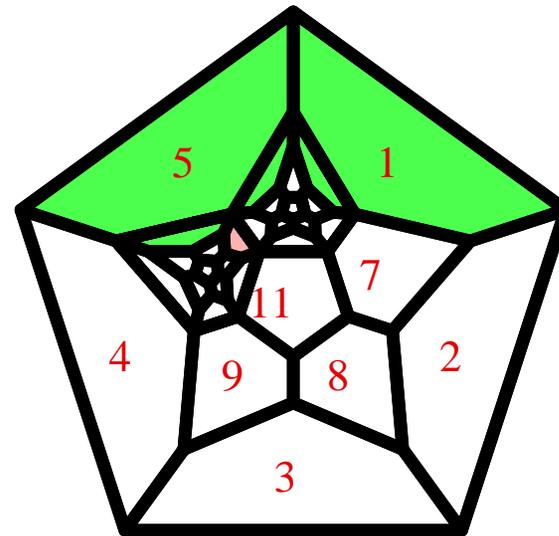
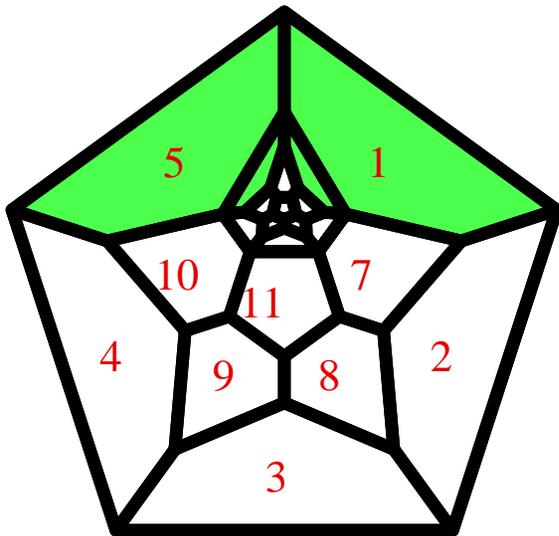
i.e. forbidden to
reflect in the plane
of the wall

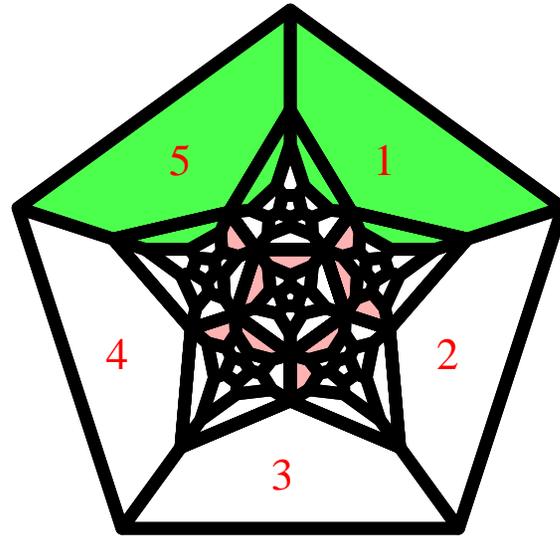
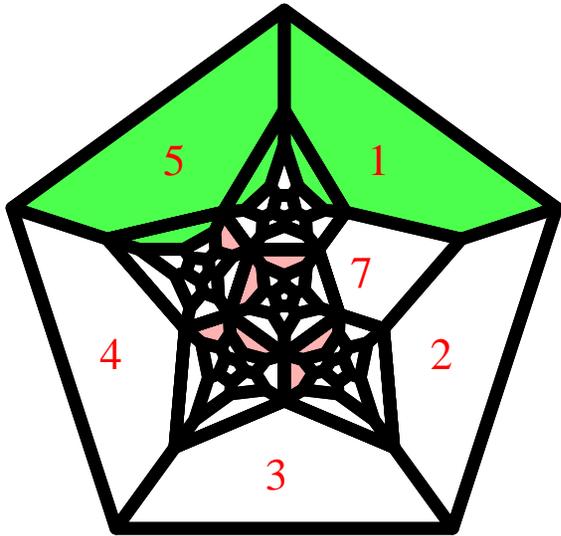
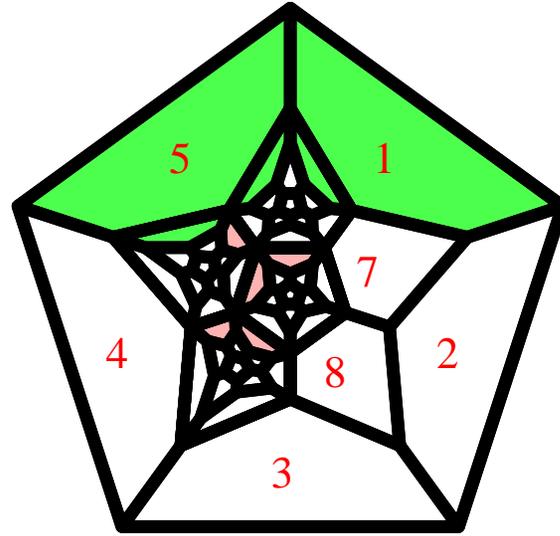
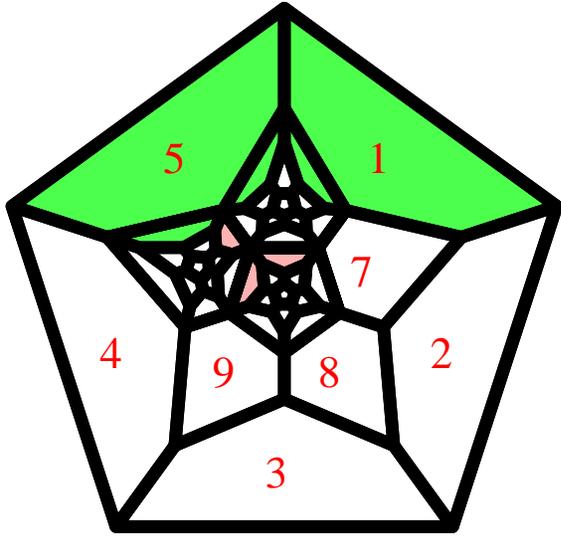


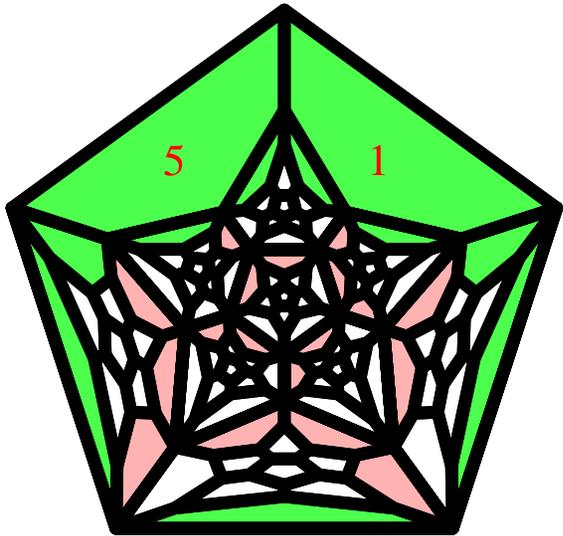
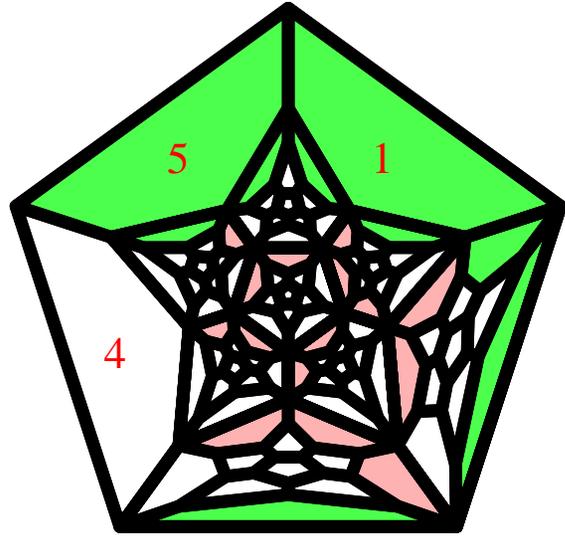
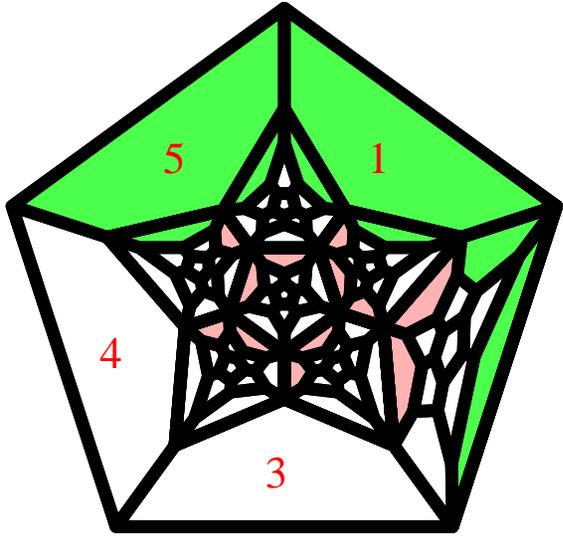
why the shadowing principle:



the splitting process:





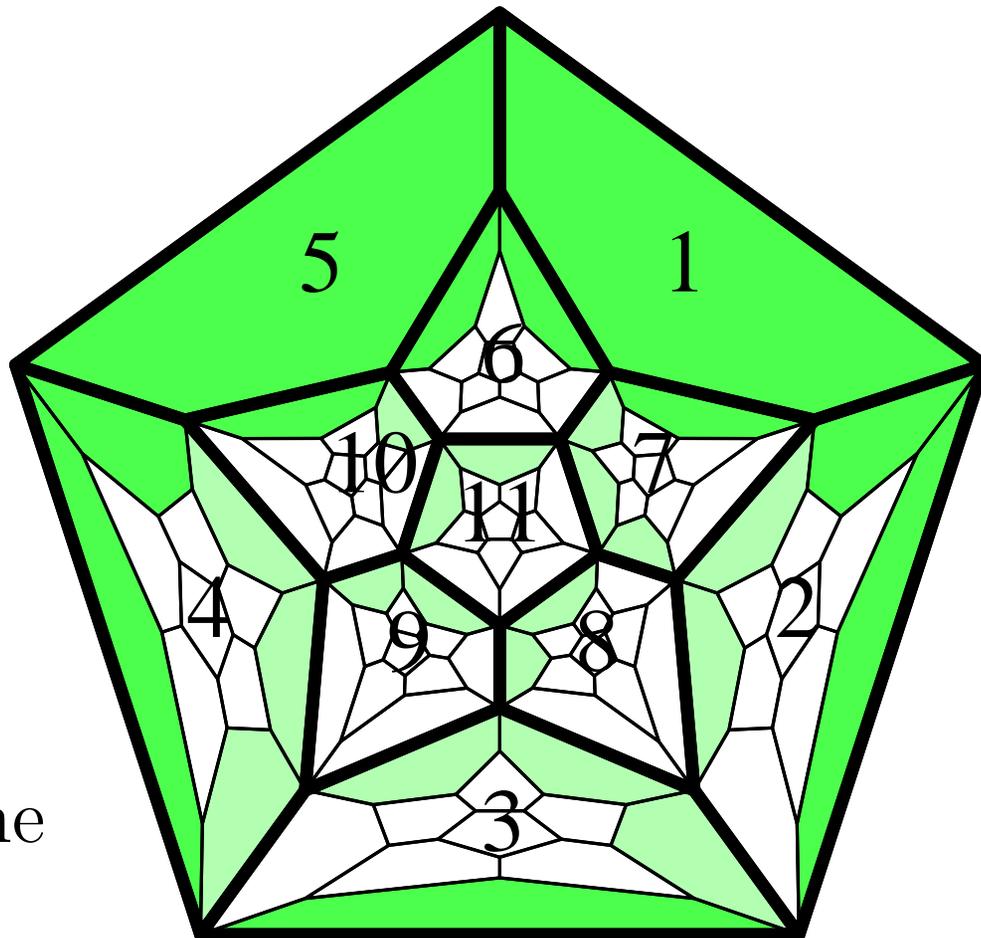


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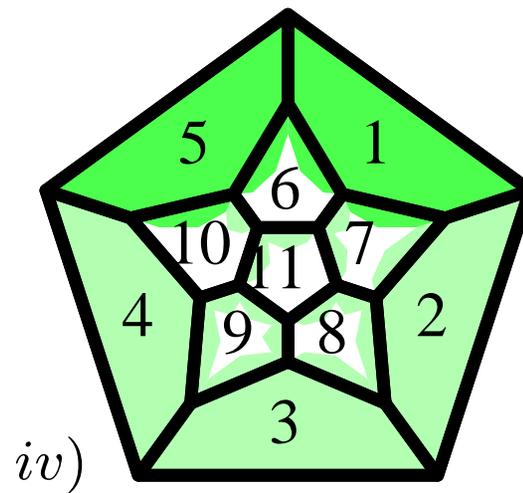
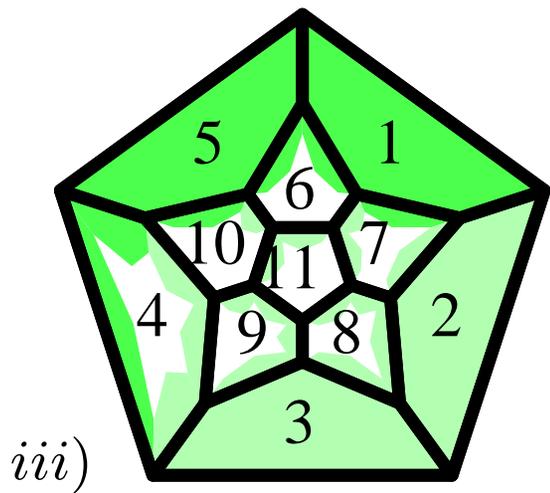
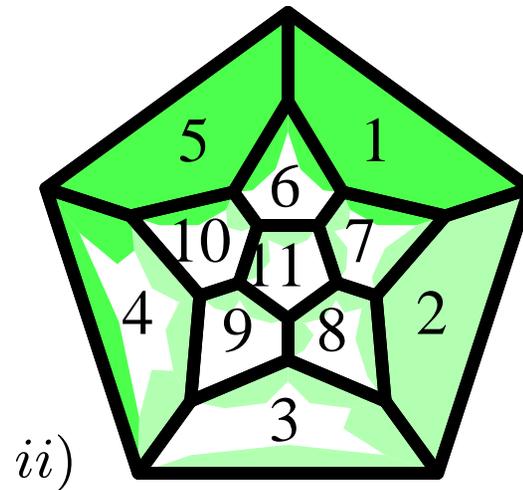
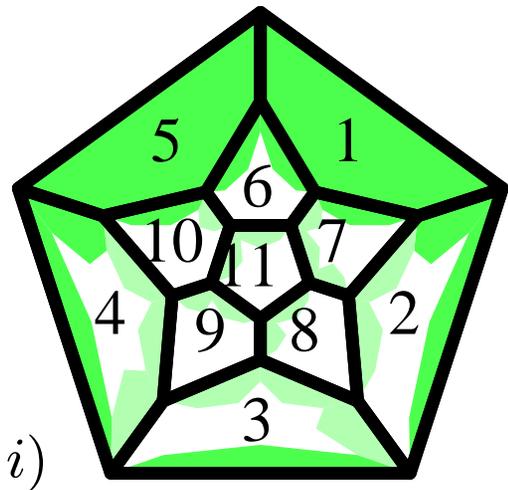
- a dodecahedron
- a corner of \mathbb{H}^3

green and blue
colours represent
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i.e. forbidden to
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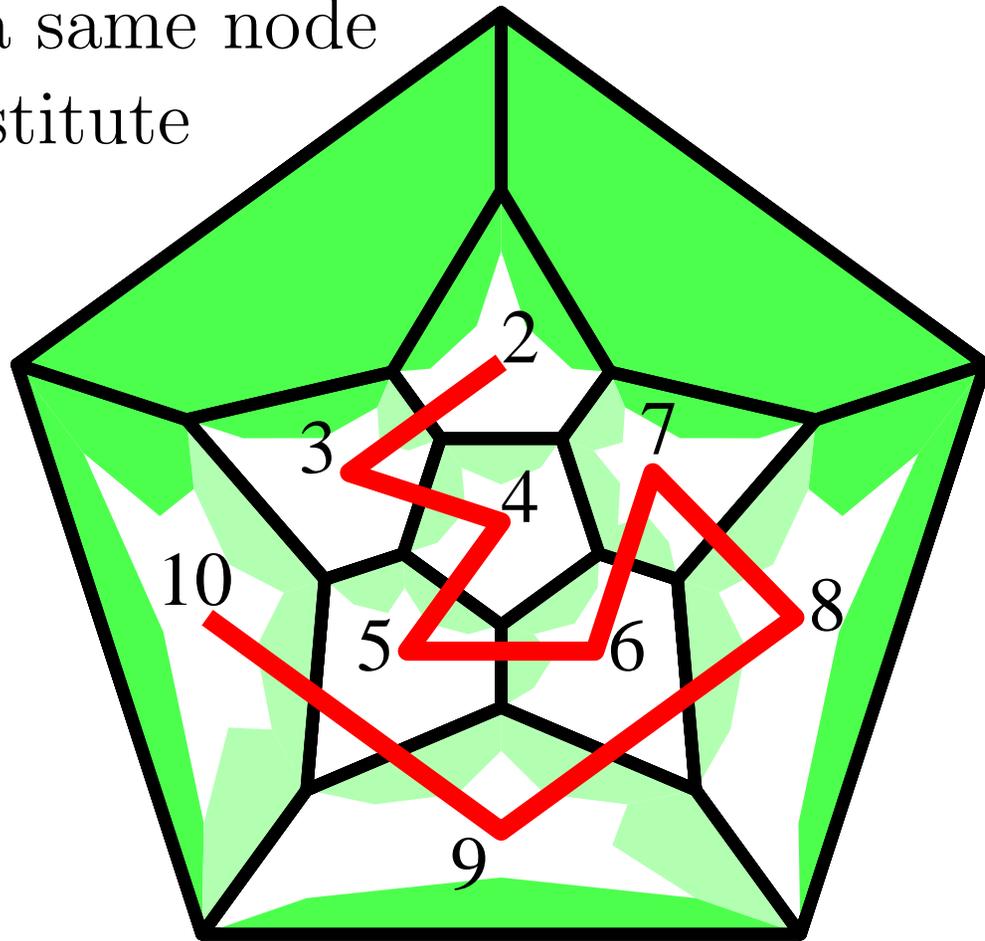


splitting a corner of \mathbb{H}^3 :

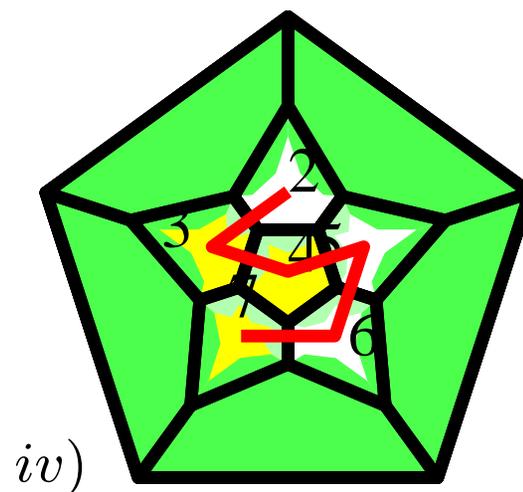
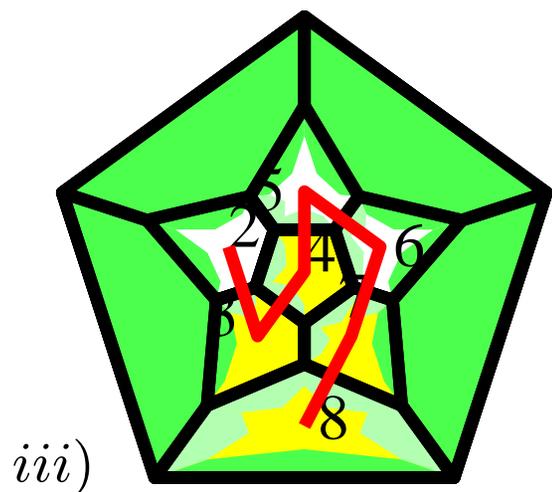
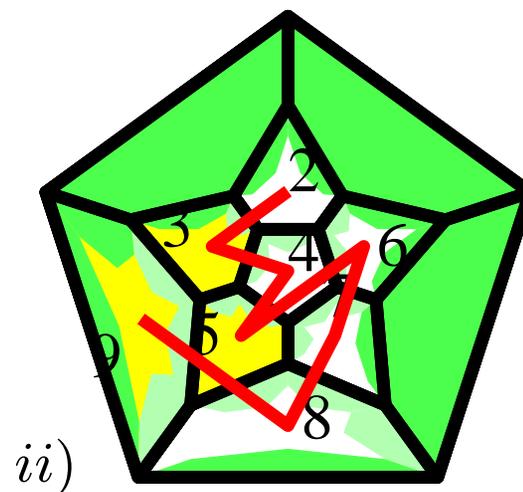
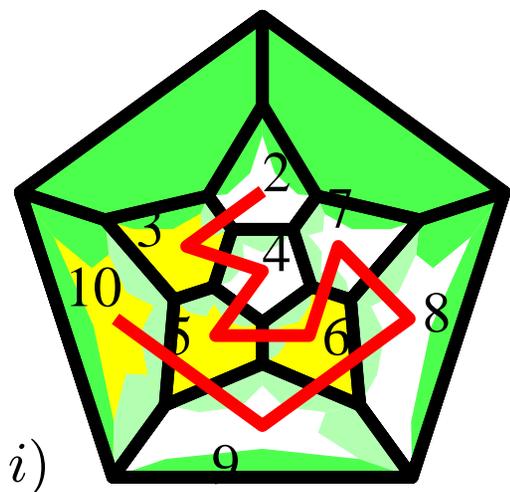


linking the different partial splittings:

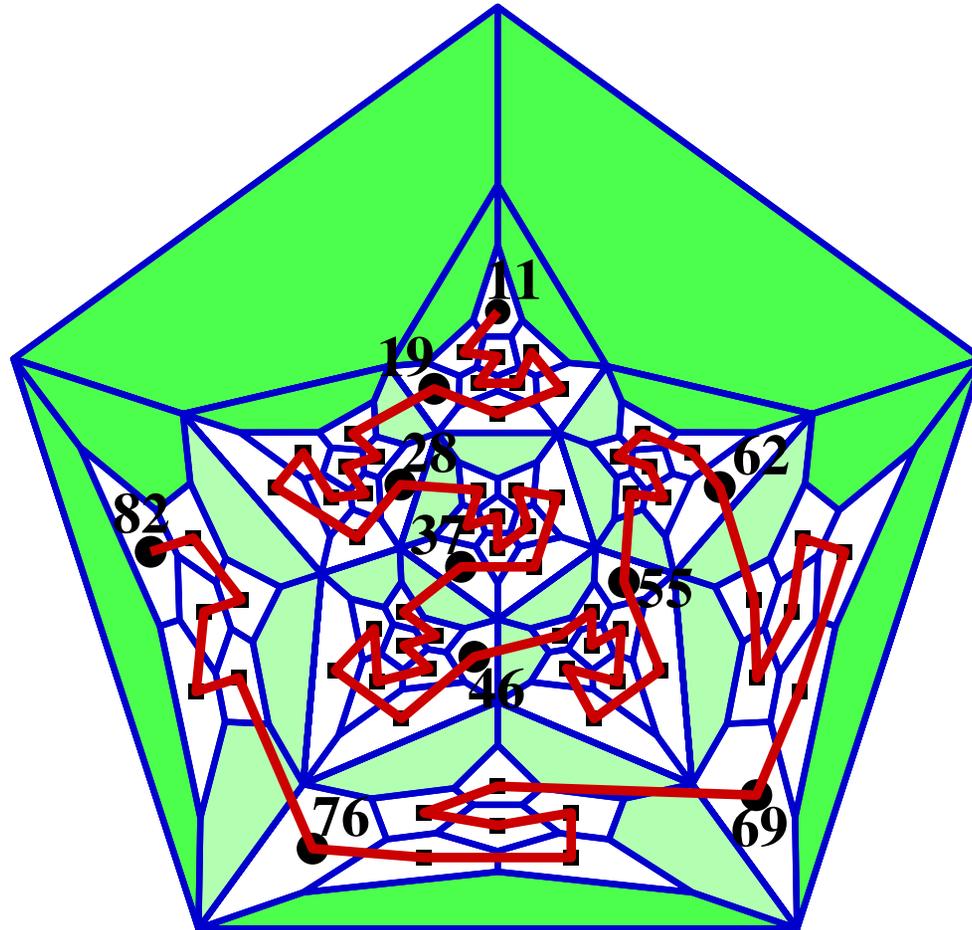
for the sons of a same node
allowing to constitute
local maps



linking the different partial splittings:



same as previously on two generations:



3. implementing cellular automata in the pentagrid

an answer to the problem of privileged direction:

take five trees around a central cell

in each tree, the tree structure provides a **locally privileged** direction:

for each node, that of the **father**

exception with the central cell

also solutions where there is no central cell:

absolutely any cell has a father (book I)

an answer to the coordinate problem:

applying the technique of the splitting method:

numbering the nodes of the spanning tree of
the splitting

then writing the numbers in their maximal
representation in the basis defined by the
greatest real root of the polynomial of the
splitting

definition of the neighbourhood:

ball of radius k around the cell:

path between two tiles T_1 and T_2 :

sequence $\{V_i\}_{i \in [1..n+1]}$, with $V_1 = T_1$,
 $V_{n+1} = T_2$, V_i and V_{i+1} sharing an edge for
 $i \in [1..n]$

distance between two tiles T_1 and T_2 :

length of a **shortest** path between T_1 and T_2

transition function:

local numbering:

number the edges of a tile:
counter clock-wise from 1 up to α , $\alpha = 5$ or
 $\alpha = 7$

n_i is the neighbour sharing the edge i
by definition, n_1 is the father

format of a rule:

$$\eta_1, \eta_2, \dots, \eta_\alpha, \eta \rightarrow \eta'$$

an interesting example:

the propagation of the Fibonacci tree structure

key point for the coordinate system, the Fibonacci tree structure can be installed by a rotation invariant CA

basic notion: **the extended status of a tile**

status of a node == number of its sons

Black node: 2 sons, **White** node: 3 sons

extended status = status + that of the father

values of the extended status:

black node: Bb, Bw

white node: Wb, W_{wm}, W_{wr}

principle of the propagation in the pentagrid:

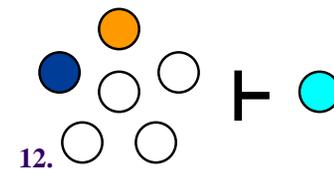
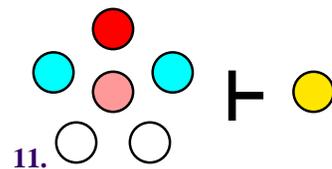
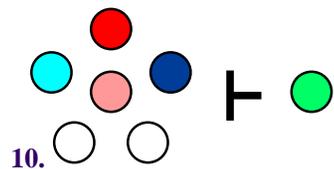
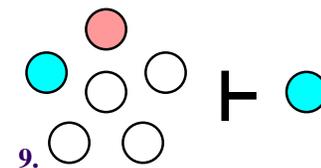
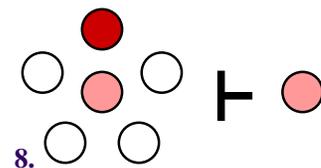
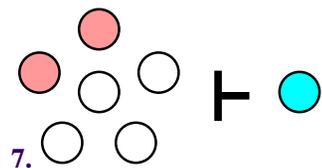
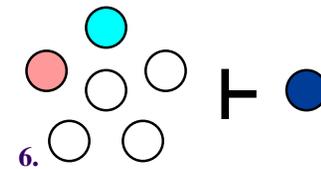
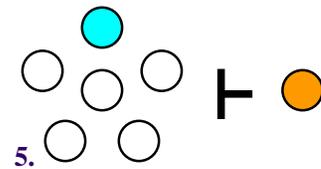
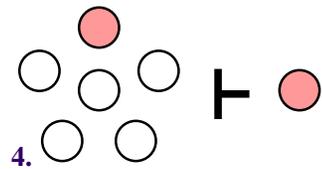
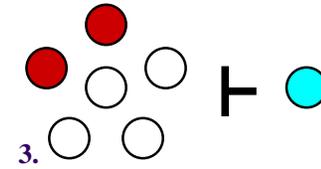
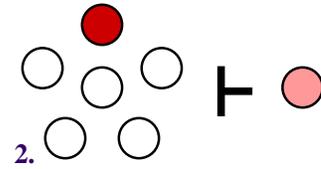
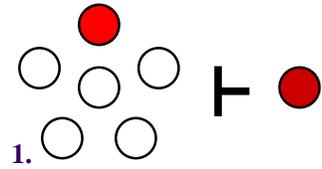
the central cell, special, initialize the five roots

when a blank cell sees one non-blank cell, it is
white

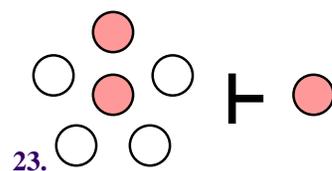
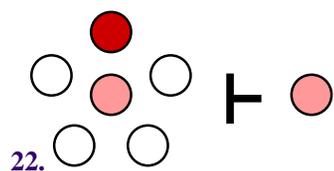
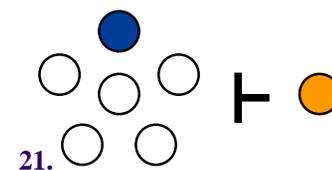
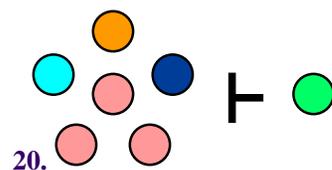
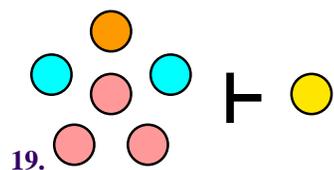
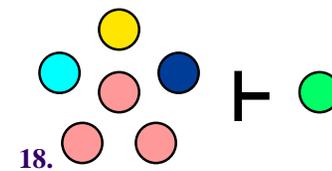
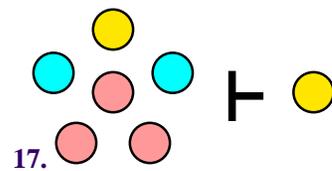
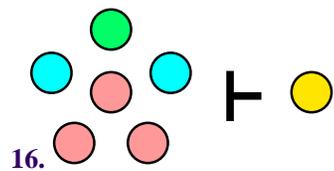
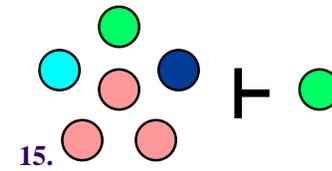
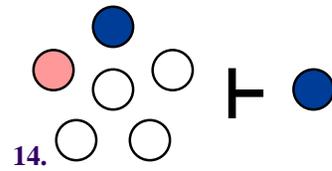
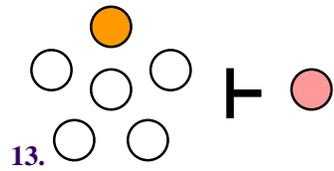
when a blank cell sees two non-blank cells, it is
black

tuning for the white cells: extended status in two
steps

the rules for the pentagrid



the rules for the pentagrid



4. universality for cellular automata in hyperbolic spaces

results :

a 22 state (weakly) universal CA in the pentagrid (FH-MM, TCS, 2002).

a 5 state (weakly) universal CA in the dodecagrid (hyperbolic 3D, MM, JCA, 2006)

a 5 state (weakly) universal and reversible CA in the hyperbolic plane (KI-ChI-KM, JCA, 2006)

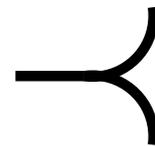
using a railway simulation

known from Ian Stewart, 1994

circuit simulating a Turing machine (IS) or
a register machine (MM) based on tracks,
switch points, crossings and a unique loco-
motive

the switch points

three kinds of points:



fixed switch

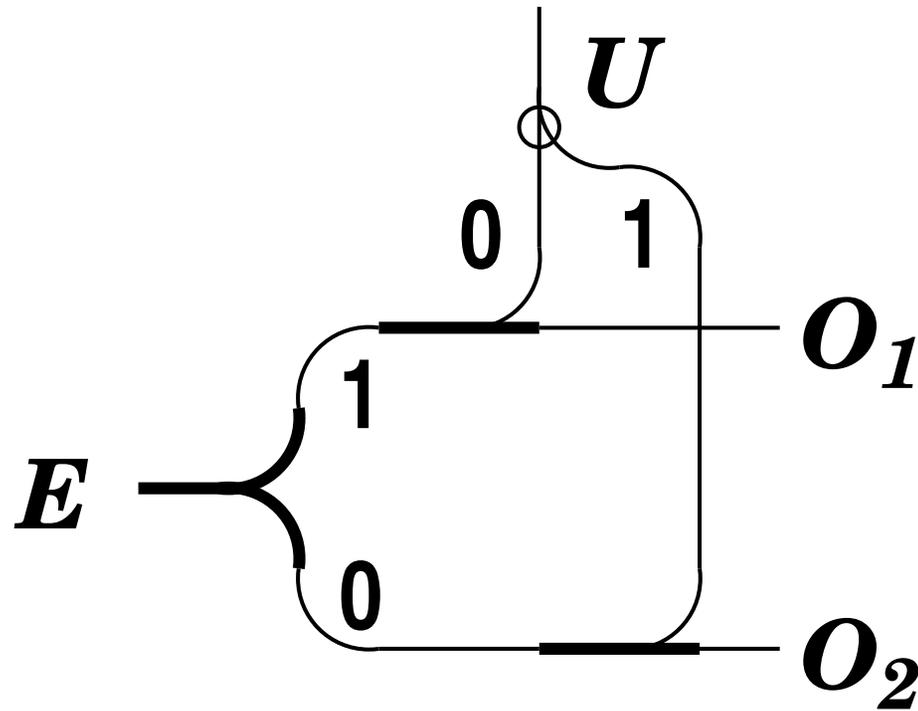
flip-flop switch

memory
switch

flip-flop is one-way: active crossing only and
after, change of the way marked for exit

memory switch: the way of passive crossing indi-
cates the next way of exit for active crossing

the basic element

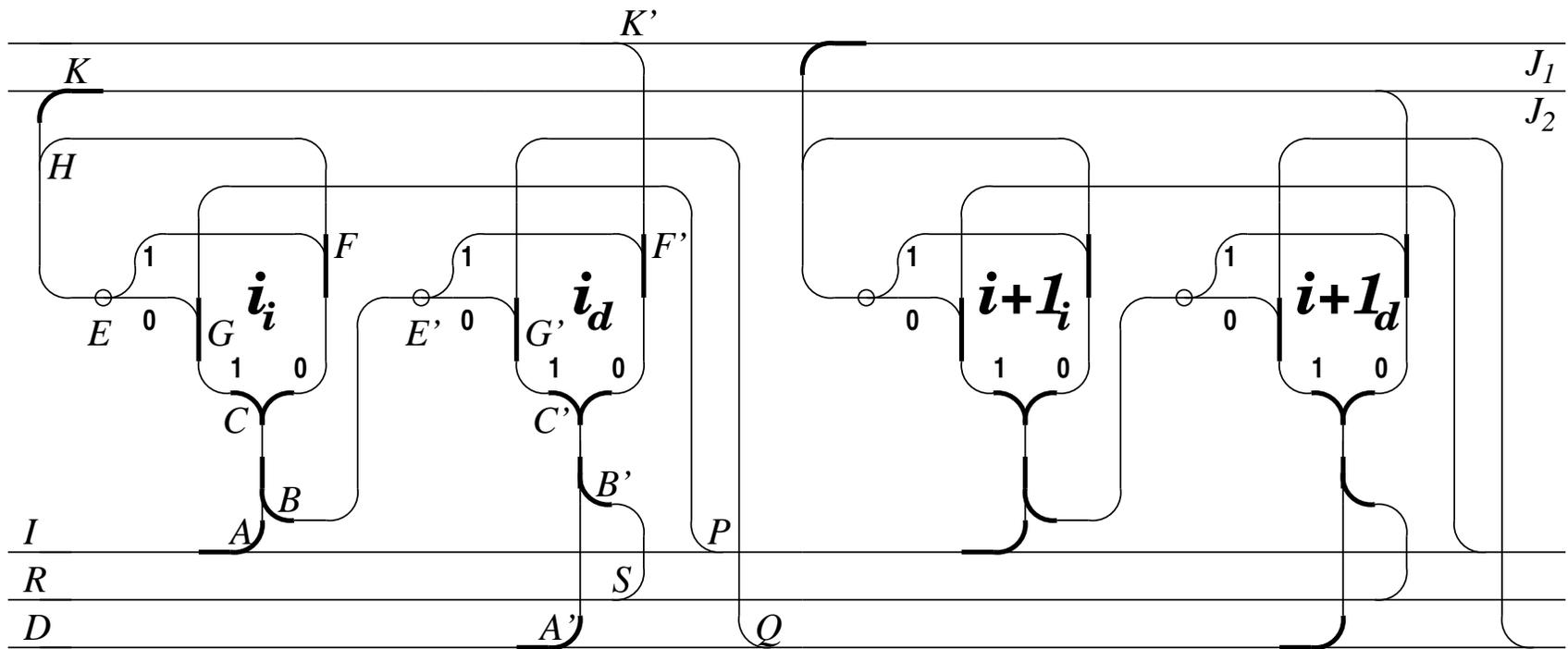


E is the **reading** entry

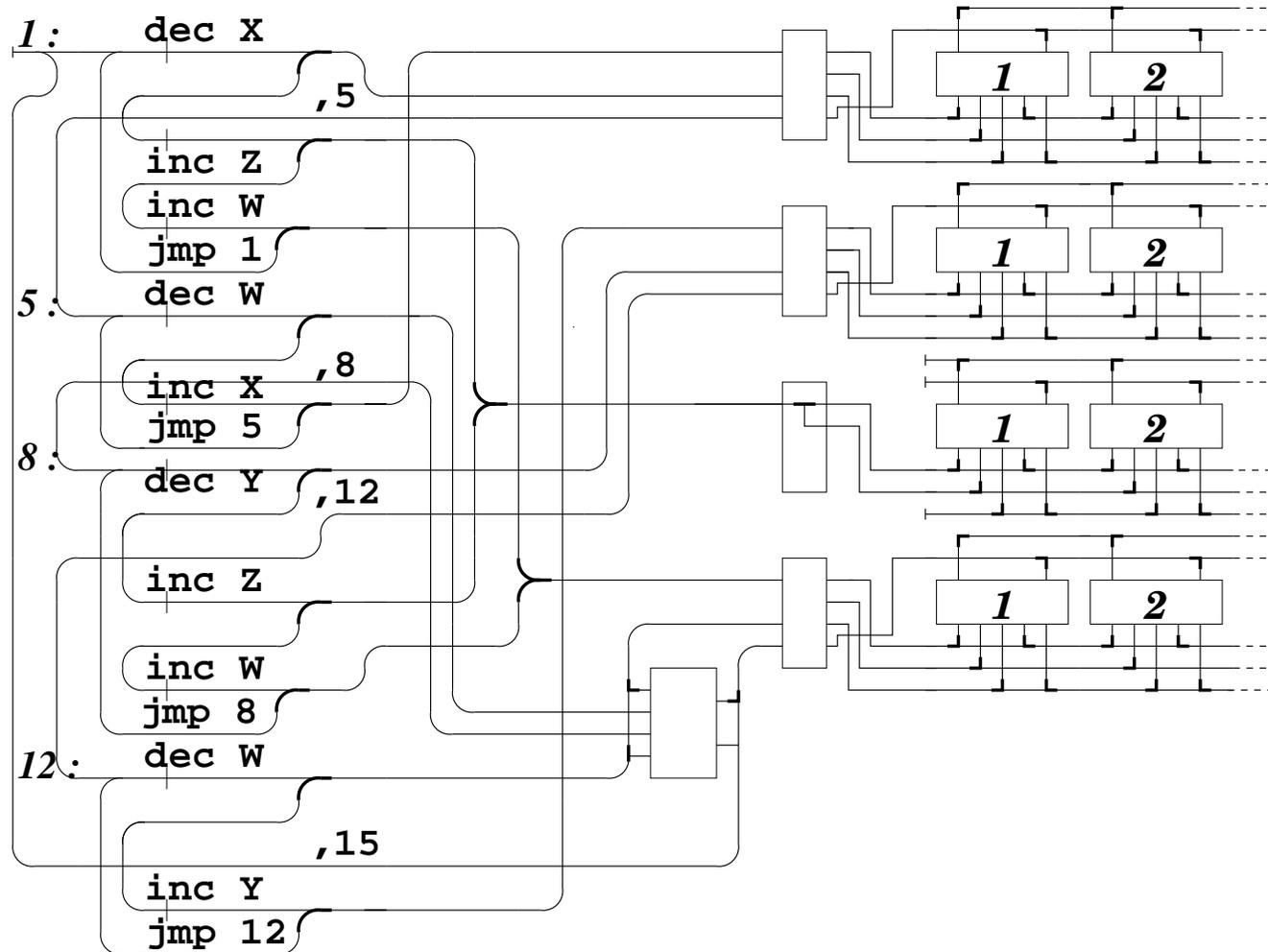
U is the **writing** entry: it changes the content of the element

combining elements allows us to build registers or Turing tape

the unit circuit:



an example :



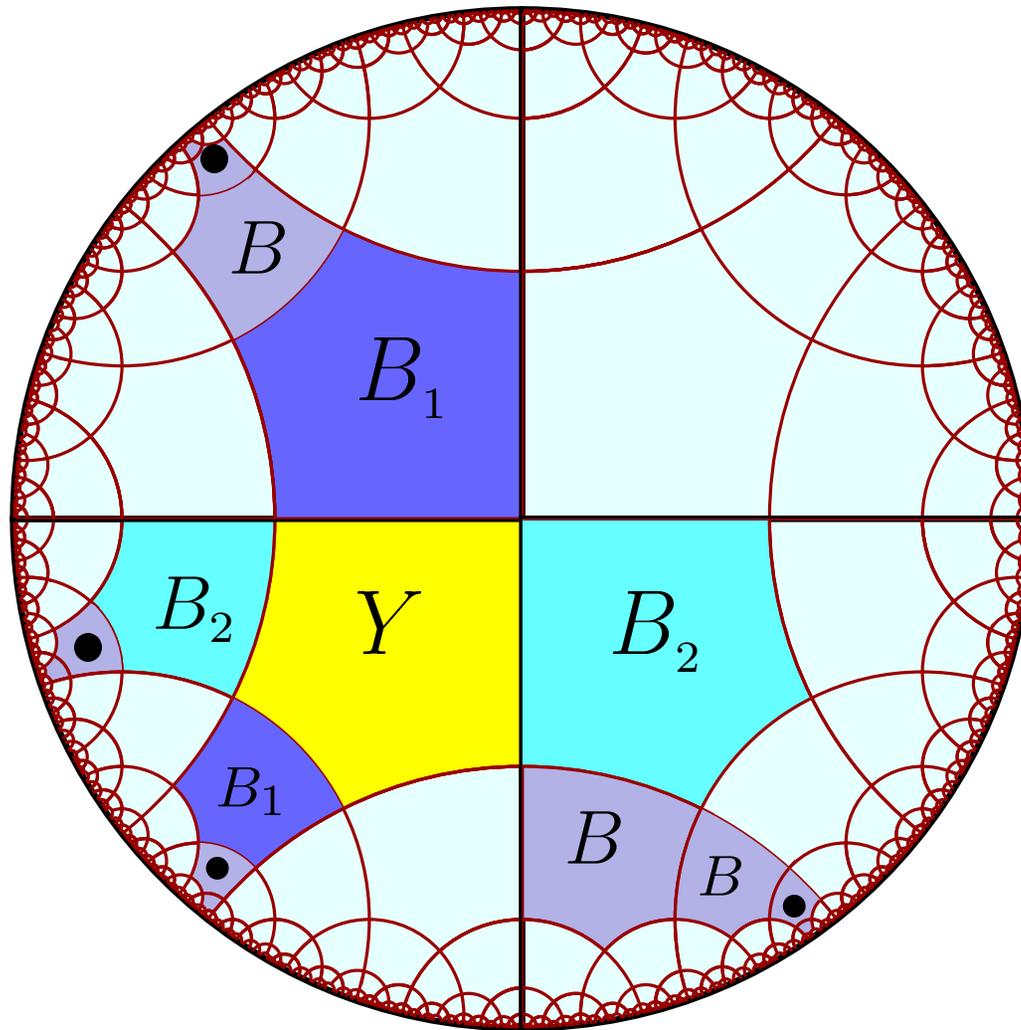
Implementation of such circuits:

in the pentagrid:

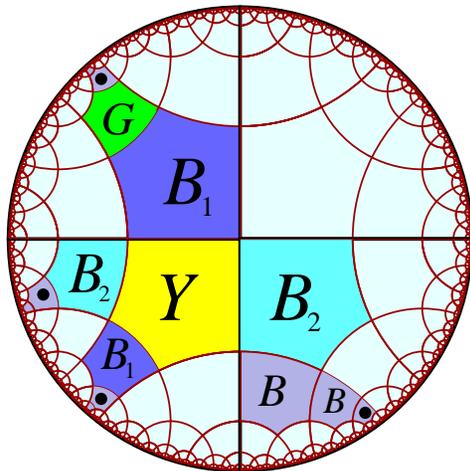
Herrmann-Margenstern (TCS 2003)

solution with 22 states

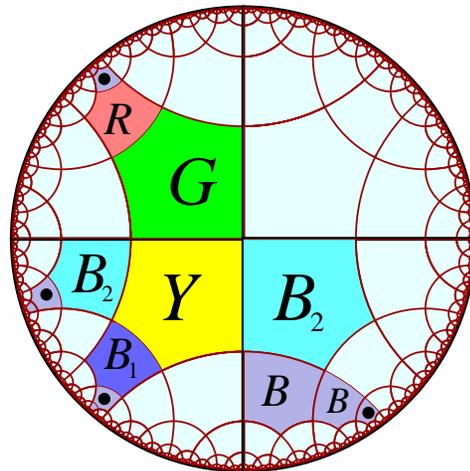
example of the stable configuration at a crossing:



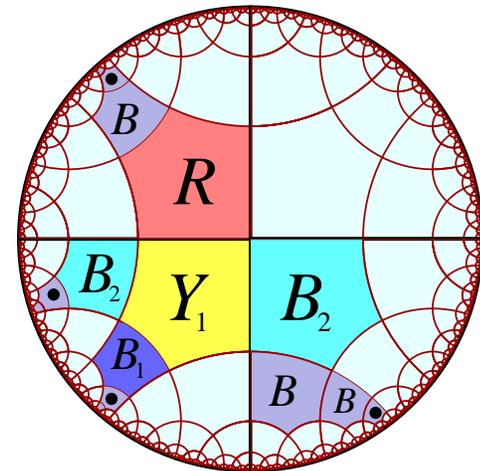
locomotive going through a crossing:



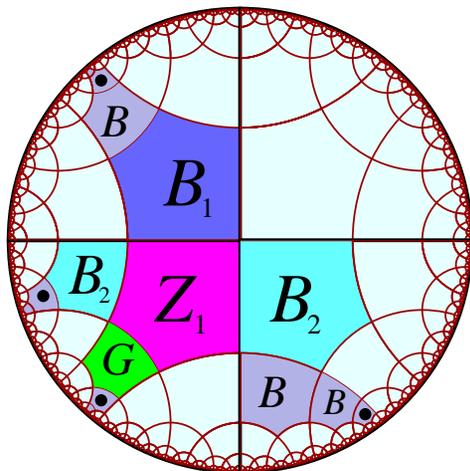
a



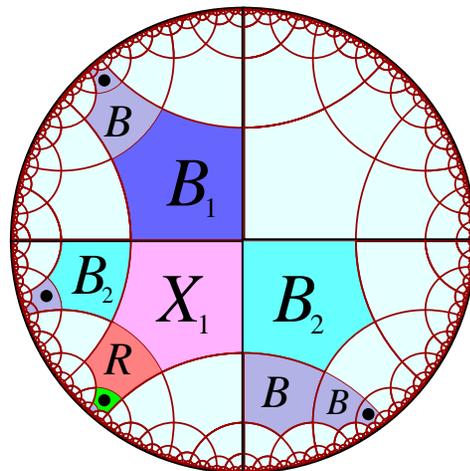
b



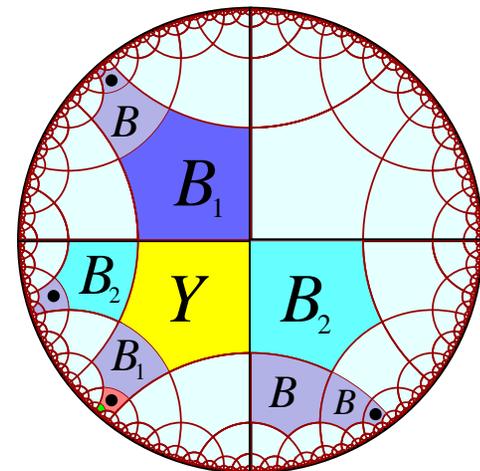
c



d

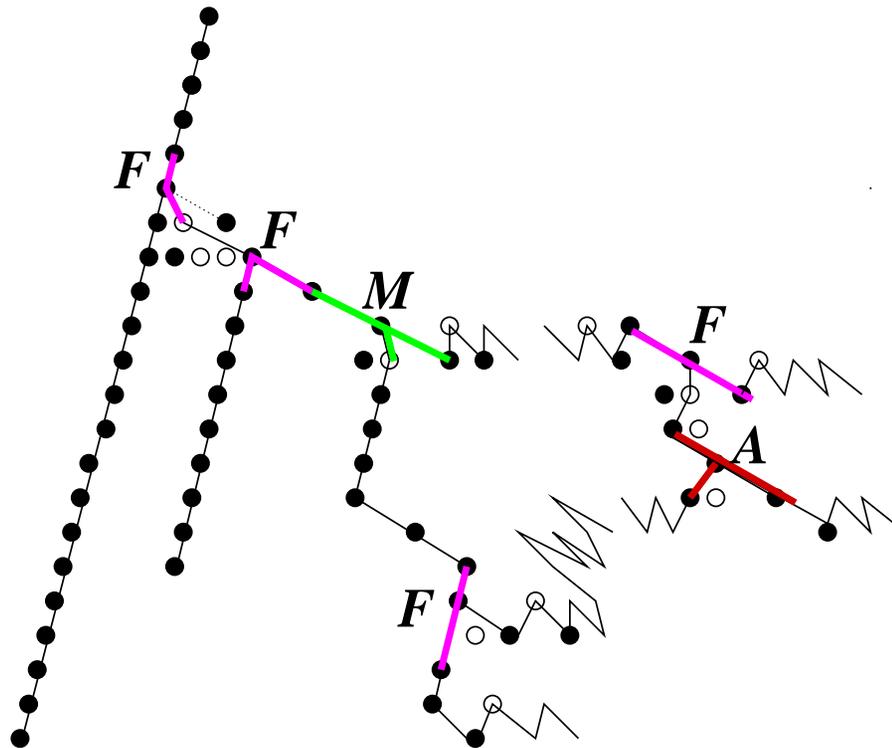


e

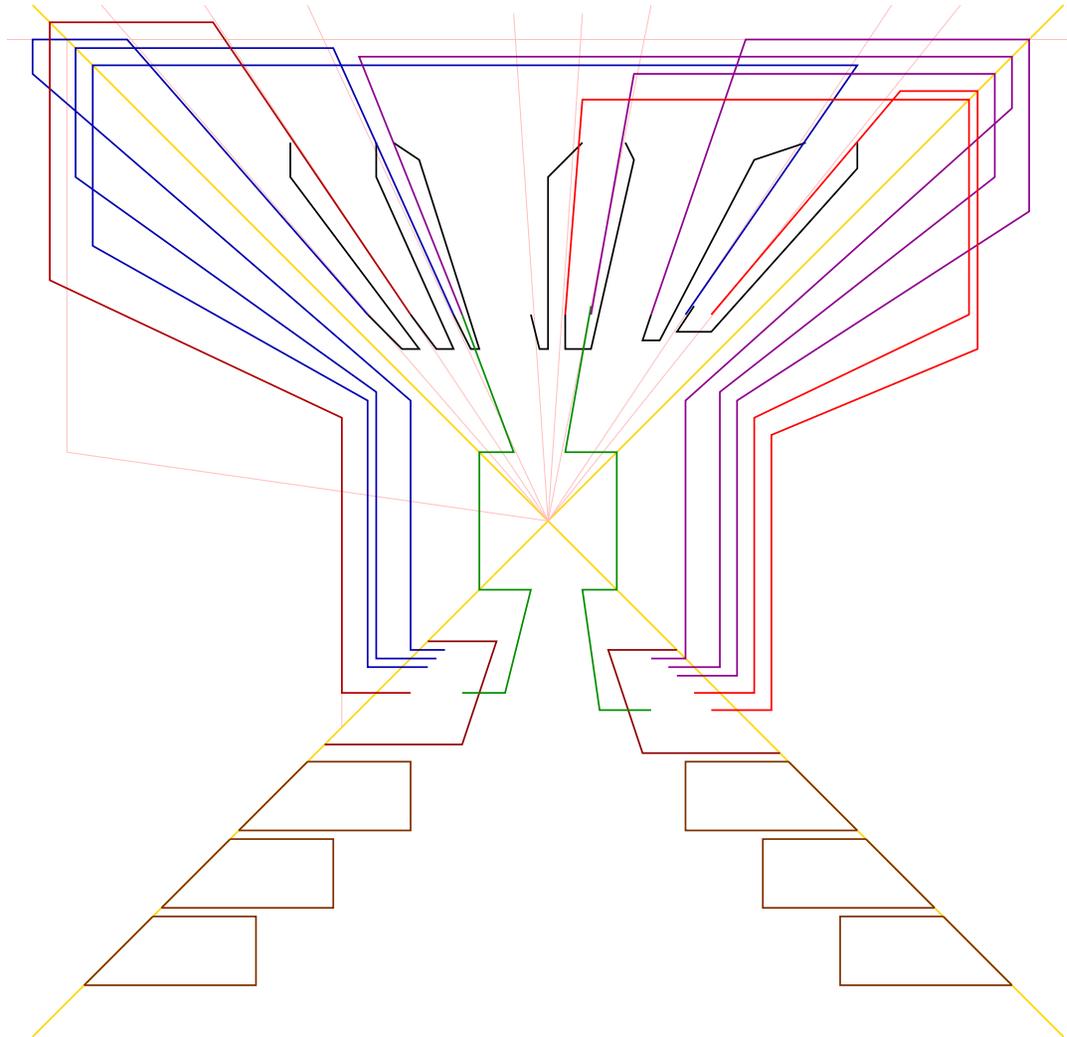


f

example of implementation: the elementary unit



implementation of the above example:



***3D* implementation:**

first idea:

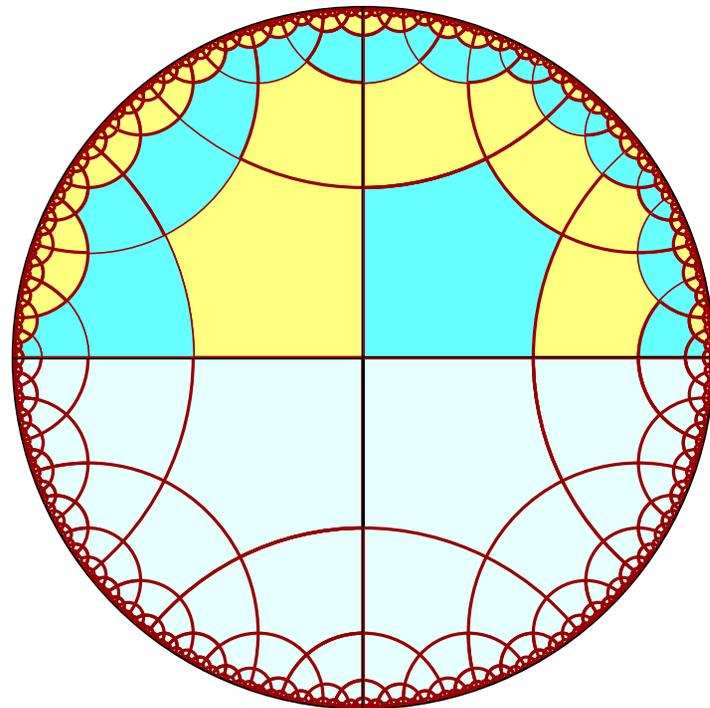
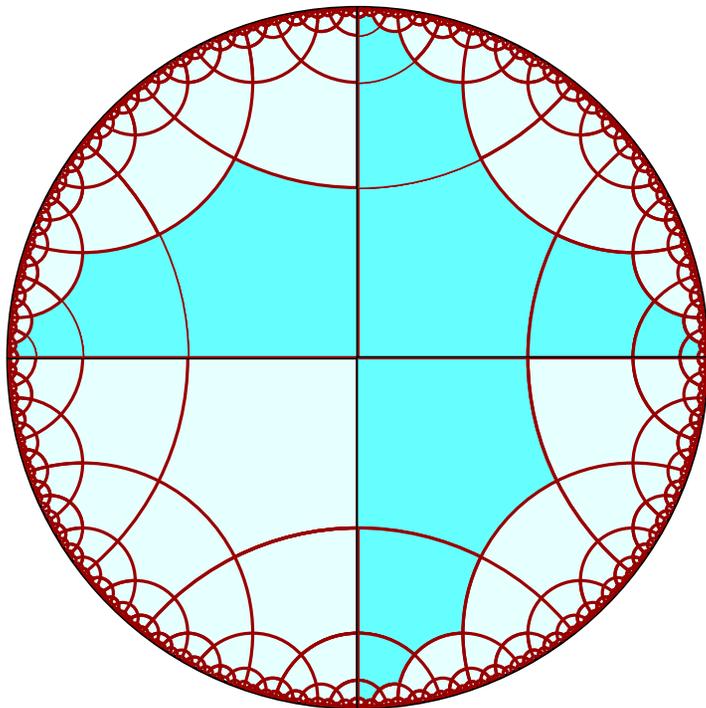
embed previous implementation in a plane
of \mathbb{H}^3

better solution:

thanks to *3D*, remove crossings

crossings:

in $3D$ space, crossings replaced by bridges:



the points with 5 colours

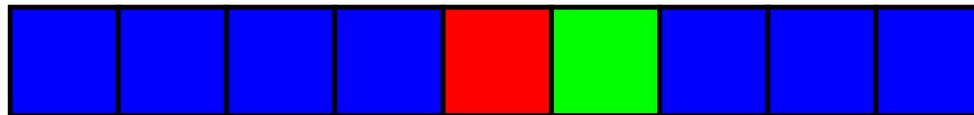
points realised by a single cell called **central**

- three 2-way tracks to the central cell
- signalling cells around the central cell
specific pattern of neighbours
- possibly differentiation of the tracks
using previous signalling cells and additional **sensors**: for memory and fixed point

motion rules

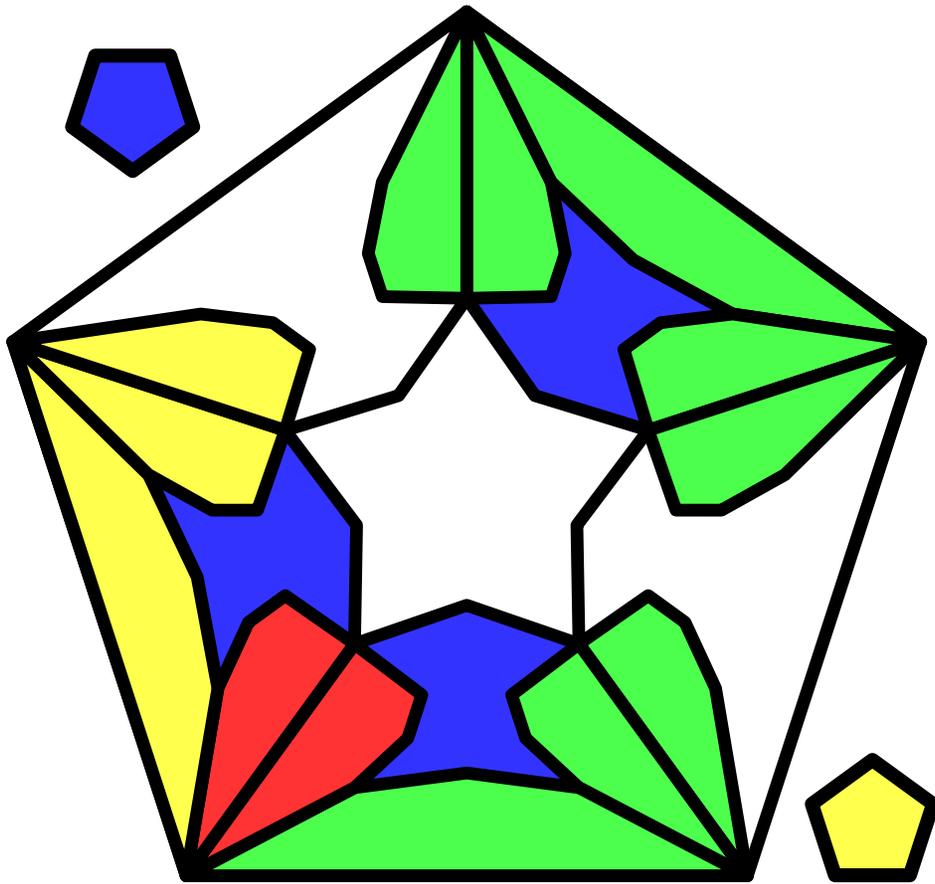
locomotive: two cells on a track,
green for the front, red for the rear

$$\begin{array}{ll} BGW^{10}; B \Rightarrow G & BRW^{10}; G \Rightarrow R \\ BGW^{10}; R \Rightarrow B & BRW^{10}; B \Rightarrow B \end{array}$$

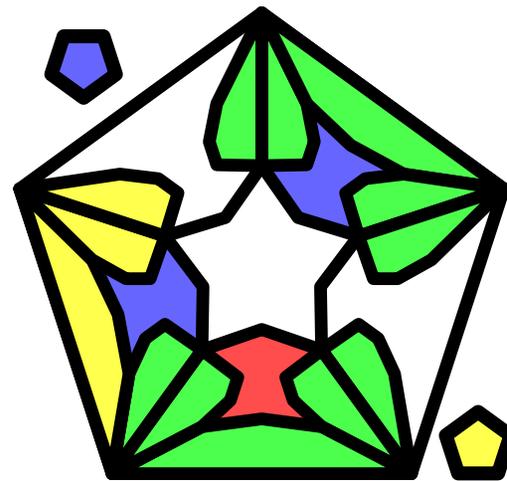
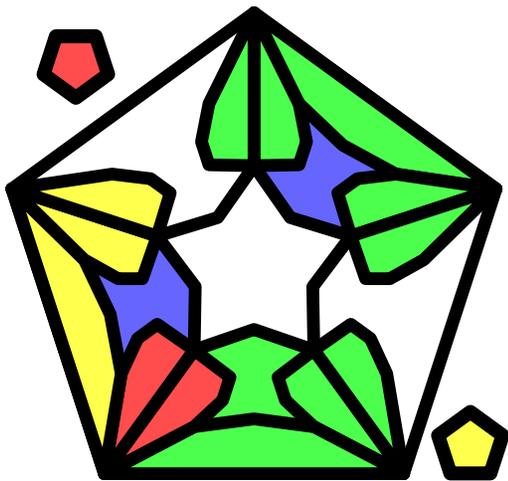
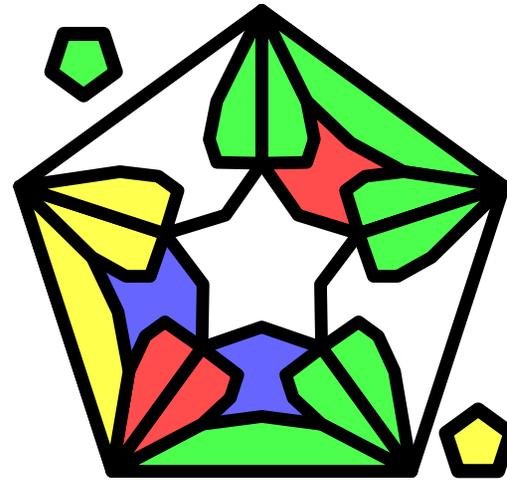
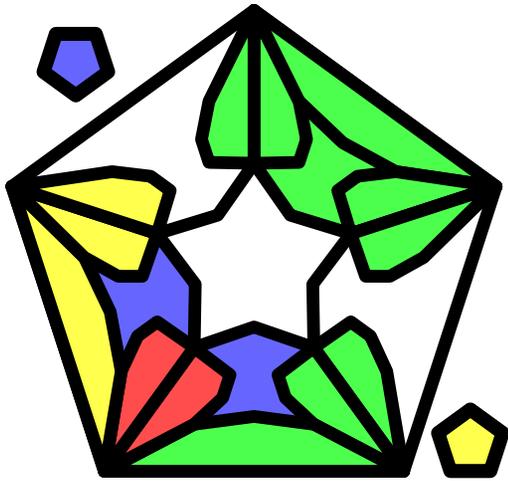


memory switch point:

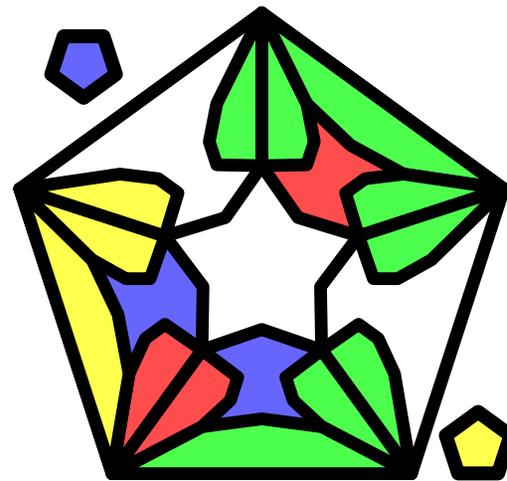
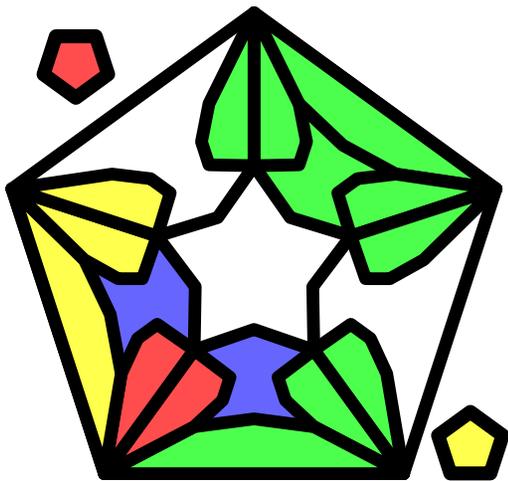
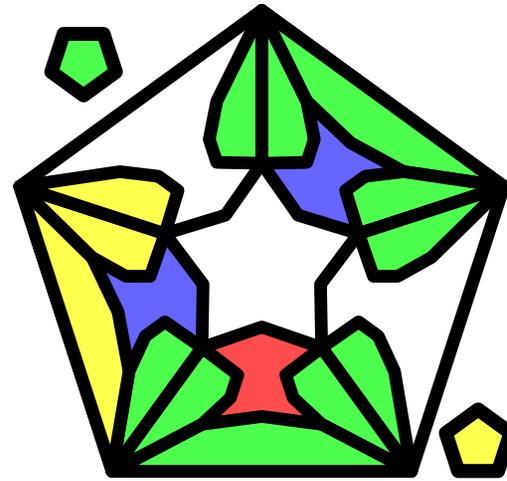
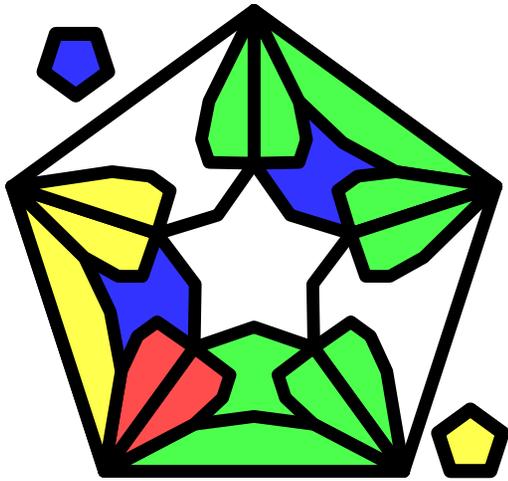
stable configuration



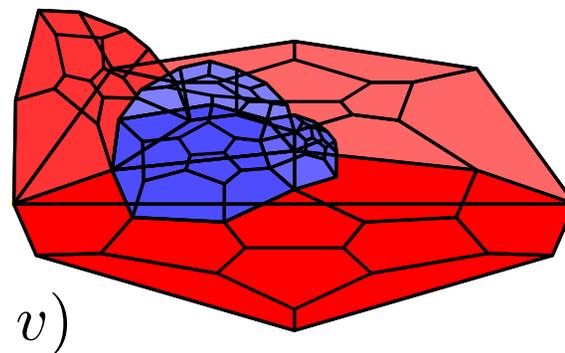
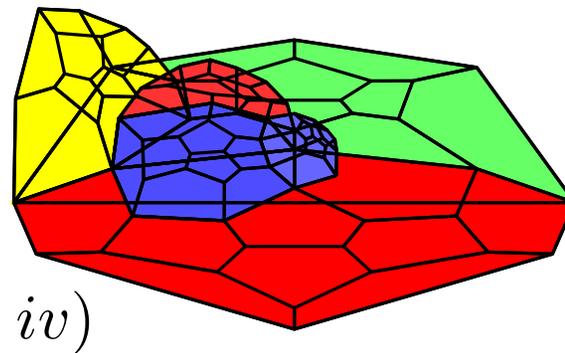
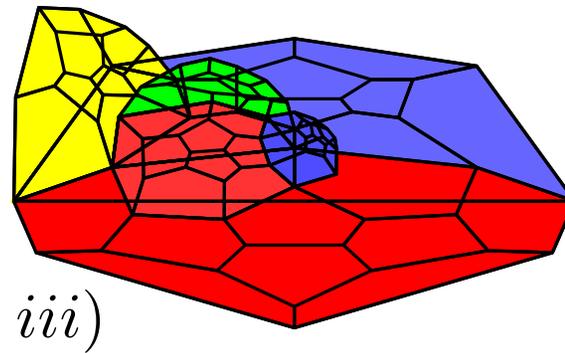
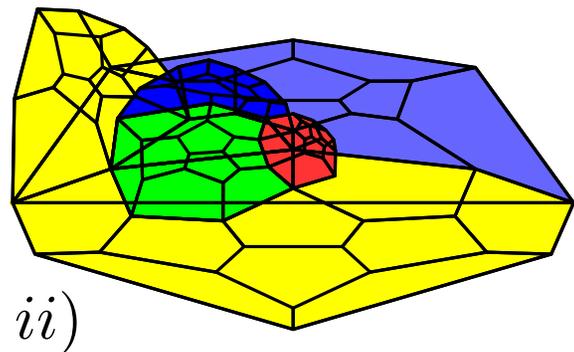
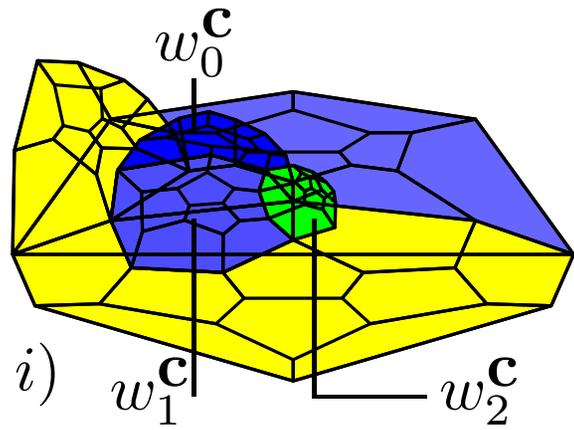
memory switch: from (A)



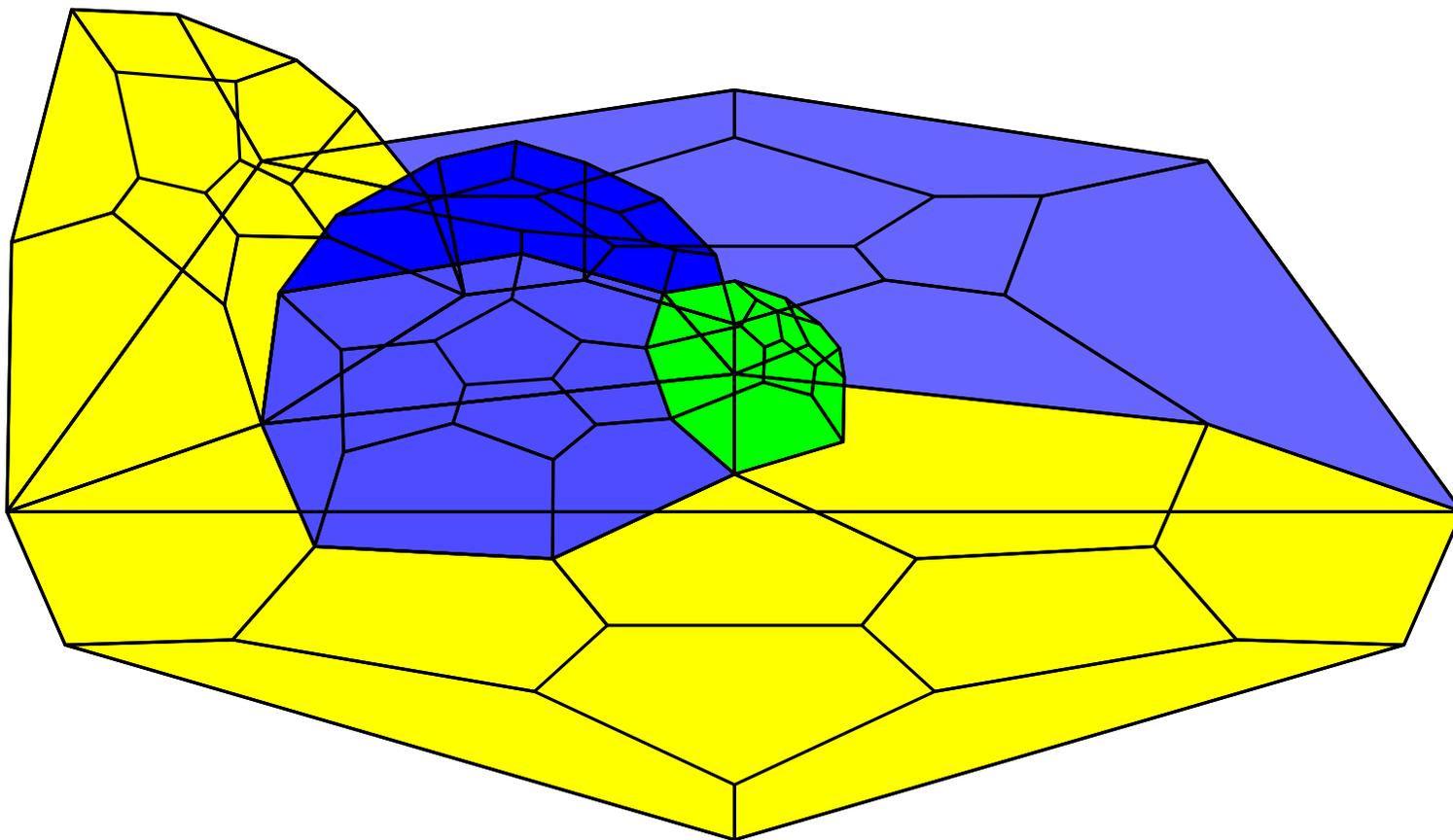
memory switch: from (B)



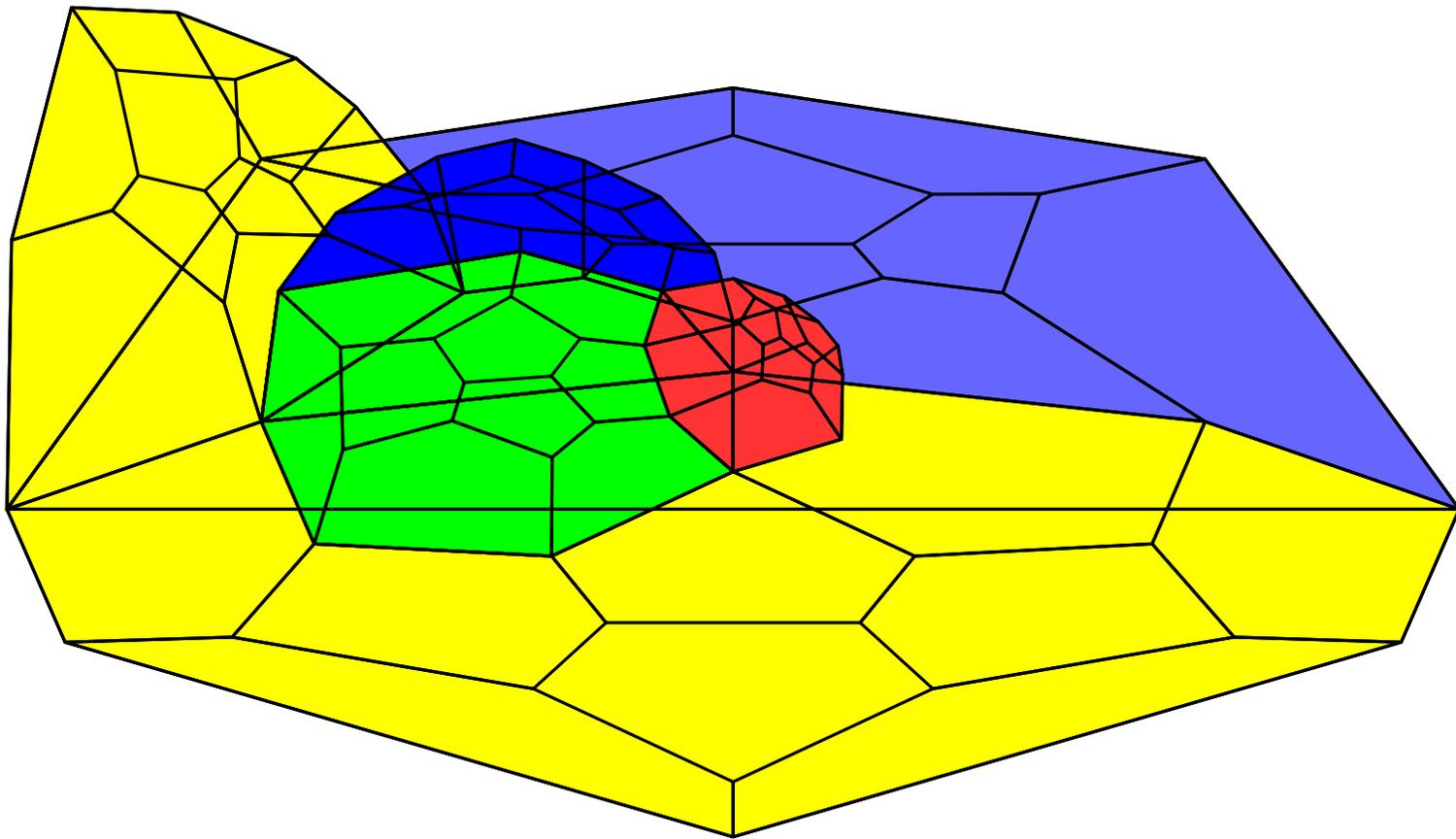
memory switch:
near 5 and (C)



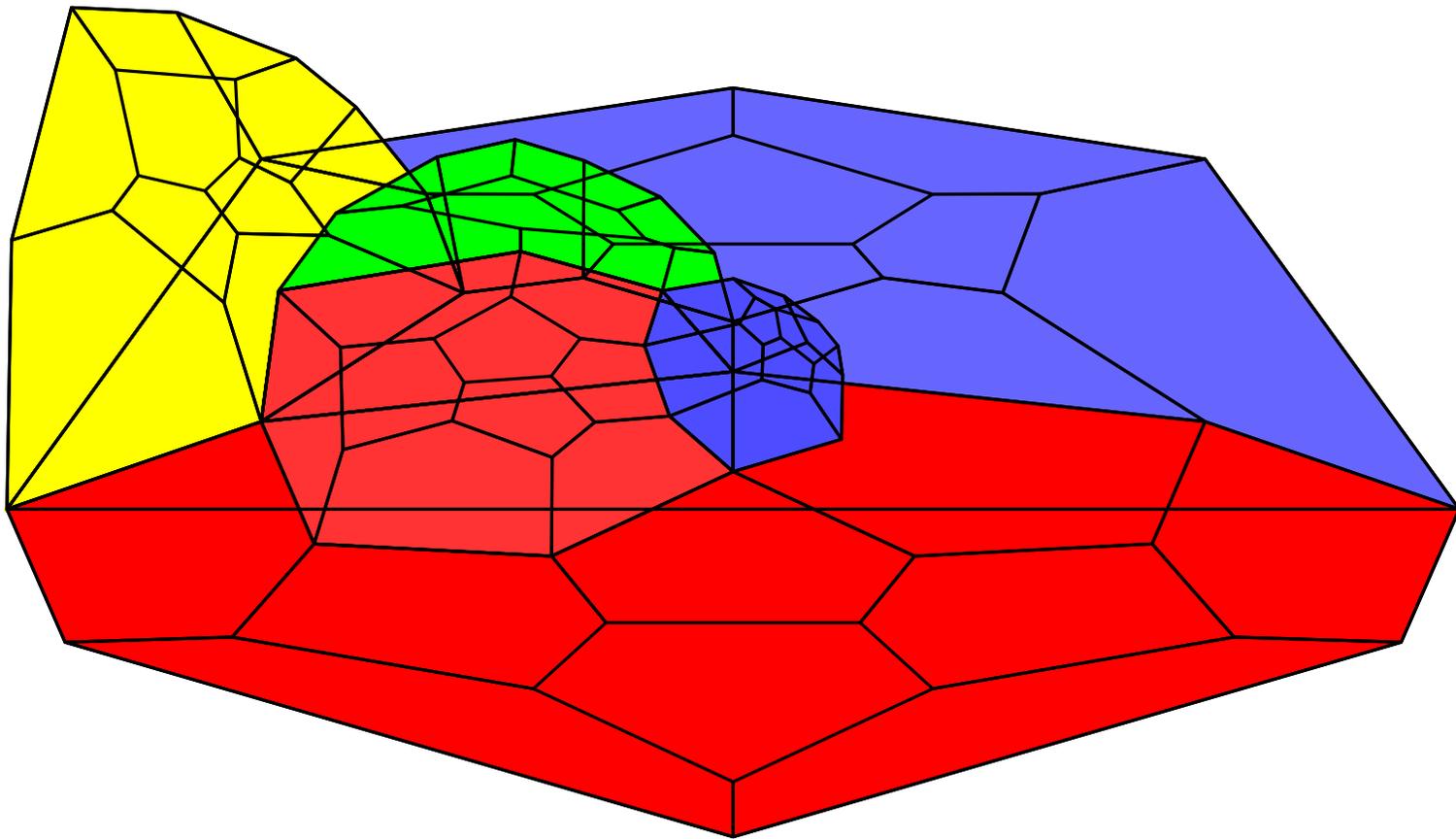
memory switch: near 5 and (C)



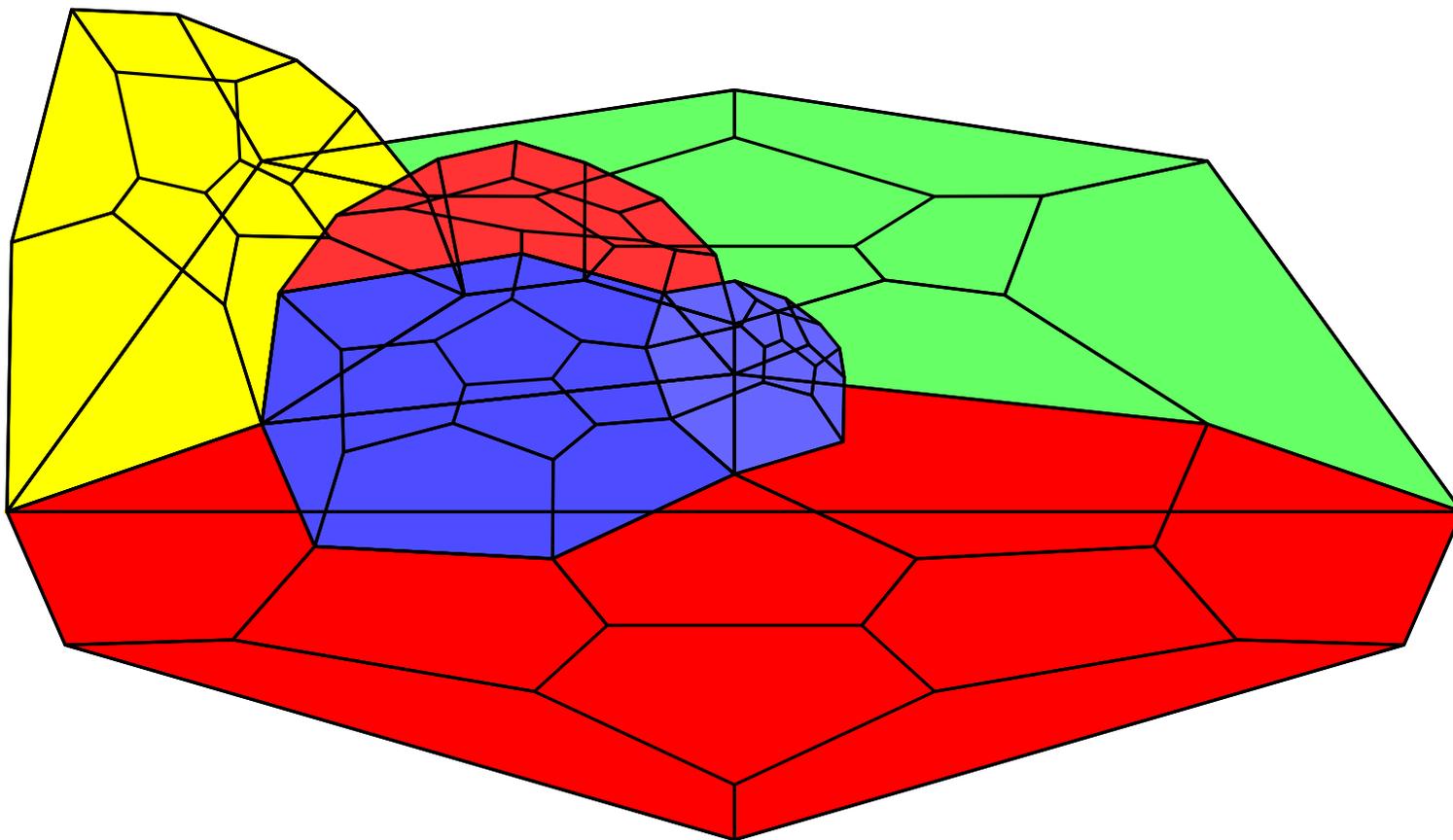
memory switch: near 5 and (C)



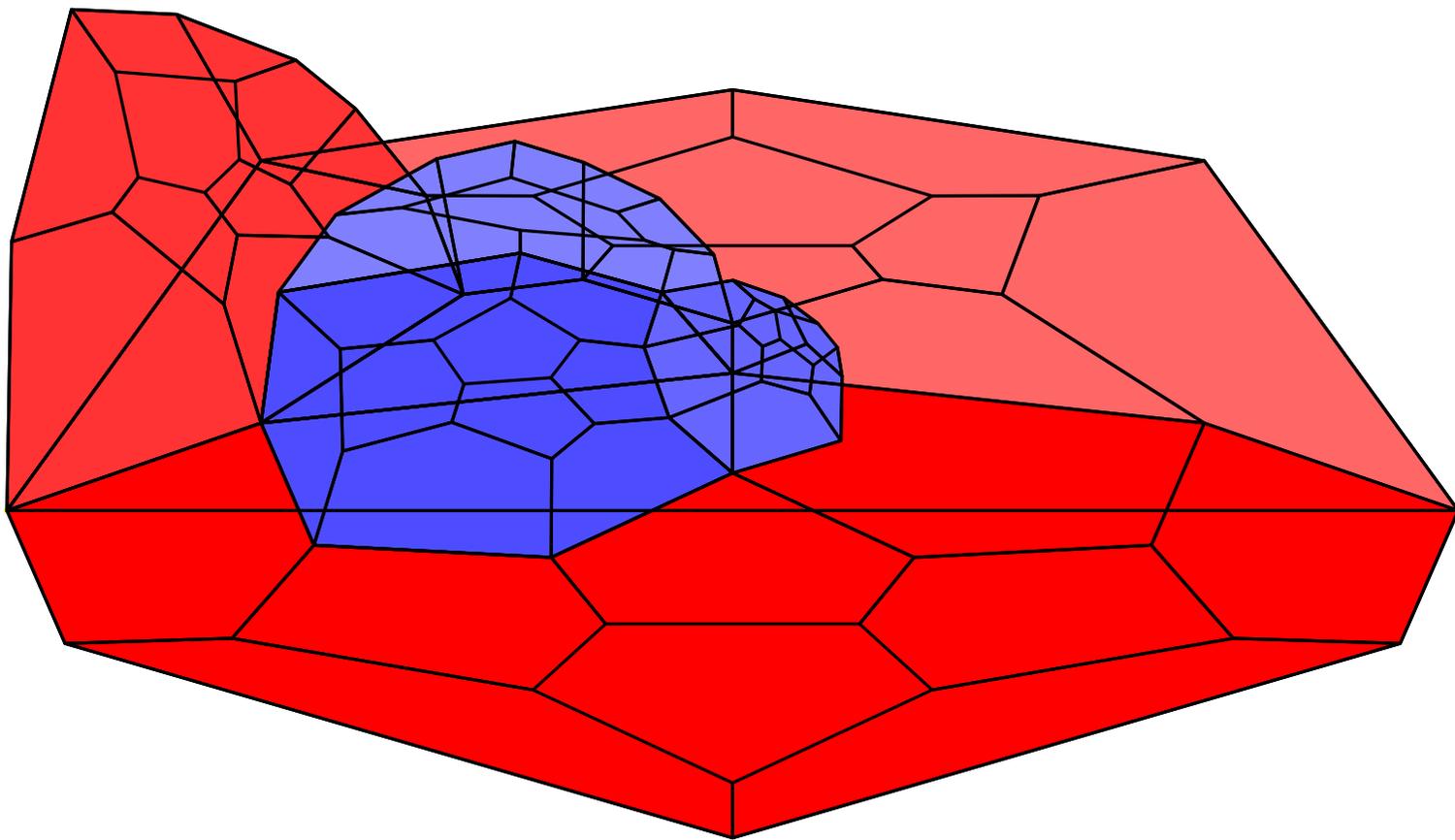
memory switch: near 5 and (C)



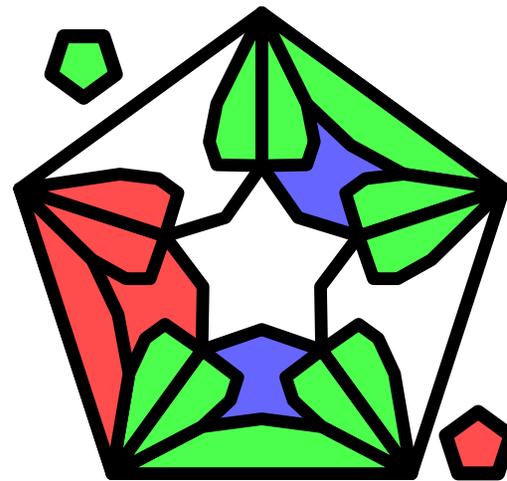
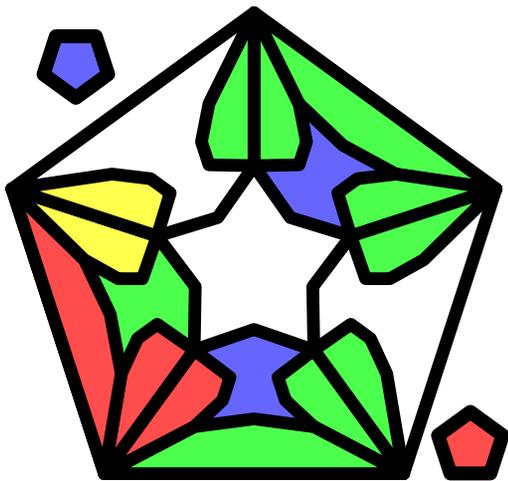
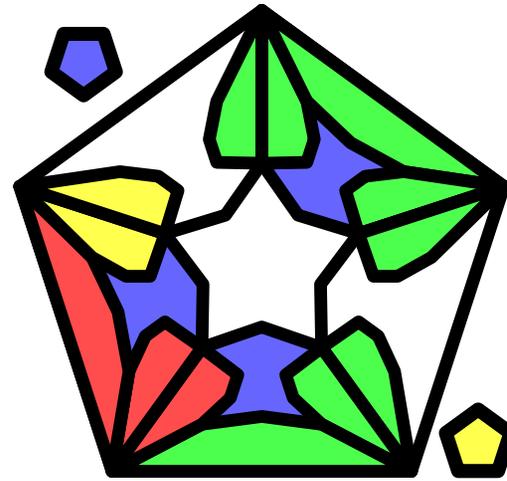
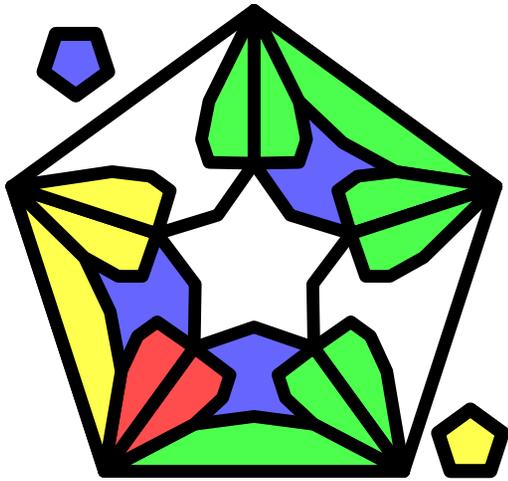
memory switch: near 5 and (C)



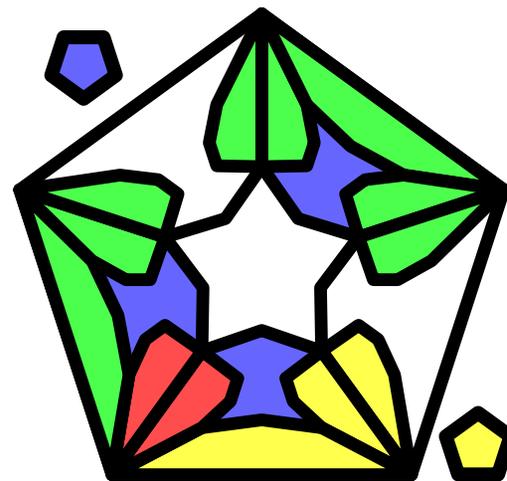
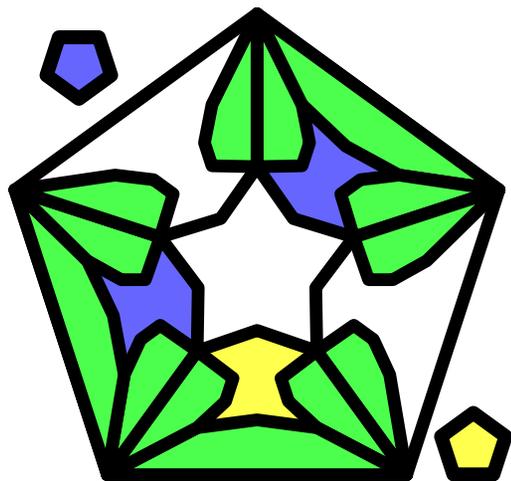
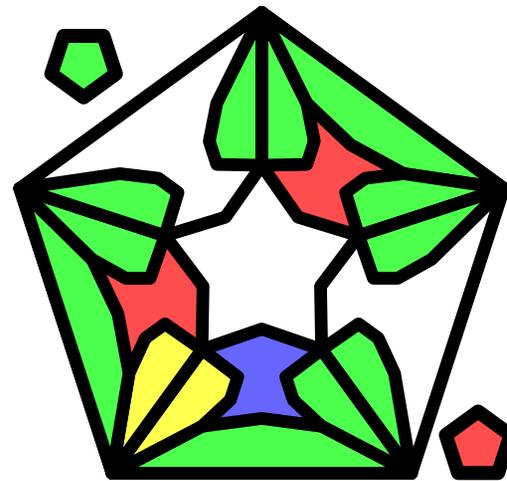
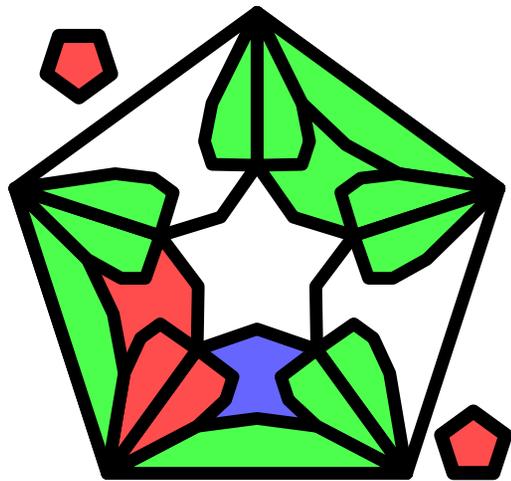
memory switch: near 5 and (C)



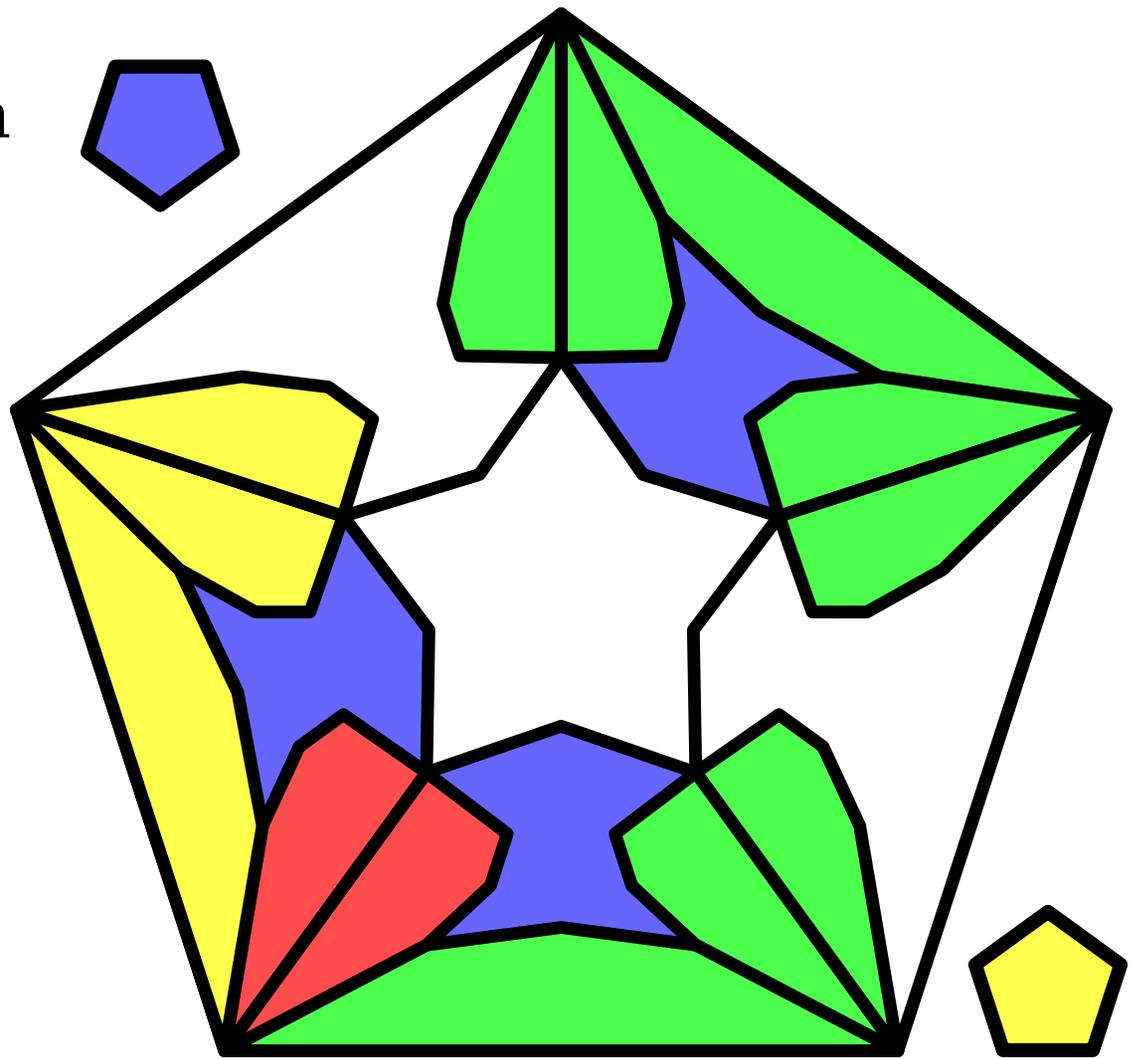
memory switch: from (C)



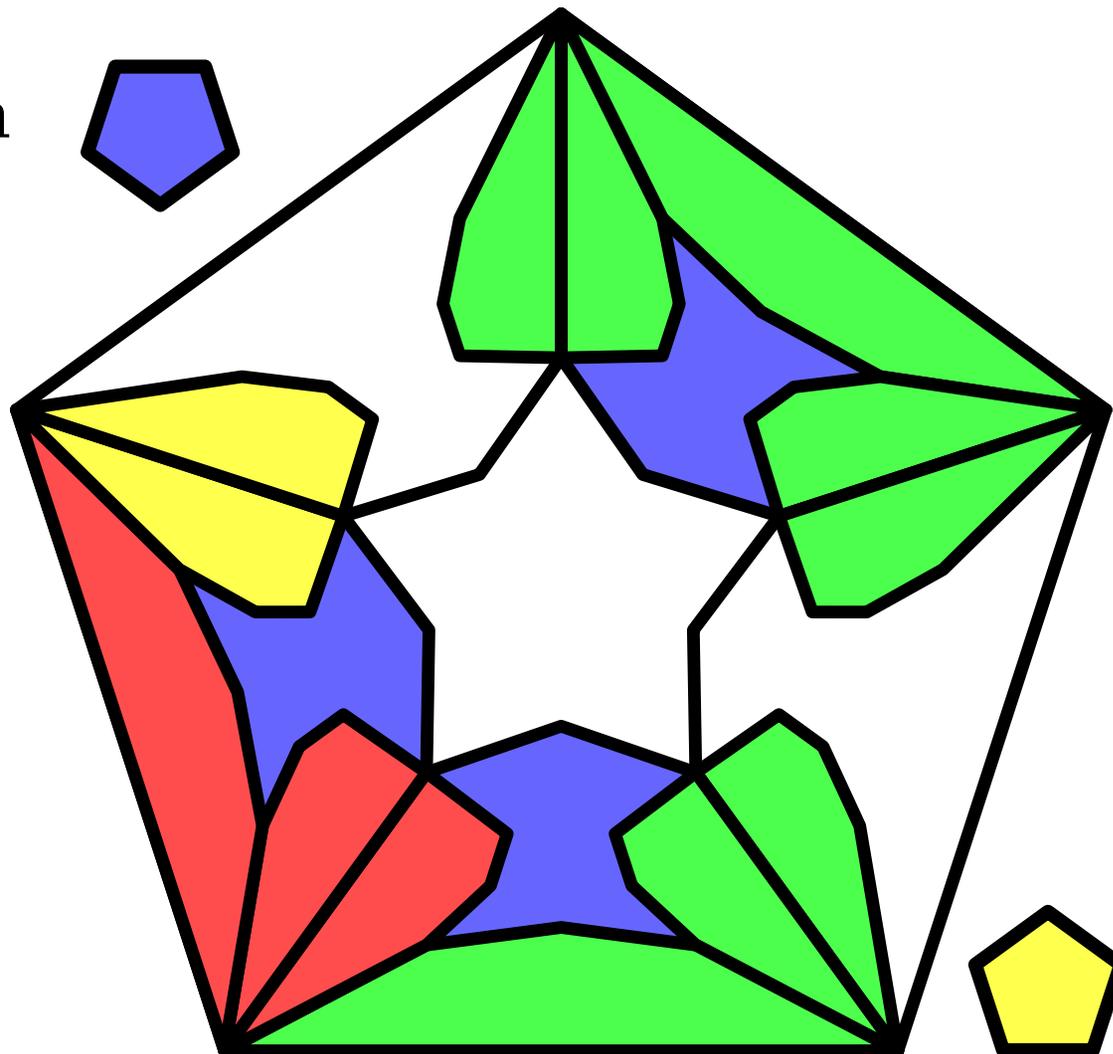
memory switch: from (C), continuation



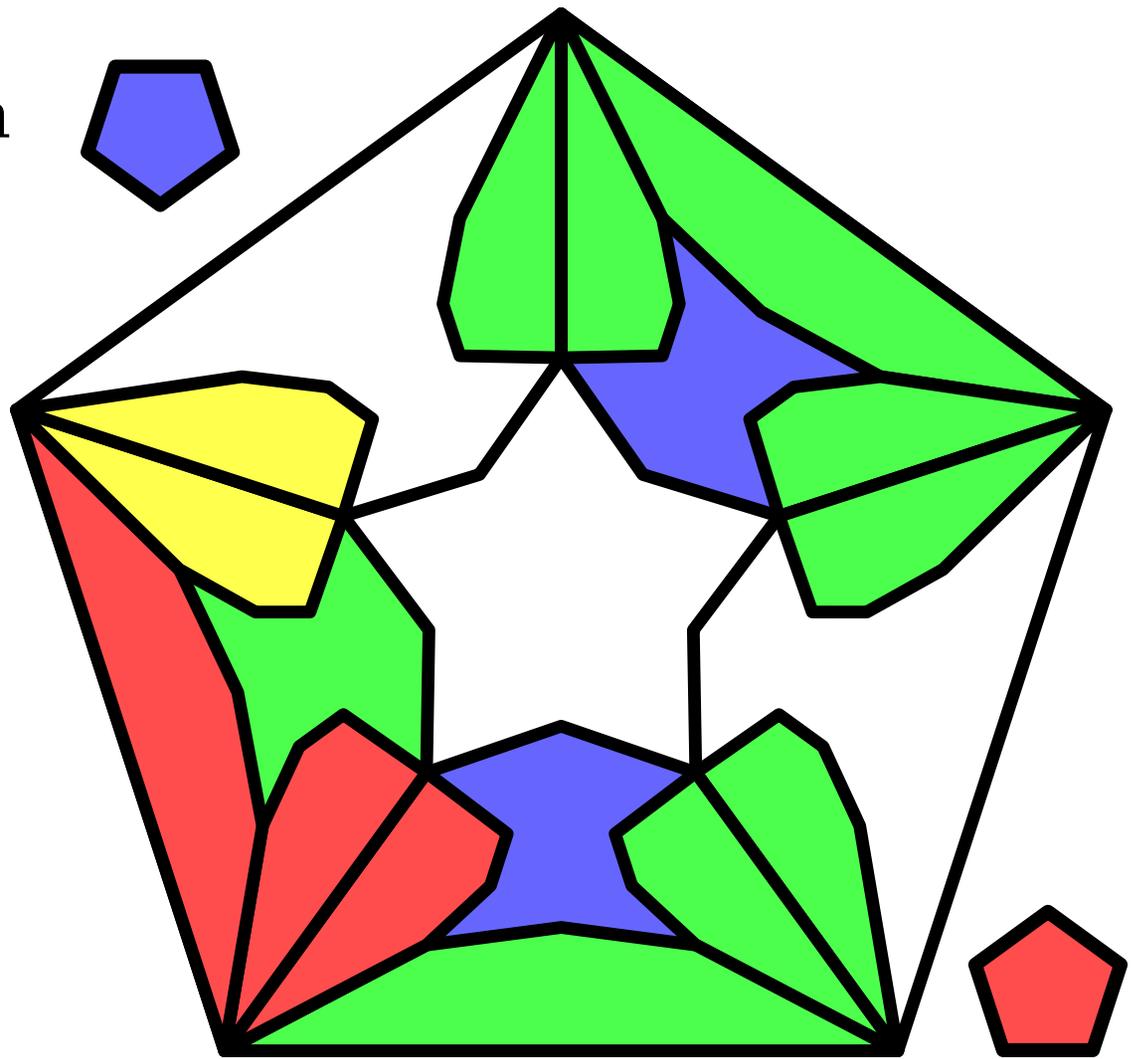
memory switch:
from (C),
continuation



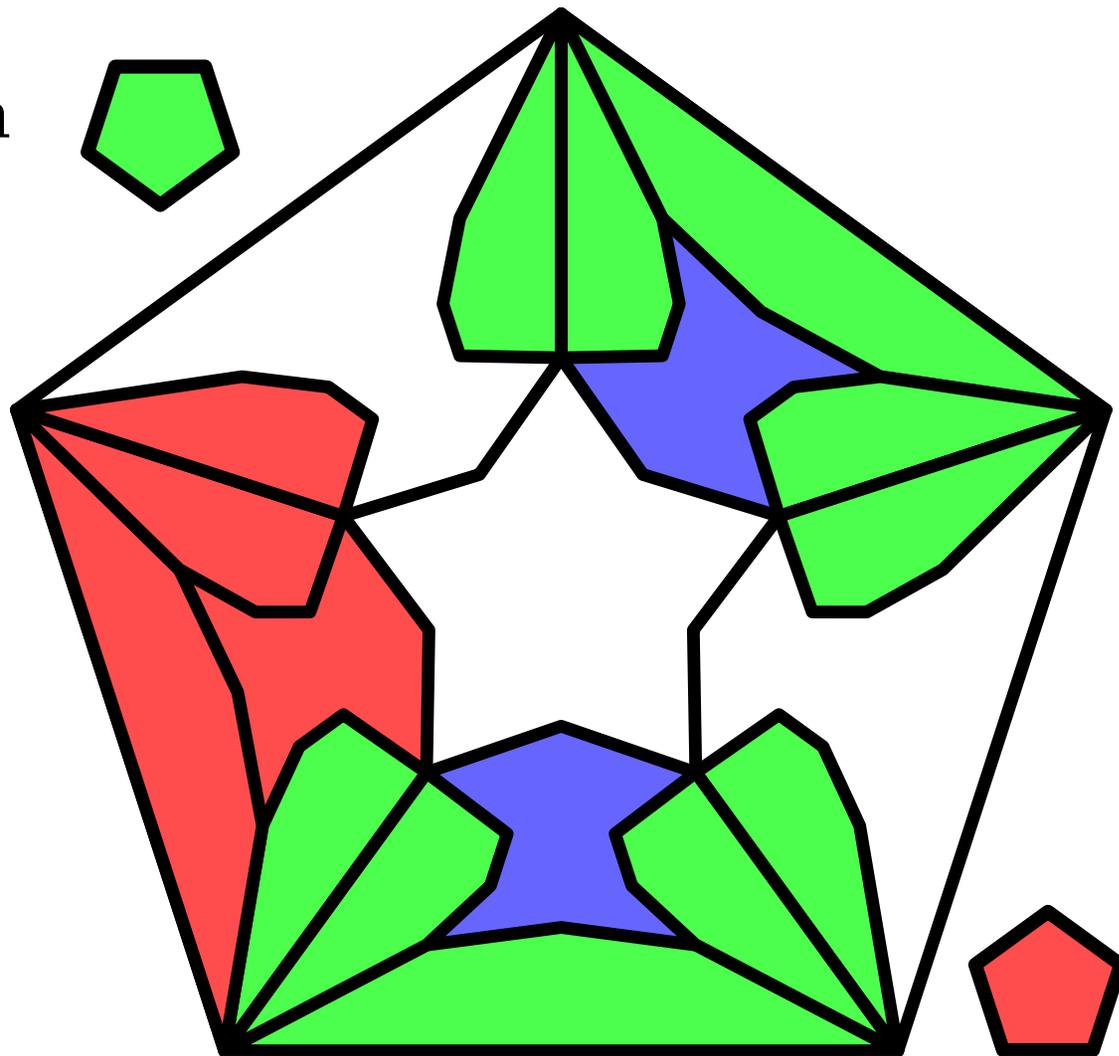
memory switch:
from (C),
continuation



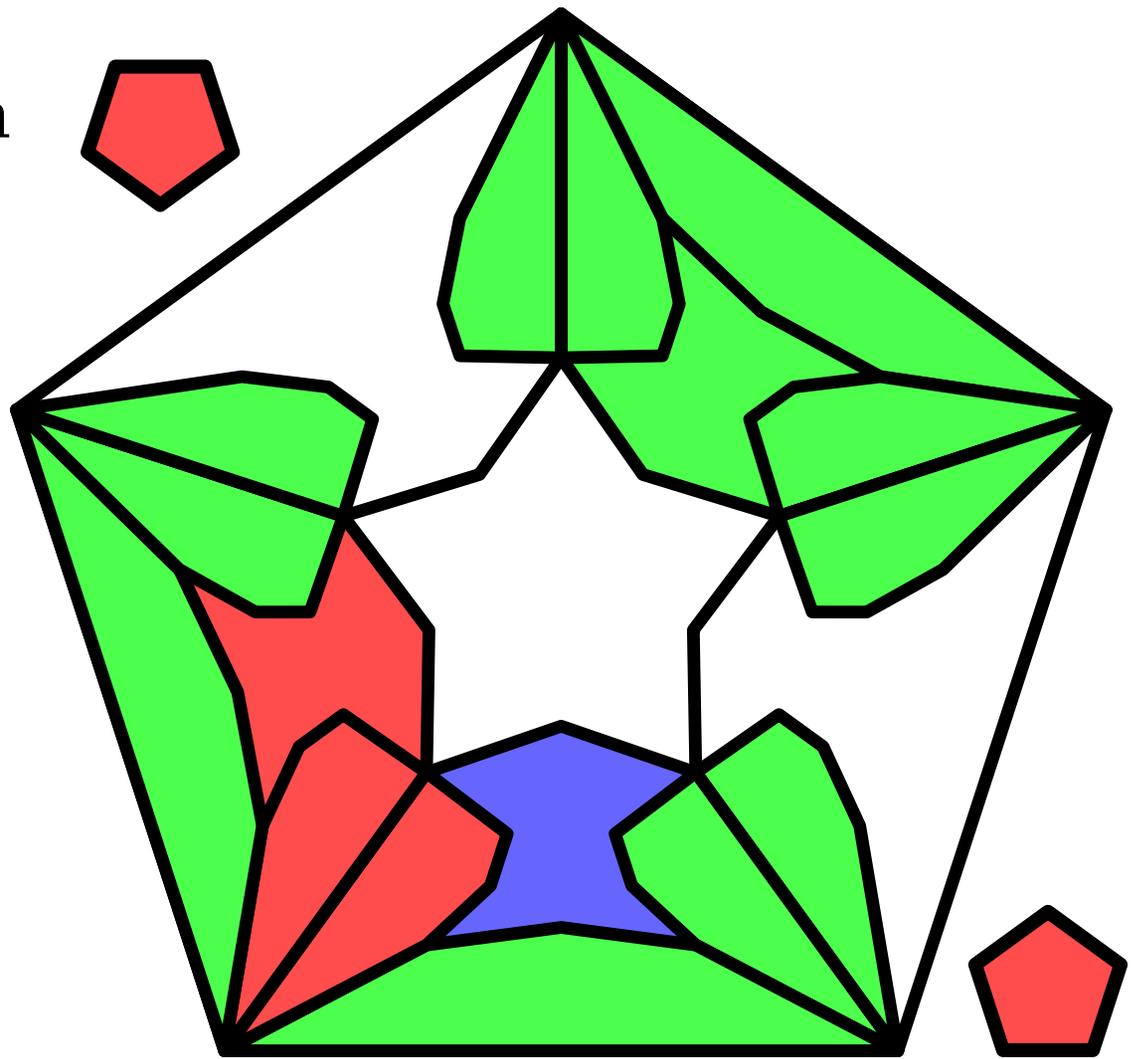
memory switch:
from (C),
continuation



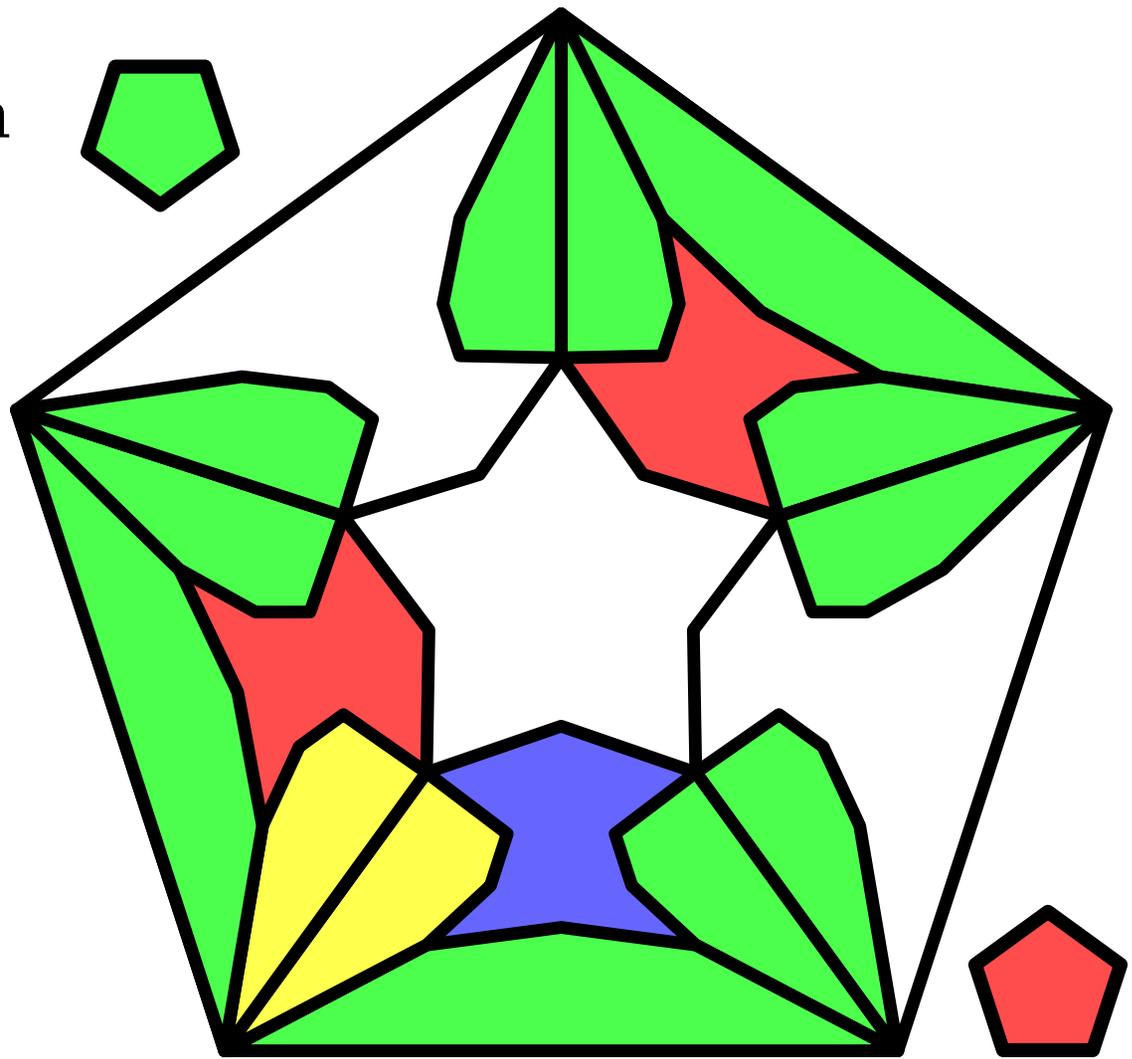
memory switch:
from (C),
continuation



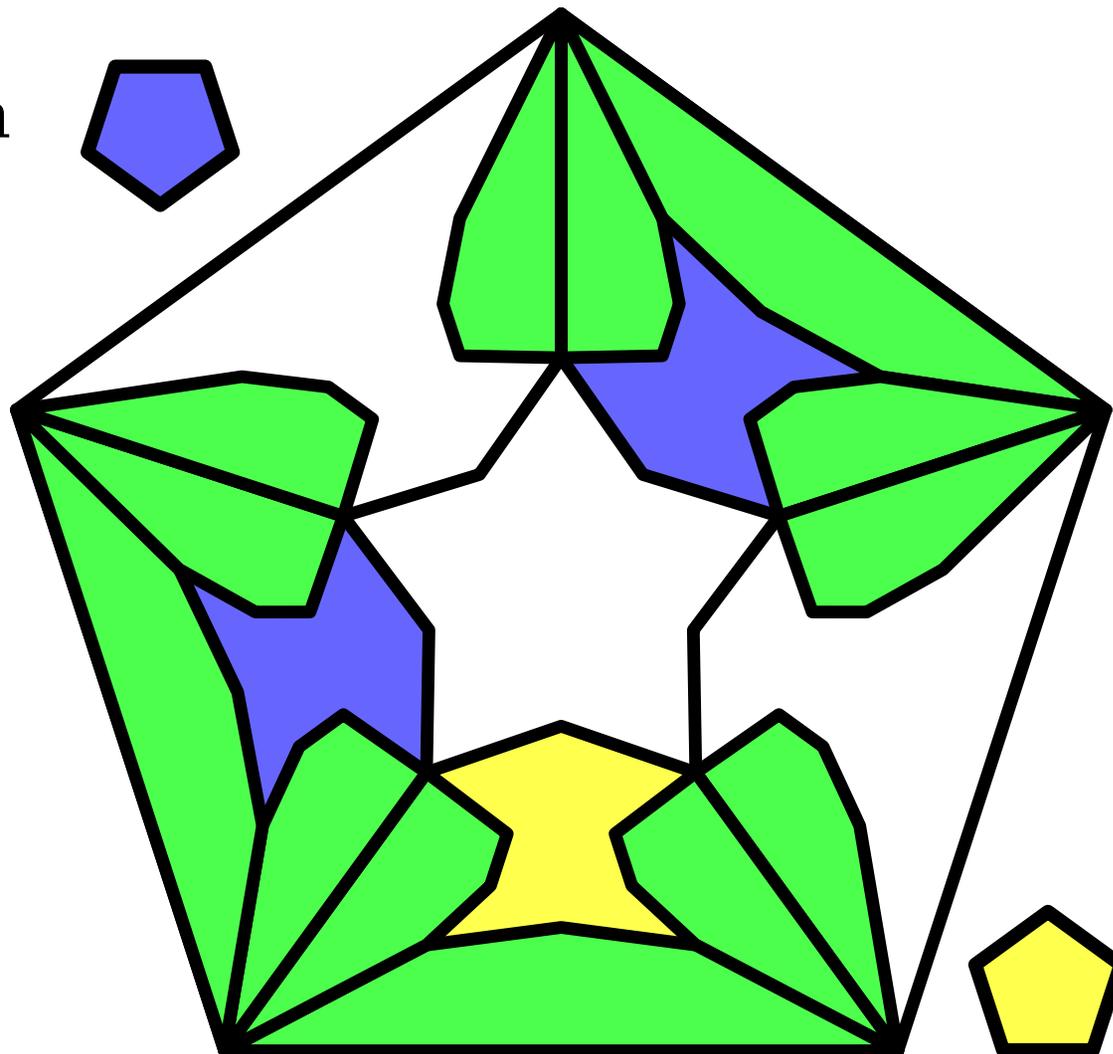
memory switch:
from (C),
continuation



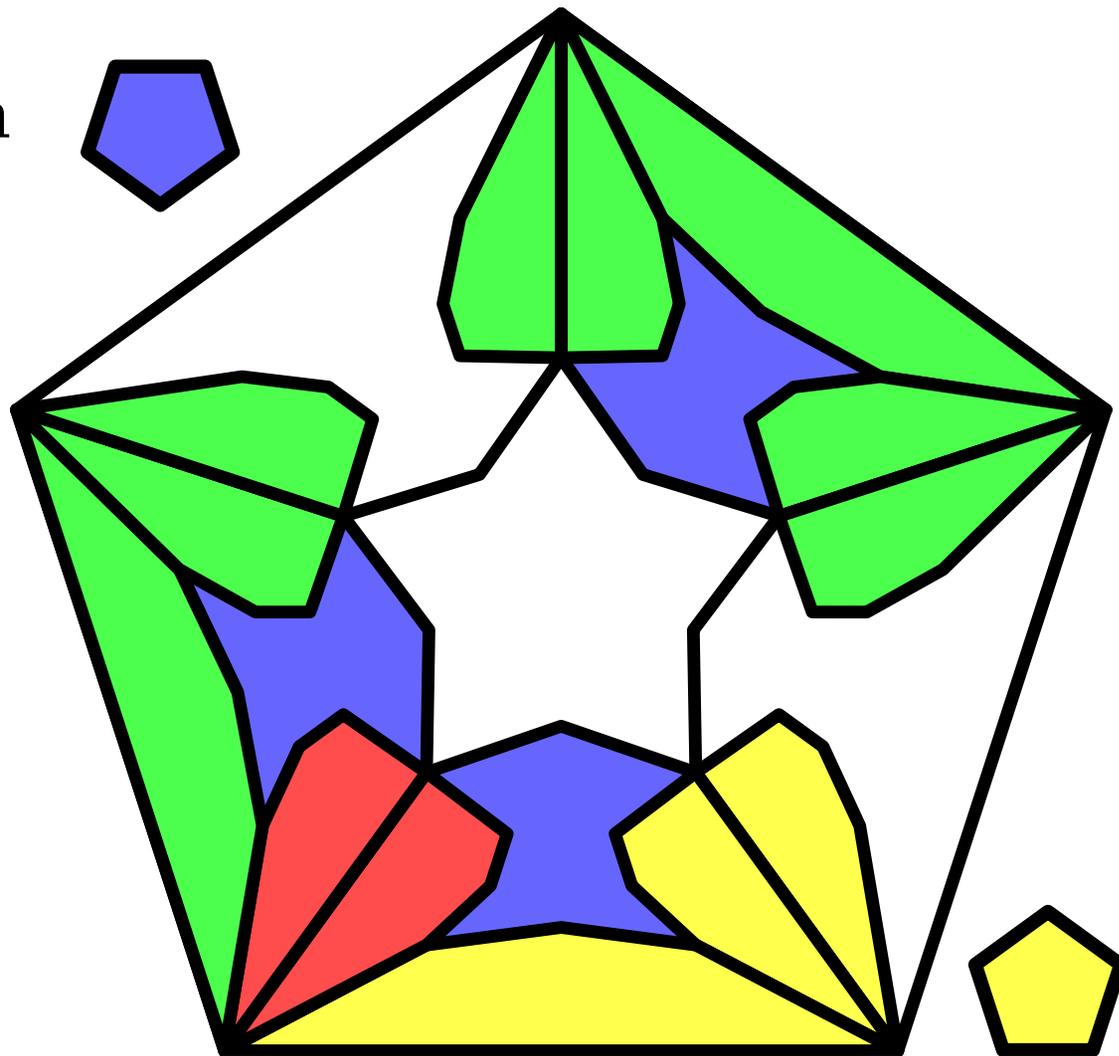
memory switch:
from (C),
continuation



memory switch:
from (C),
continuation

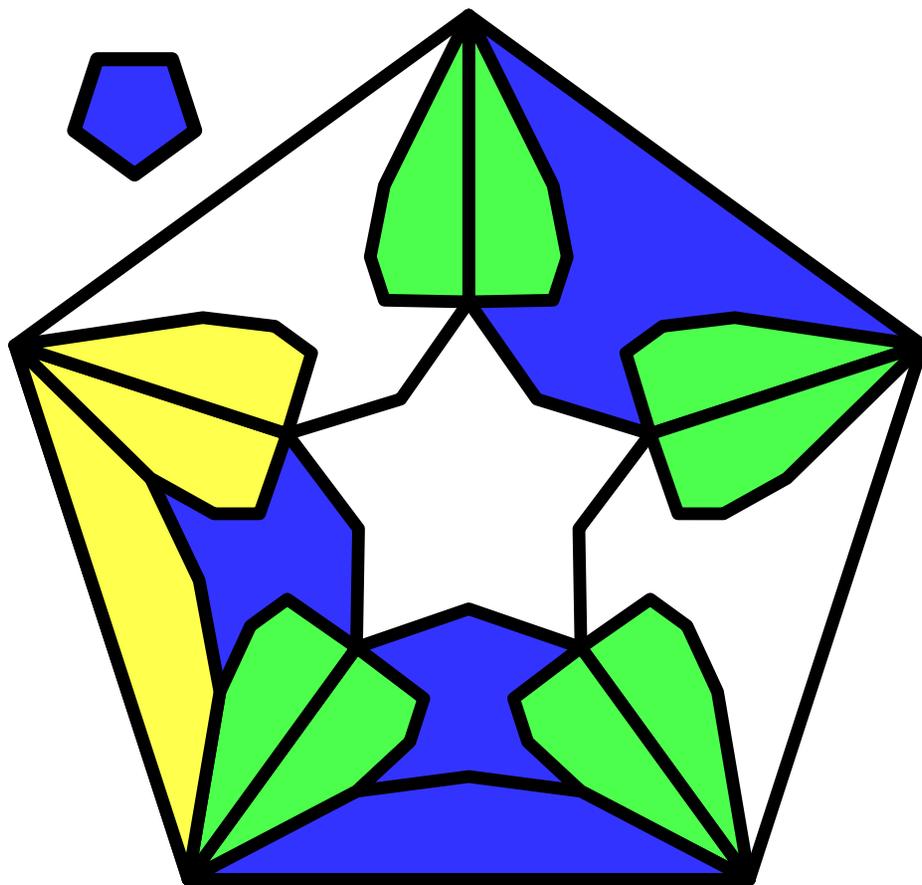


memory switch:
from (C),
continuation

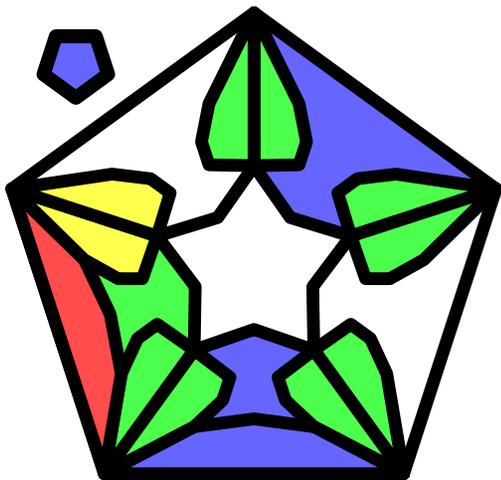
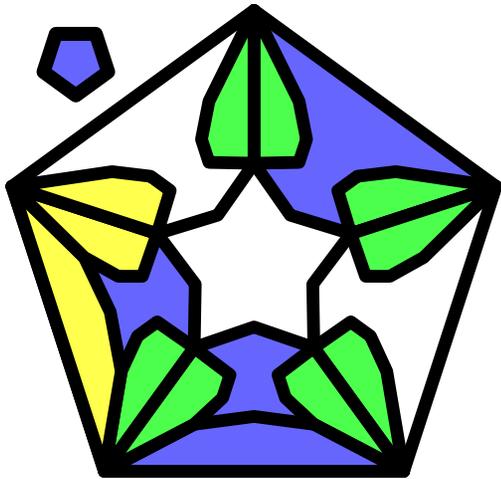


fixed switch point:

stable configuration



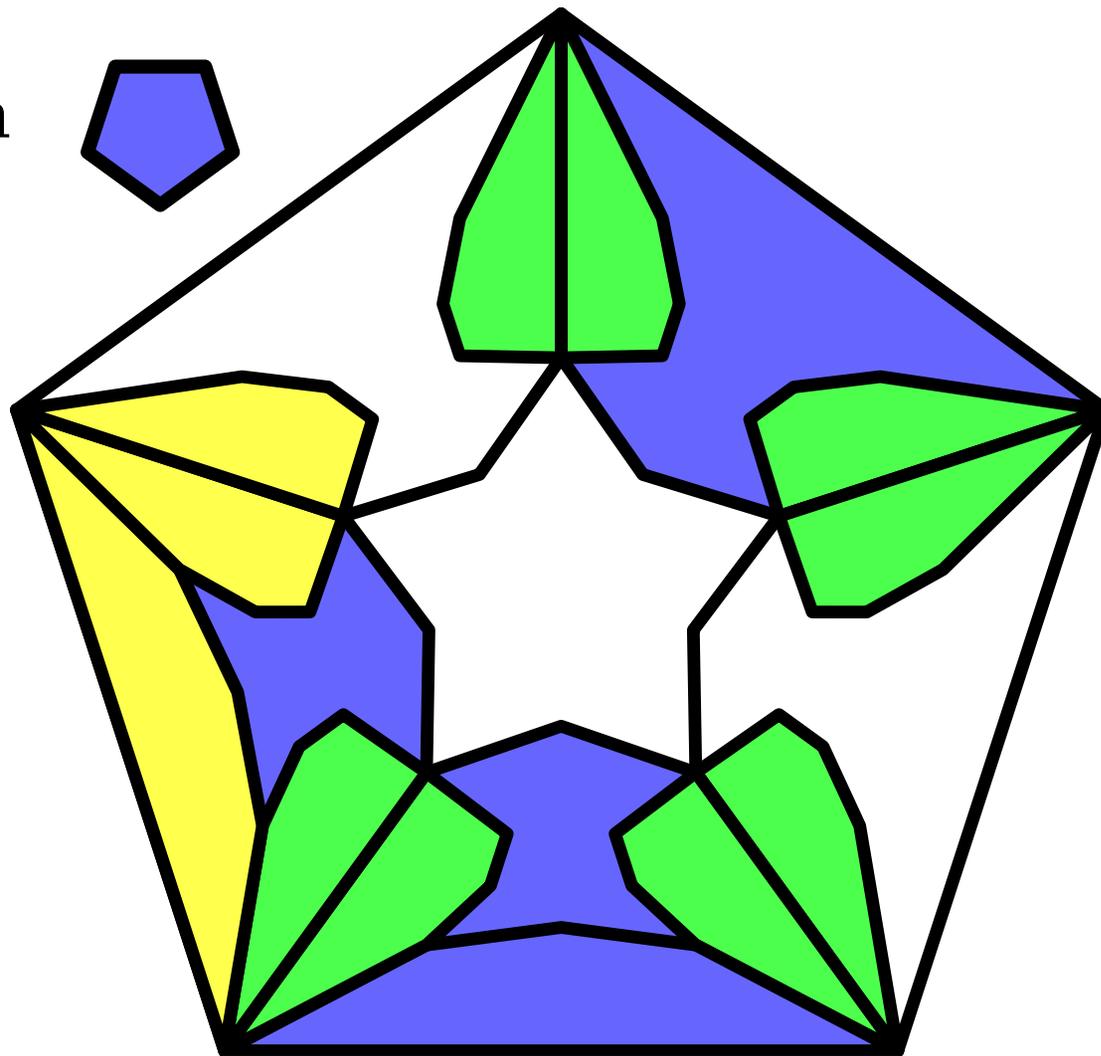
fixed switch: from (C)



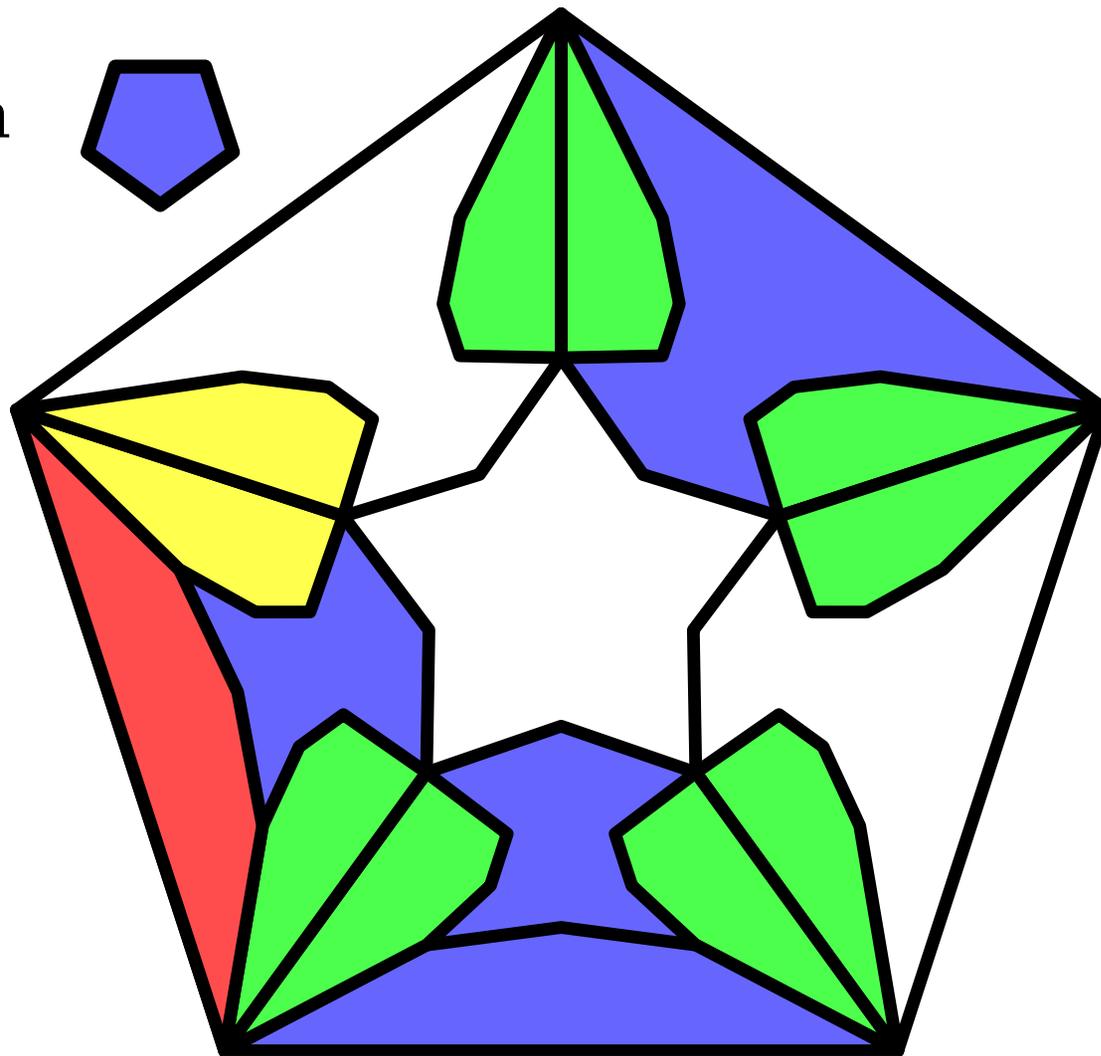
fixed switch: from (C), continuation



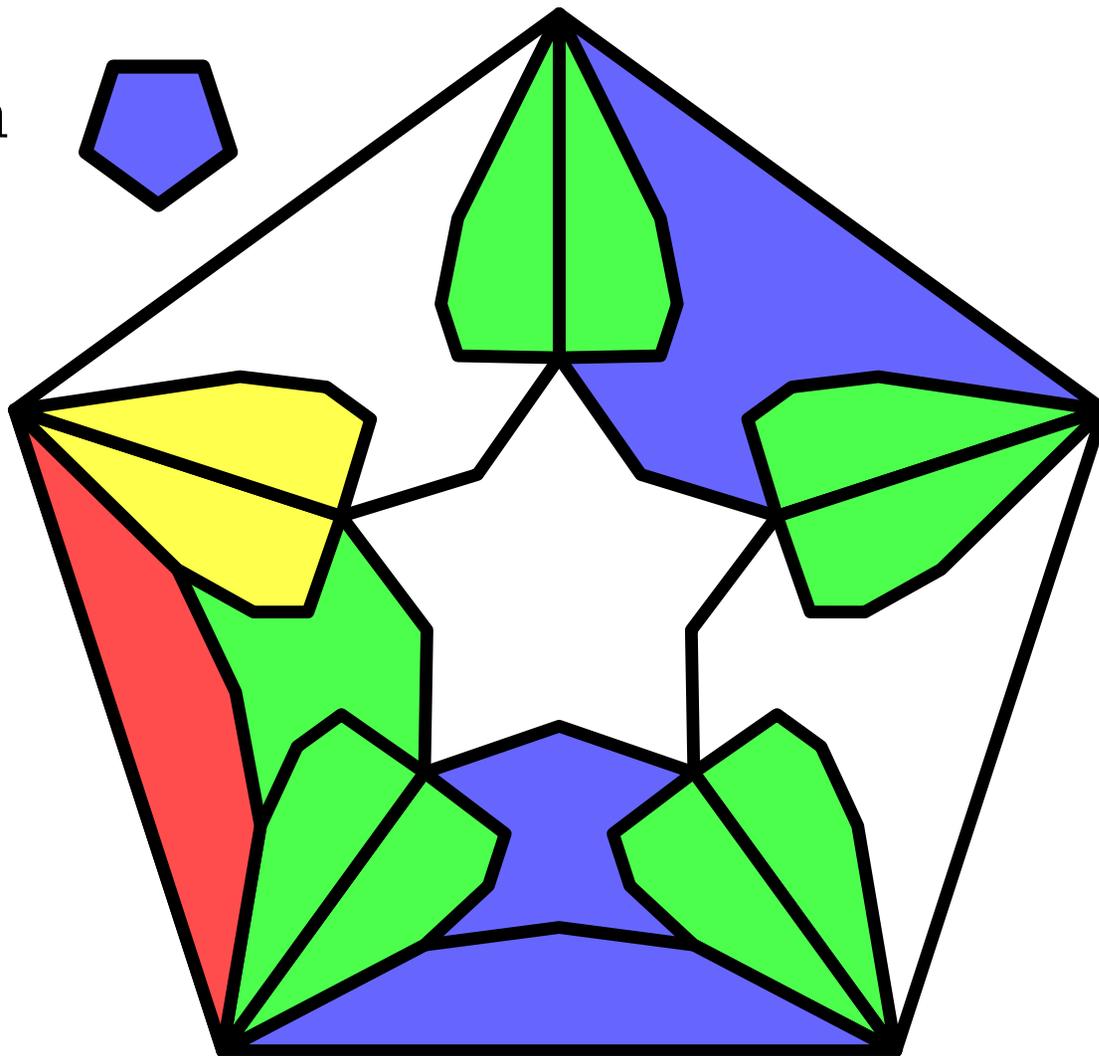
fixed switch:
from (C),
continuation



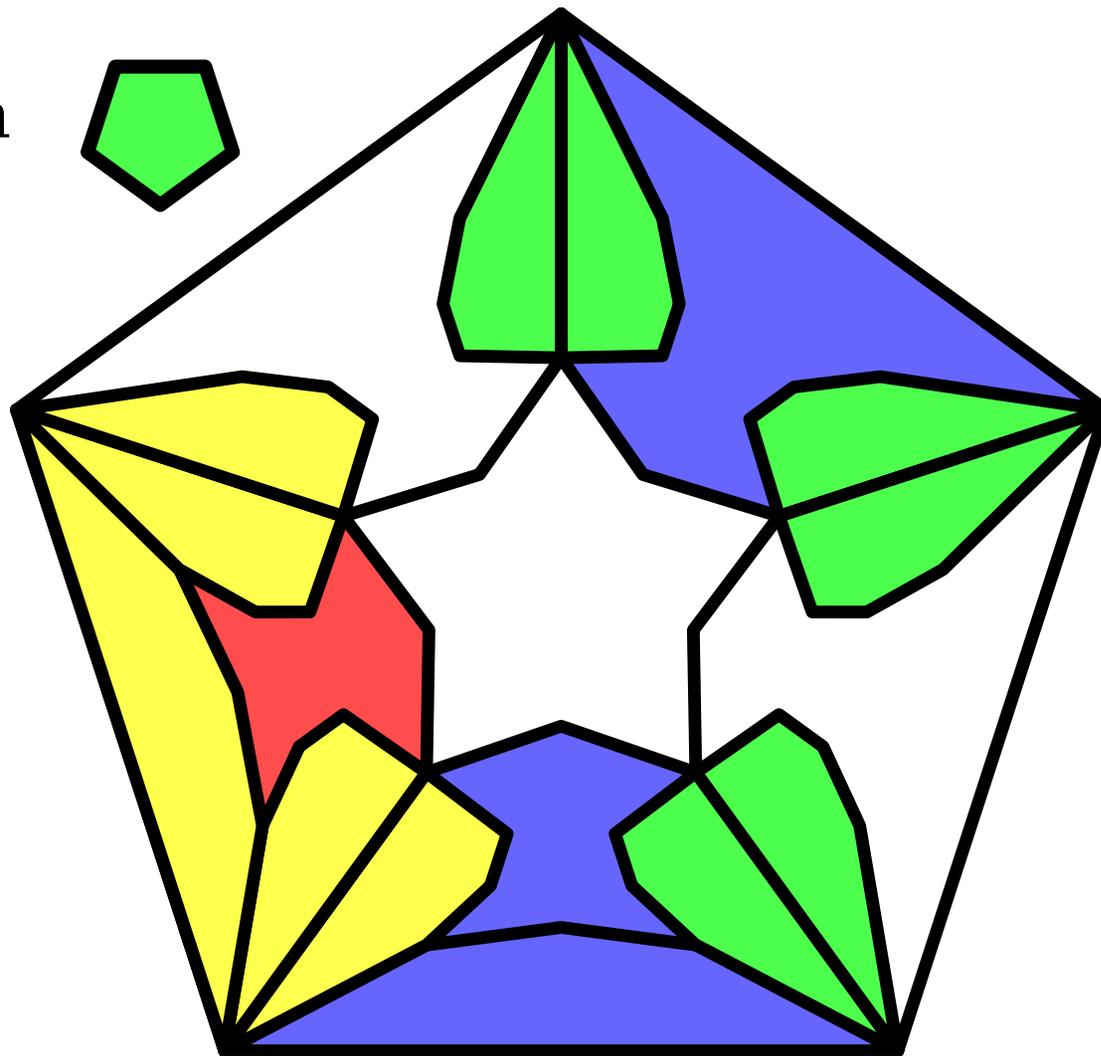
fixed switch:
from (C),
continuation



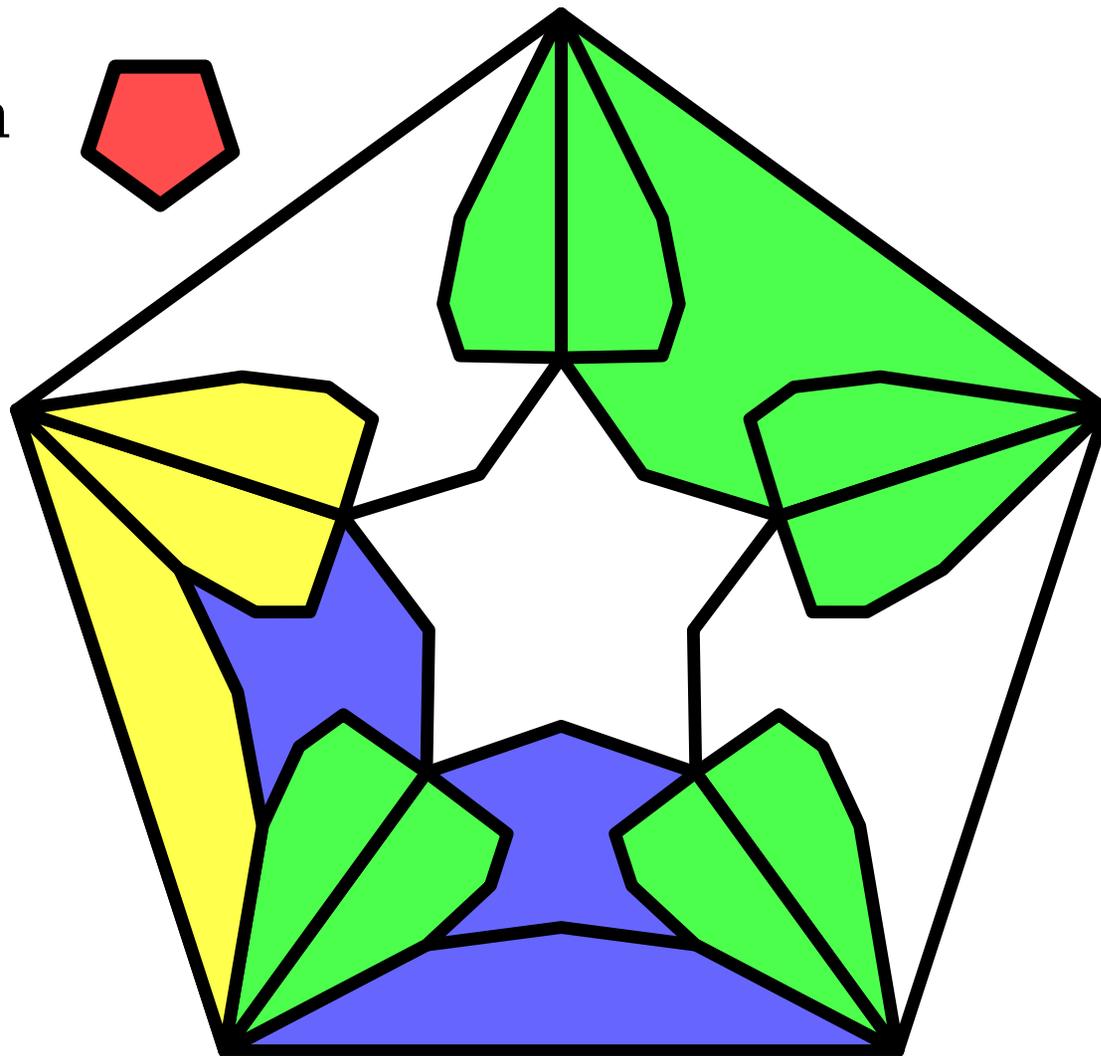
fixed switch:
from (C),
continuation



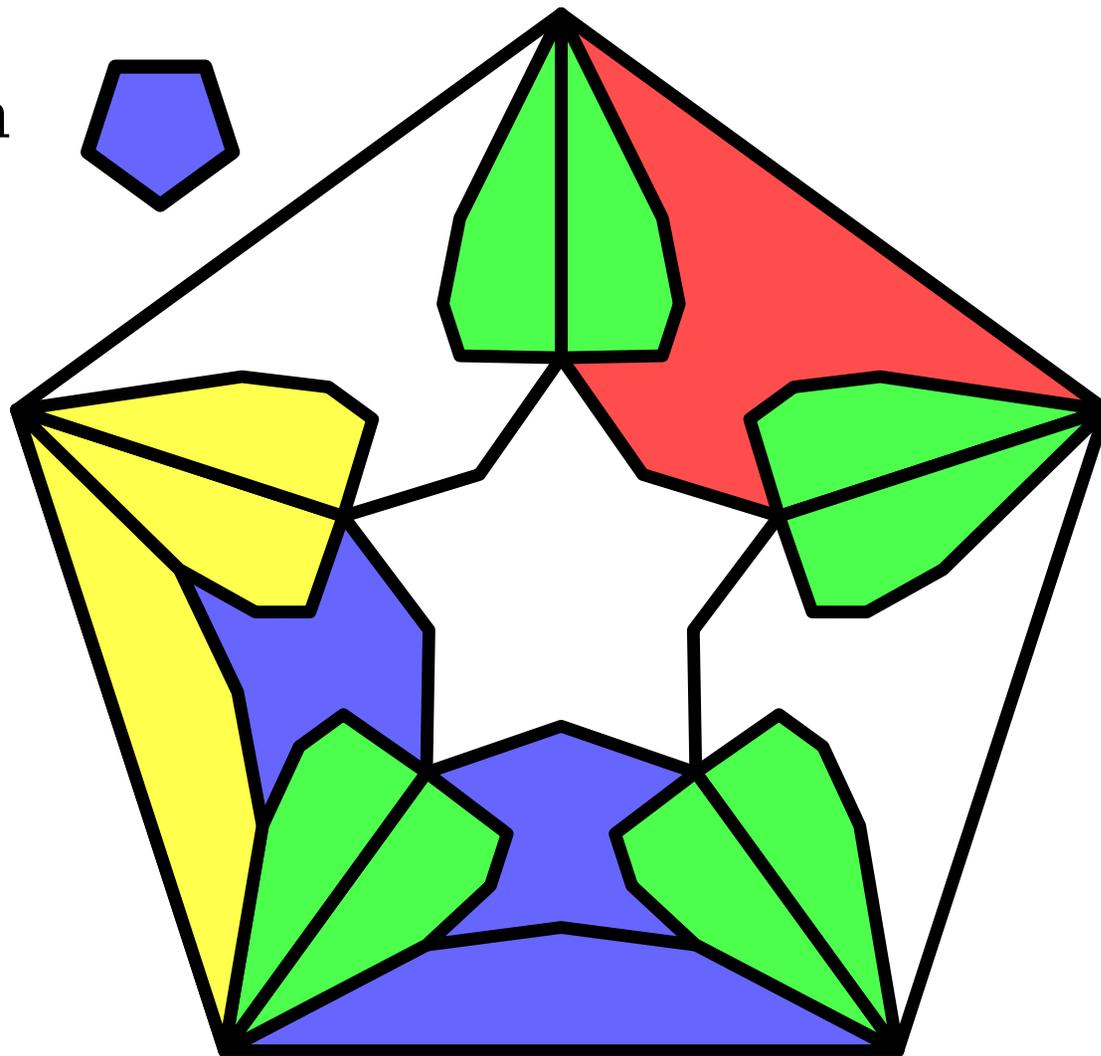
fixed switch:
from (C),
continuation



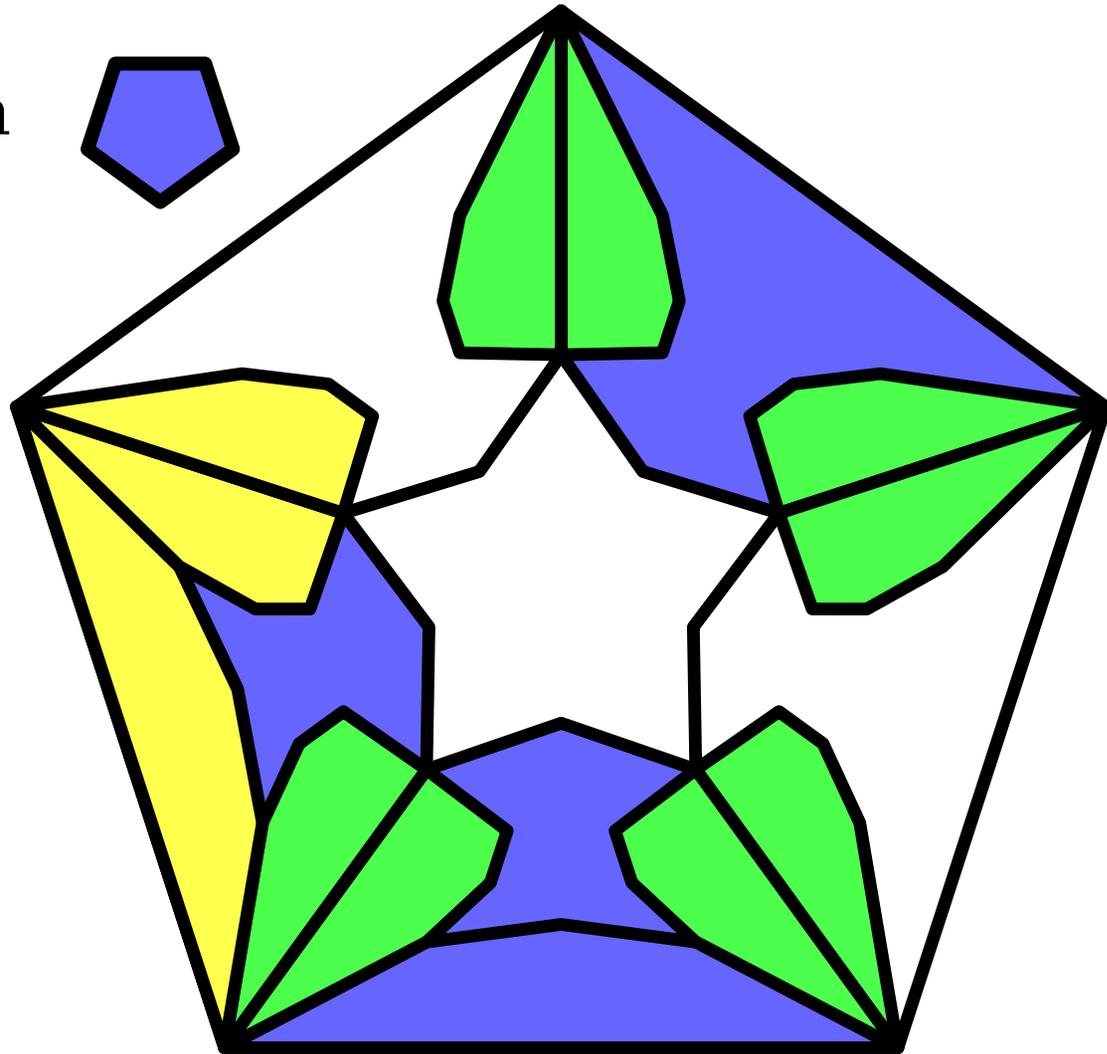
fixed switch:
from (C),
continuation



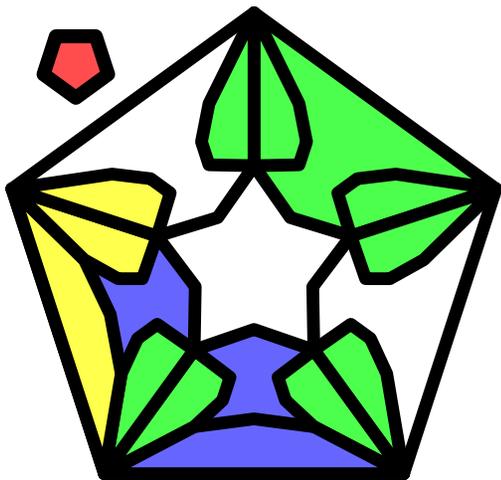
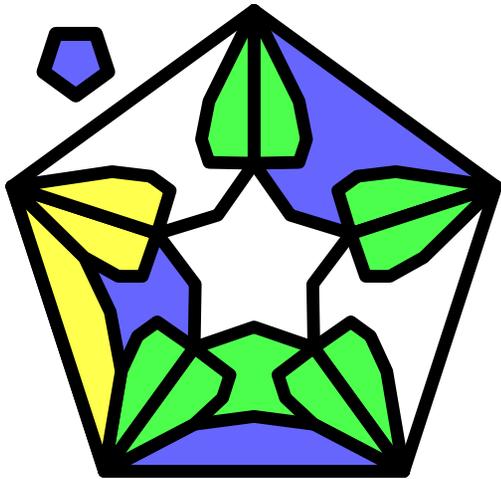
fixed switch:
from (C),
continuation



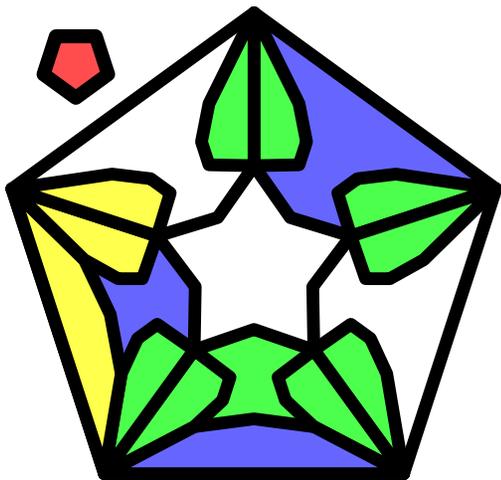
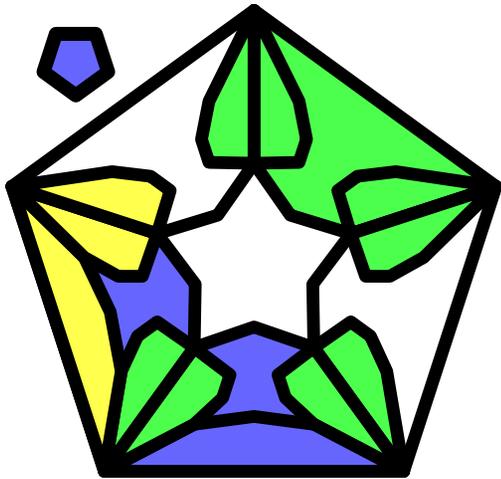
fixed switch:
from (C),
continuation



fixed switch: from (B)

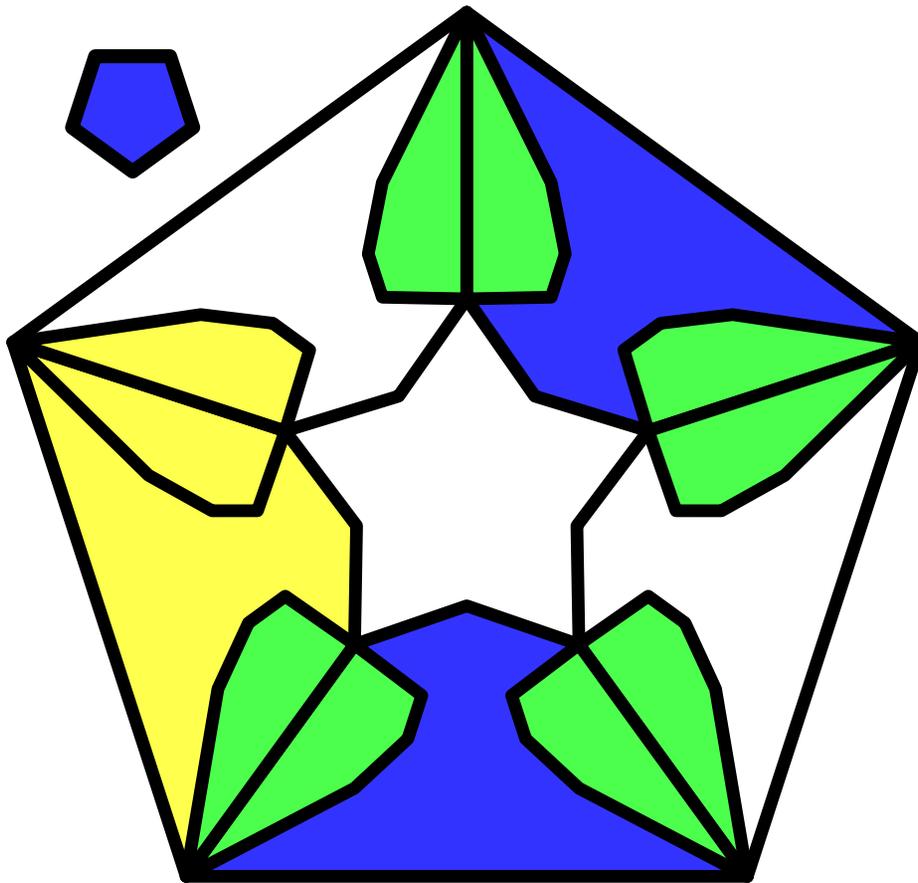


fixed switch: from (A)

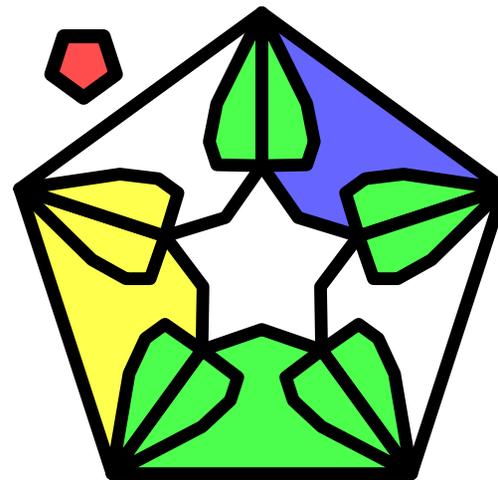
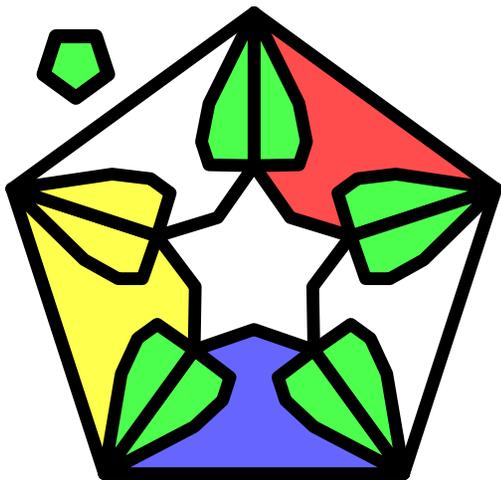


flip-flop switch point:

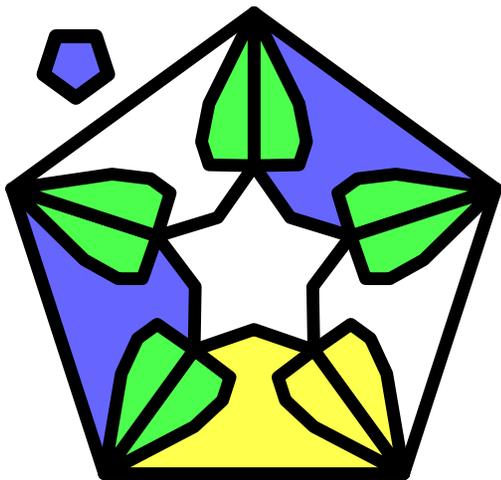
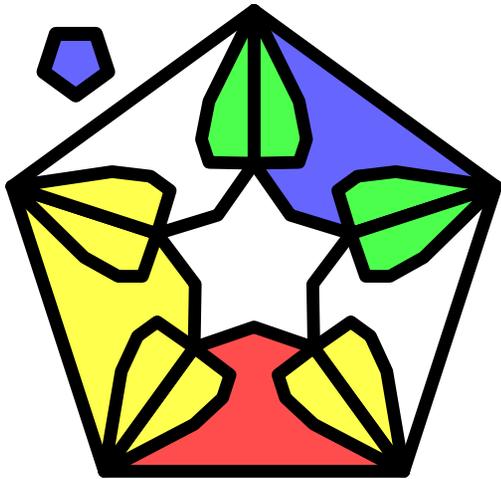
stable configuration



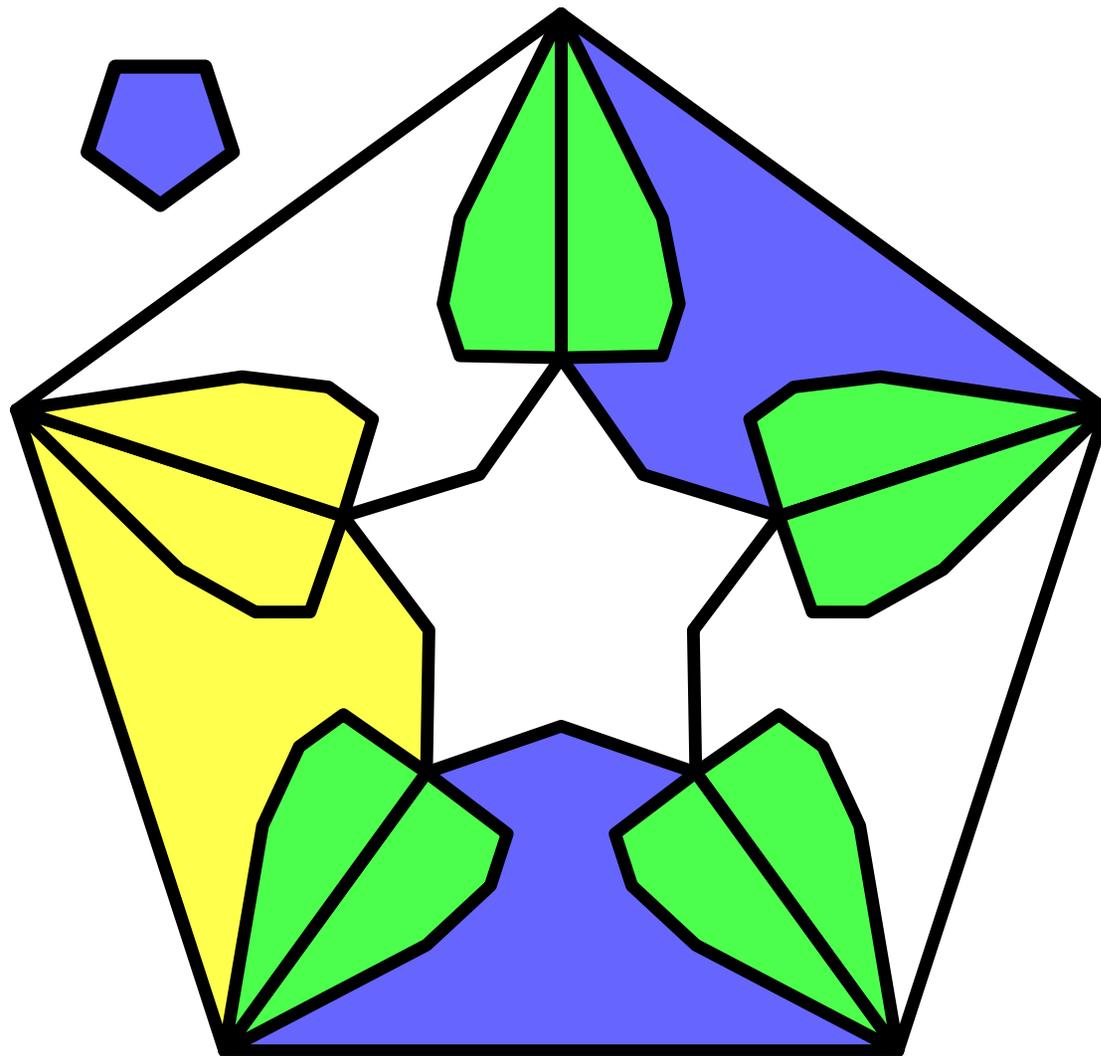
flip-flop switch: of course, from (A)



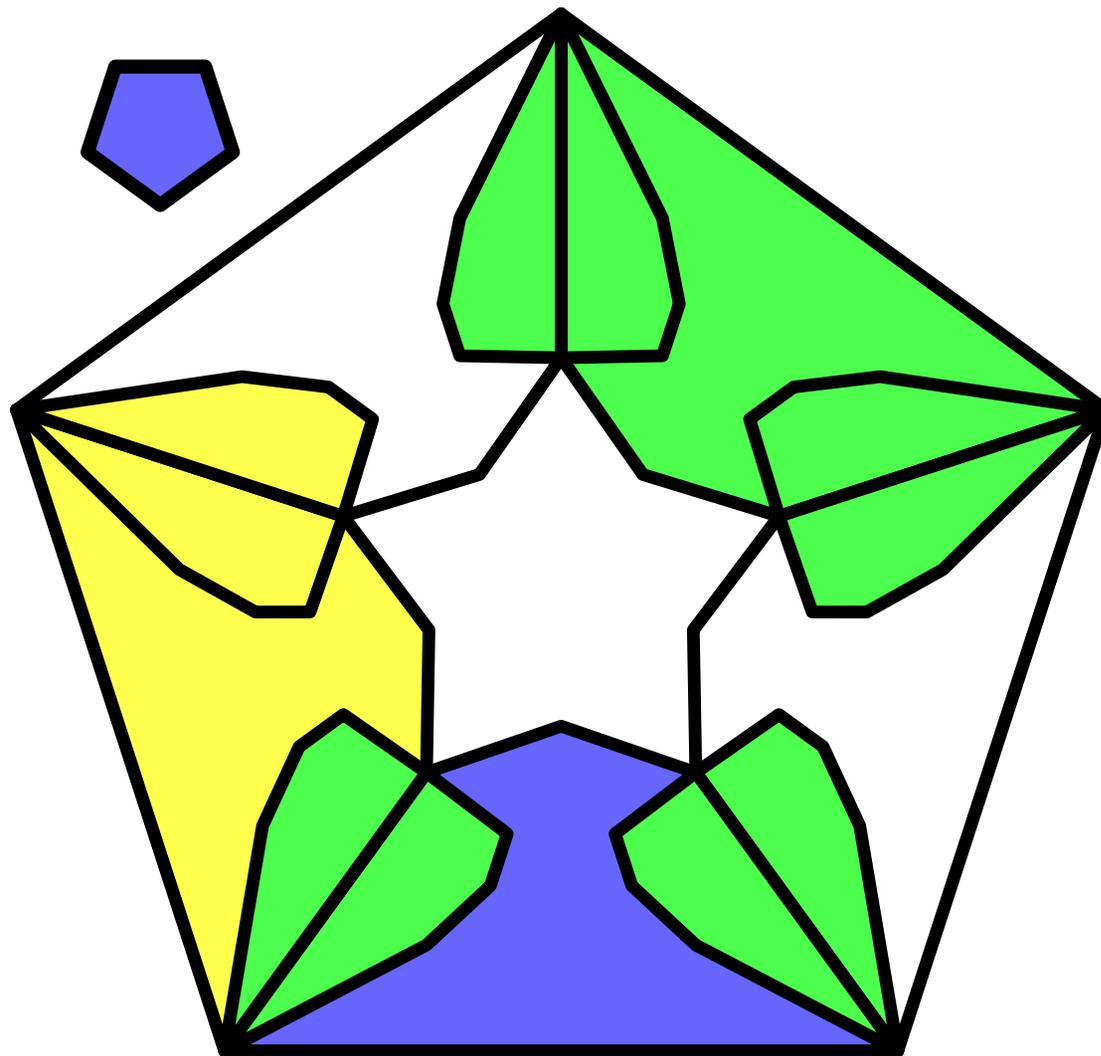
flip-flop switch: from (A), continuation



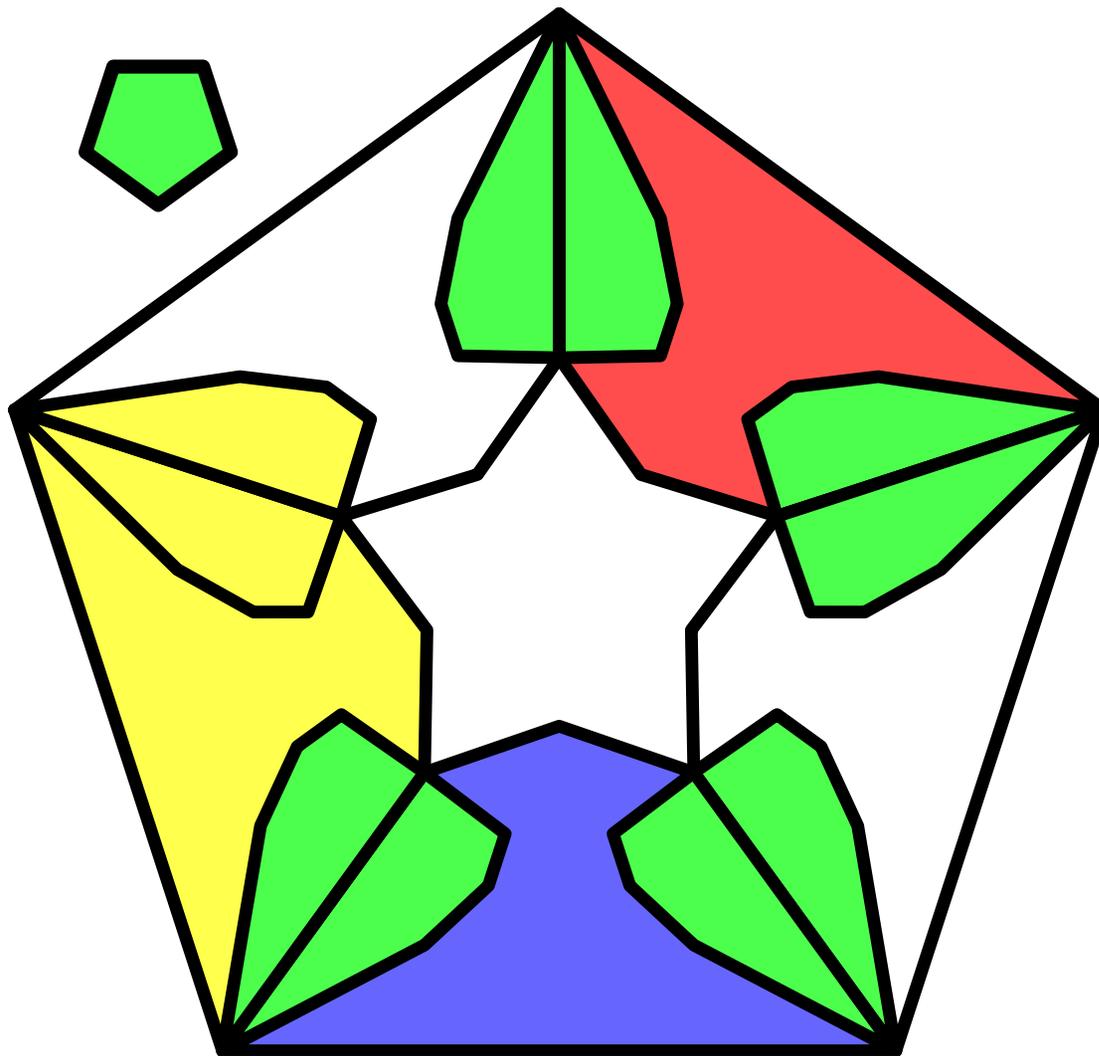
flip-flop switch:
(from (A))



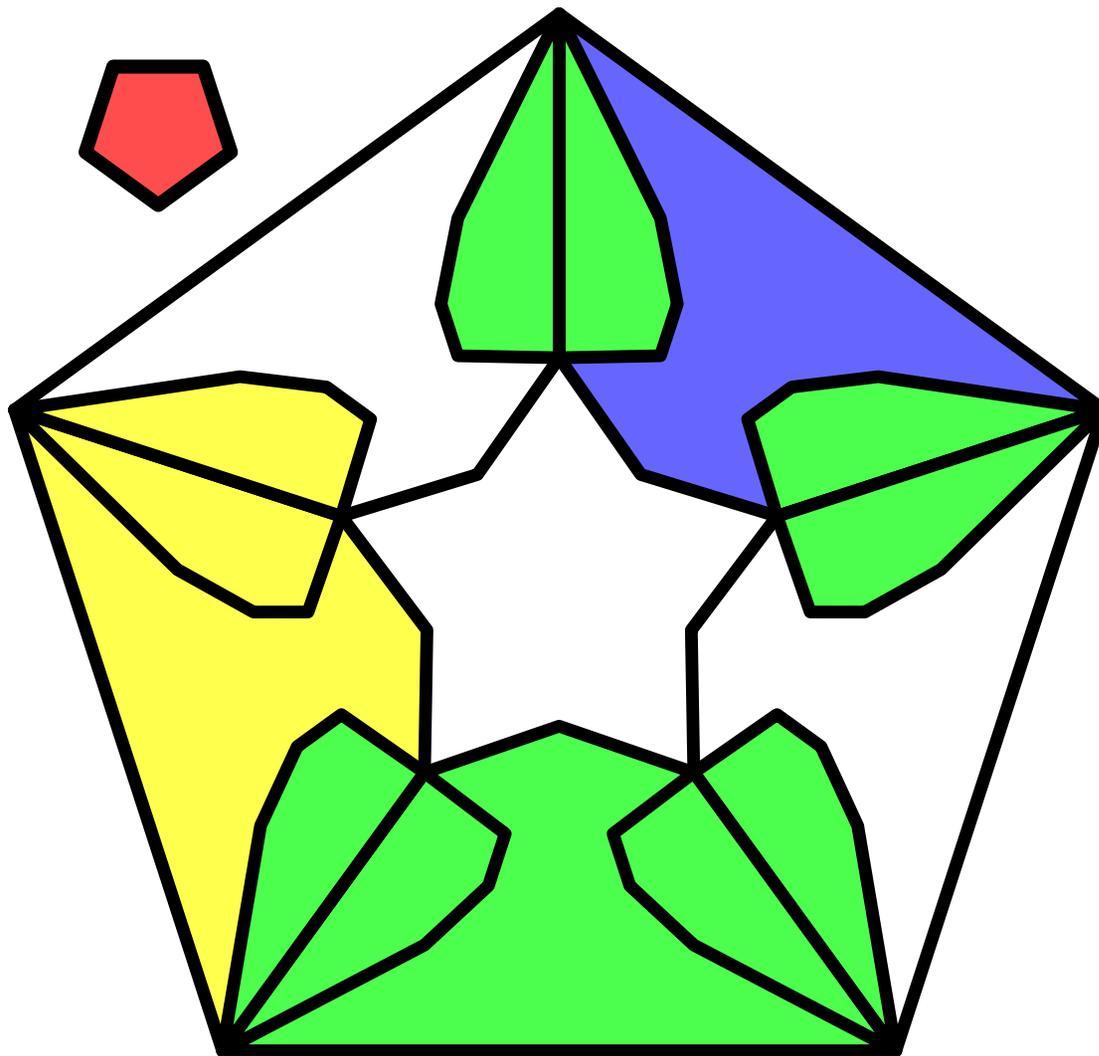
flip-flop switch:
(from (A))



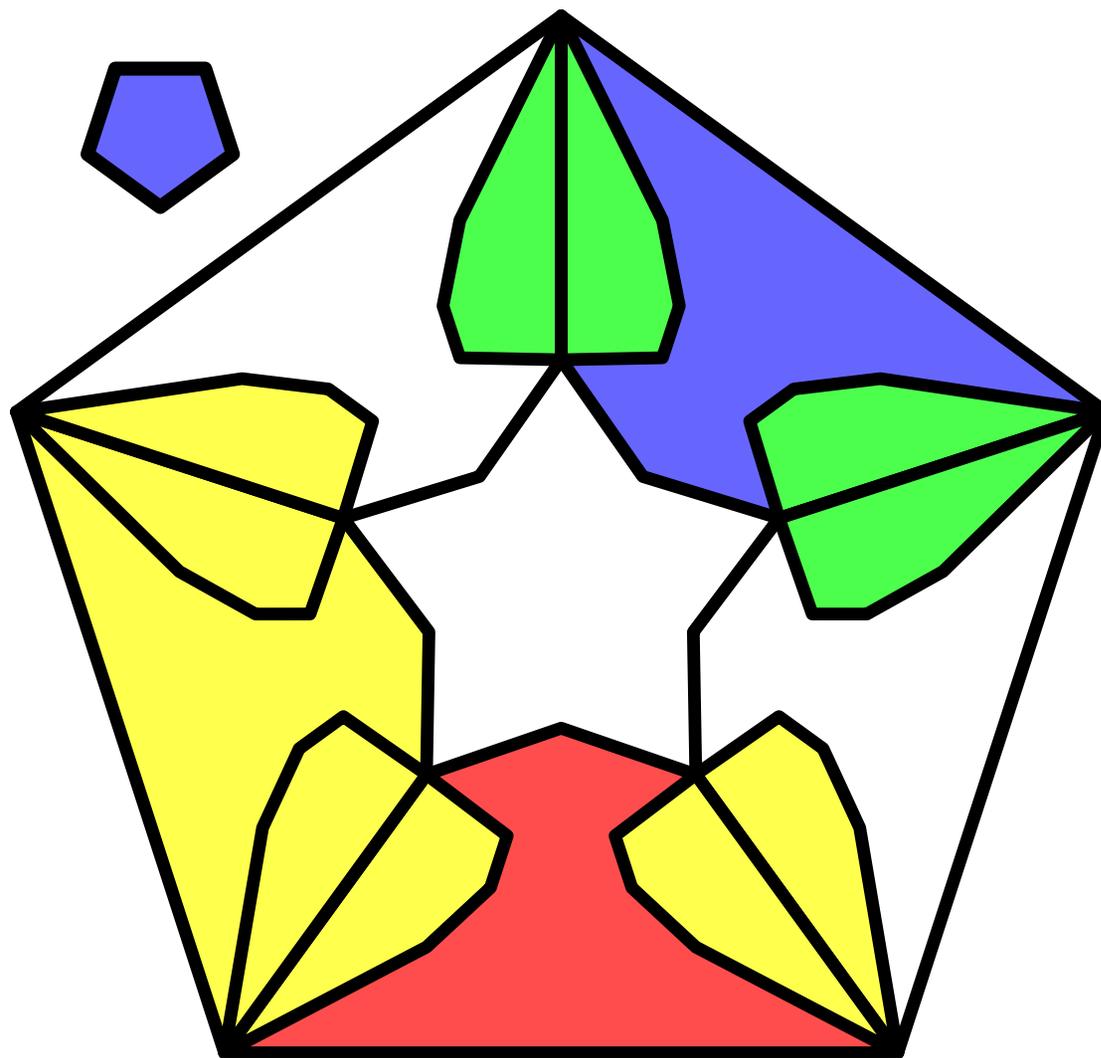
flip-flop switch:
(from (A))



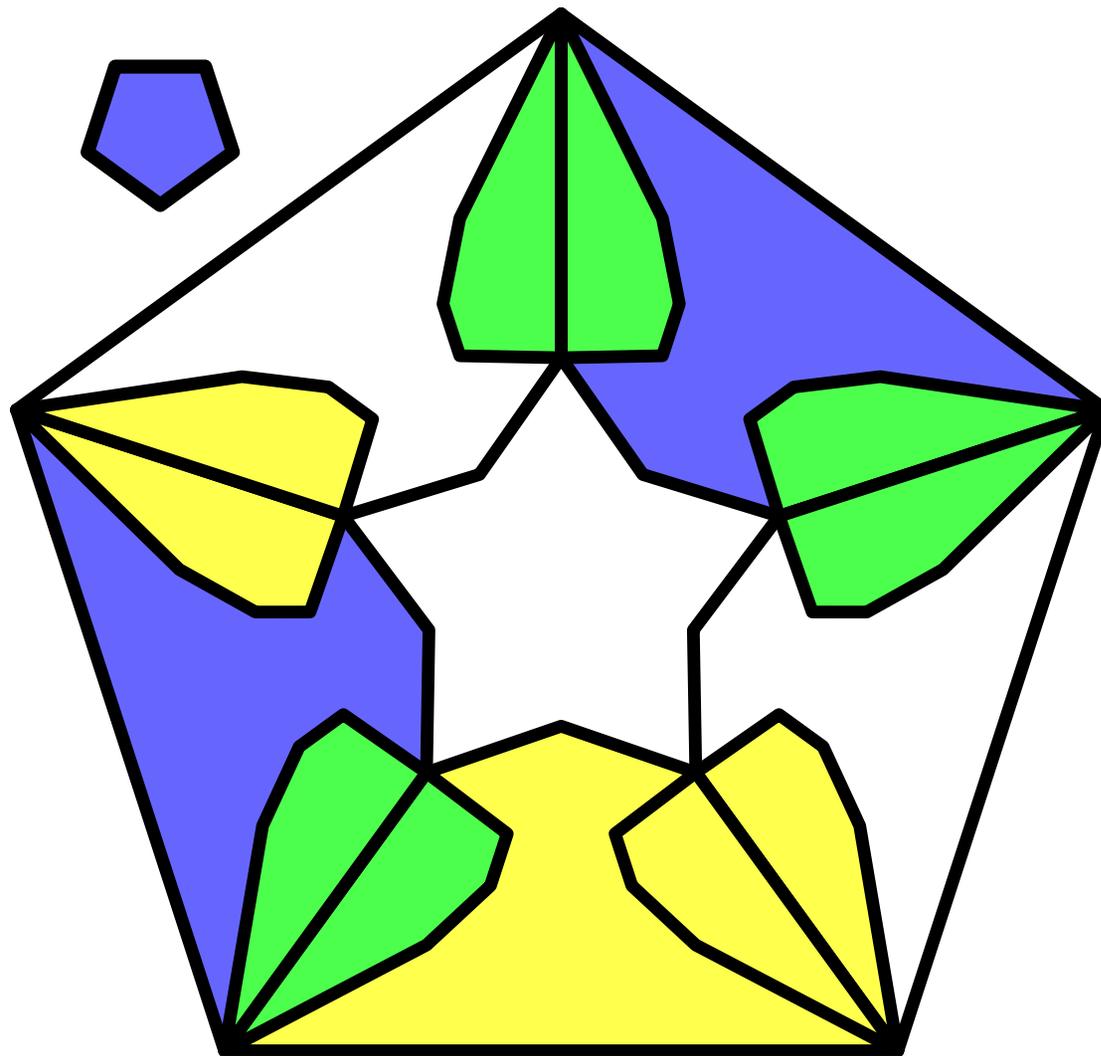
flip-flop switch:
(from (A))



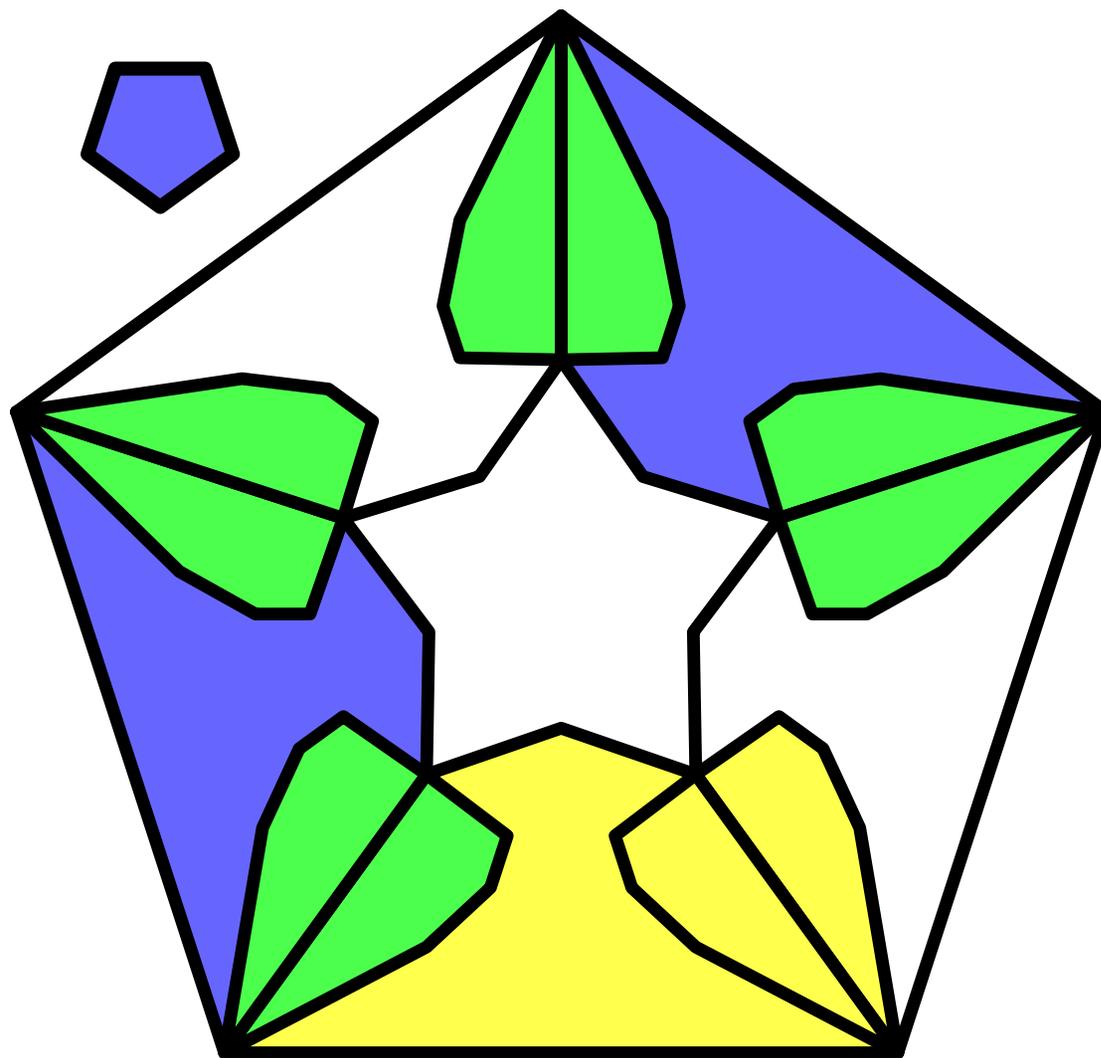
flip-flop switch:
(from (A))



flip-flop switch:
(from (A))



flip-flop switch:
(from (A))



the rules

format

$$B^a G^b R^c Y^d W^e; O \Rightarrow N$$

$$a, b, c, d, e \in \{0..12\}, a + b + c + d + e = 12$$

not only rotation invariant rules:

avoiding difference of **patterns**, using only
difference of **composition**

stability rules

invariance of the background

$$W^{12}; W \Rightarrow W$$

tracks

linear sequence of blue cells ; each one has exactly two blue neighbours, except when arriving on a switch

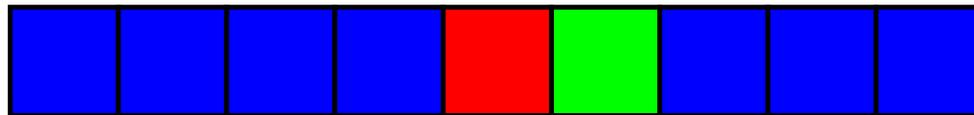
hence:

$$BW^{11}; W \Rightarrow W \quad B^2W^{10}; B \Rightarrow B$$

motion rules

locomotive: two cells on a track,
green for the front, red for the rear

$$\begin{array}{ll} BGW^{10}; B \Rightarrow G & BRW^{10}; G \Rightarrow R \\ BGW^{10}; R \Rightarrow B & BRW^{10}; B \Rightarrow B \end{array}$$



switch rules

simplified notation:

rules grouped by **invariant patterns**, omitted in the rule

distinction between:

stability rules:

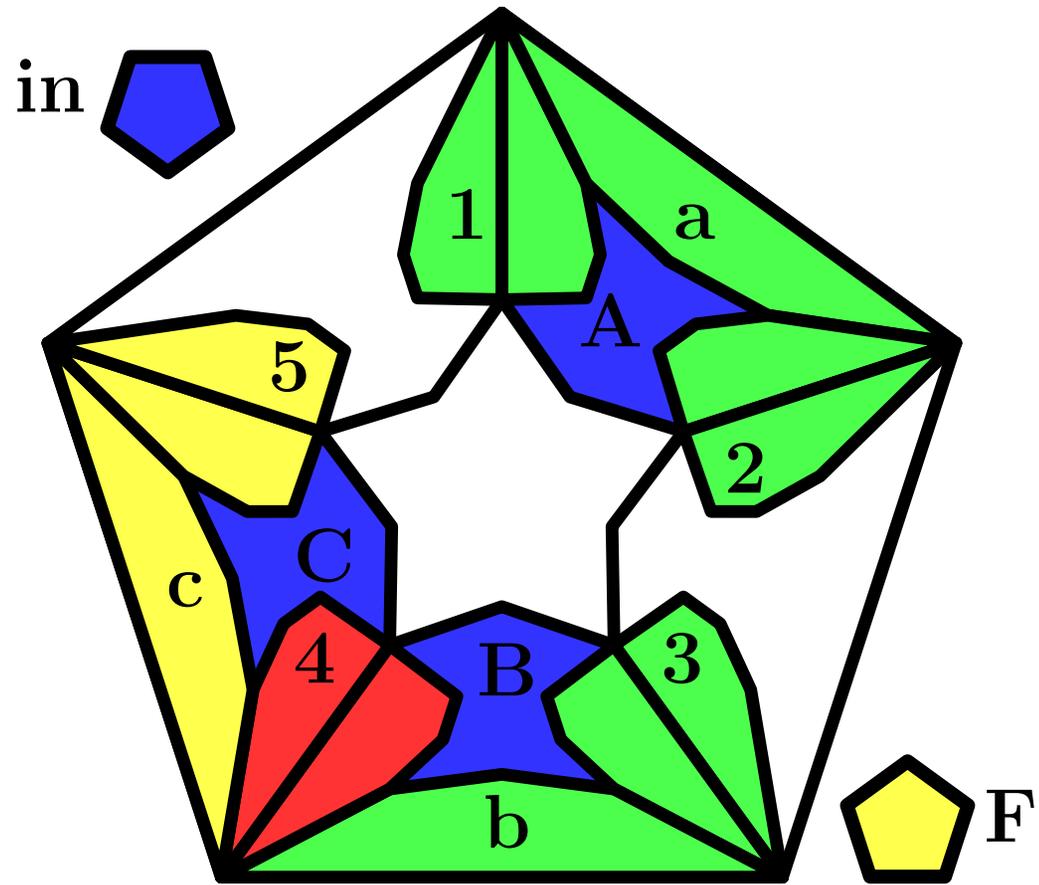
when no locomotive in or near the switch

and

action rules:

when a locomotive crosses the switch

faces,sensors and captors



tables of the patterns

first for faces **A**, **B**, **C** and **F** and for **in**, the cell itself:

	A	B	C	F	in
memory	W^7	$B^2W^3Y^2$	$B^2W^3Y^2$	W^3Y^5	R^5W^3
fixed	W^8	W^5Y^2	$G^2W^3Y^2$		W^9
flip-flop	W^8	W^8	W^8		W^9

F is not white only for memory switches

tables of the patterns, continuation 1

next, for sensors **1**, **2**, **3**, **4** and **5**:

	1	2	3	4	5
memory	W^{11}	W^{11}	W^8Y^3	W^6Y^4	W^8Y^3
fixed	W^{11}	W^{11}	W^8Y^3	W^{10}	Y^{11}
flip-flop	W^{11}	W^{11}	W^7Y^4	W^{10}	W^7Y^4

these patterns are present for all switches

last table for **captors** of faces **B** and **C** which are specific to memory and fixed switch points:

	a	b	c
memory	G^5W^4	W^4Y^5	W^4Y^5
fixed		W^5Y^4	$G^2R^4W^3$

a present only for memory switches ;

b and **c** also missing in flip-flops

enough to organise checking by patterns ;
however, we display the rules logically

for memory switches:

A:	$B^2G^3; BB$	$BG^4; BG$	$BRG^3; GR$	$G^5; RB$
	$BRG^3; BB$	$BG^4; RB$		
B:	$B^2G^2R; BB$	$BG^4; BB$	$BRG^2R; BB$	$BG^3Y; BY$
	$B^2G^3; YB$	$B^2RY^2; BB$	$BG^3R; BG$	$BG^2R^2; GR$
	$BG^4; RB$	$BG^2R^2; BB$		
C:	$B^2RY^2; BB$	$BGR^2Y; BG$	$BR^3Y; GR$	$BG^2R^2; RR$
	$BG^3Y; RB$	$B^2G^3; BB$	$B^2G^2R; BB$	$BGRY^2; BB$
	$BR^2Y^2; BB$	$B^2GY^2; BB$	$BG^2Y^2; BB$	

memory switches, continuation (1):

F:	$BG^2Y; YY$	$BG^2R; YR$	$BG^2R; RR$	$G^3R; RR$
	$G^4; RY$	$BG^3; YY$	$G^3Y; YY$	$G^2RY; YY$
in:	$B^3Y; BB$	$B^2GR; BG$	$B^2R^2; GR$	$BGR^2; RG$
	$BR^3; GB$	$B^2Y^2; BB$	$B^2GY; BG$	$B^2RY; GR$
	$B^2GY; RB$	$B^2RY; BB$	$BR^3; GB$	$B^2Y^2; BB$
1:	$B; GG$	$R; GG$	$G; GG$	
2:	$B; GG$	$R; GG$	$G; GG$	
3:	$B; GG$	$Y; GY$	$B; YY$	$G; GG$
	$R; GG$			
4:	$B^2; RR$	$BG; RG$	$BR; GR$	$BR; RY$
	$BR; YG$	$BY; GR$		
5:	$B; YY$	$G; YR$	$R; RG$	$R; GG$

memory switches, continuation (2):

a:	$BBY; GG$	$BBR; GG$	$BGR; GG$	$BRR; GG$
	$BGY; GG$	$BR Y; GG$		
b:	$B^2Y; GG$	$B^2R; GG$	$BYY; GY$	$B^2Y; YY$
	$BGY; GG$	$BR Y; GG$	$BR Y; GG$	
c:	$BBY; YY$	$BGY; YR$	$BR Y; RR$	$BGR; RR$
	$BRR; GG$	$BBY; GG$		

as **b** and **c** have the same pattern, we check that the rules are distinct, this is why rules for **c** are also for **b** and conversely

fixed switches:

captor **a** missing ; **F** always white

A:	$B^2G^2; BB$	$BG^3; BG$	$BG^2R; GR$	$BG^3; RB$
	$BG^2R; BB$			
B:	$B^3G^2; BB$	$B^2G^3; BG$	$B^2G^2R; GR$	$B^2G^2Y; RB$
	$B^2G^2R; BB$	$B^2G^2Y; BB$		
C:	$B^2GY^2; BB$	$BG^2Y^2; BB$	$BGRY^2; BB$	$B^2Y^3; BB$
	$BG^2RY; BG$	$BGR^2Y; GR$	$BGRY^2; BB$	
in:	$B^3; BB$	$B^2G; BG$	$B^2R; GR$	$B^2G; RB$
	$B^2R; BB$			
b:	$B^2; BB$	$BG; BB$	$BR; BB$	
c:	$B^2; YY$	$BG; YR$	$BR; RR$	$BG; RY$
	$BR; YY$			

fixed switches, continuation:

2 follows the same rules as **1** ;
the same for **3** although its pattern is different from the common pattern of **1** and **2**

1:	$B; GG$	$G; GG$	$R; GG$
4:	$B^2; GG$	$BG; GY$	$BR; YG$
5:	$B; YY$	$G; YY$	$R; YY$

flip-flop switches:

no more captors, **A**, **B** and **C** same rules

A:	$B^2G^2; BB$	$BG^3; BG$	$BG^2R; GR$	$BG^3; RB$
	$BG^2R; BB$			
B:	$B^2G^2; BB$	$BG^3; BG$	$BG^2R; GR$	$BGY^2; RY$
	$BGRY; YY$	$B^2GY; YY$		
C:	$B^2GY; YY$	$BG^2Y; YY$	$BGRY; YY$	$B^2Y^2; YB$
	$B^2GY; BB$			
in:	$B^2Y; BB$	$BGY; BG$	$BRY; GR$	$BGY; RB$
	$BRY; BB$			

flip-flop switches, continuation:

2 follows the same rules as **1** ;

5 follows the same rules as **3** ;

1:	<i>B; GG</i>	<i>G; GG</i>	<i>R; GG</i>	
3:	<i>B; GG</i>	<i>G; GY</i>	<i>R; YY</i>	<i>Y; YY</i>
4:	<i>BY; GG</i>	<i>GY; GY</i>	<i>RY; YG</i>	
5:	<i>Y; YY</i>	<i>G; YY</i>	<i>B; YG</i>	<i>B; GG</i>

the compatibility between all the rules was checked by a software

hence, we proved:

theorem – *there is a universal cellular automaton with 5 states in the rectangular dodecagrid of the 3D hyperbolic space.*

of course, here universal = *weakly* universal

not best result,

possible improvements

using computer simulations

5. beyond the Turing barrier

most models for going beyond the Turing barrier either make use of mathematical real numbers or consist in manipulation on **time**

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\Rightarrow super Turing computations using only rational numbers

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one model, **black hole** computations of Jérôme Durand-Lose, also introduces geometric considerations

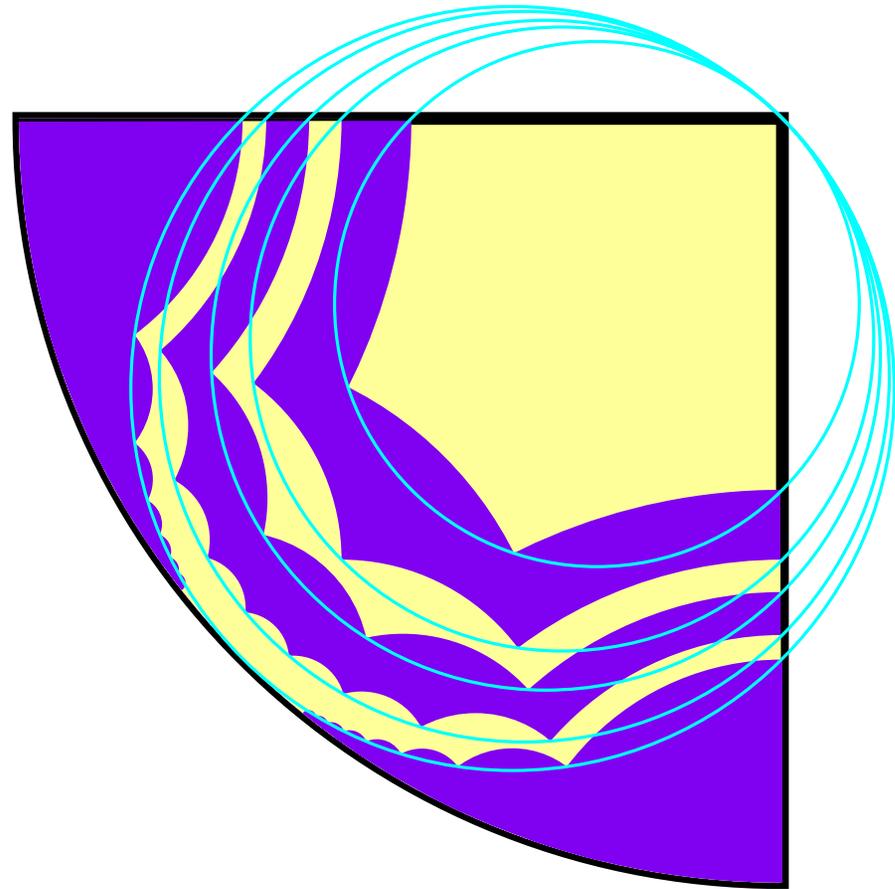
⇒ super Turing computations using only rational numbers

here, thanks to the hyperbolic plane, we get super Turing computations in finite time, using mainly **space**

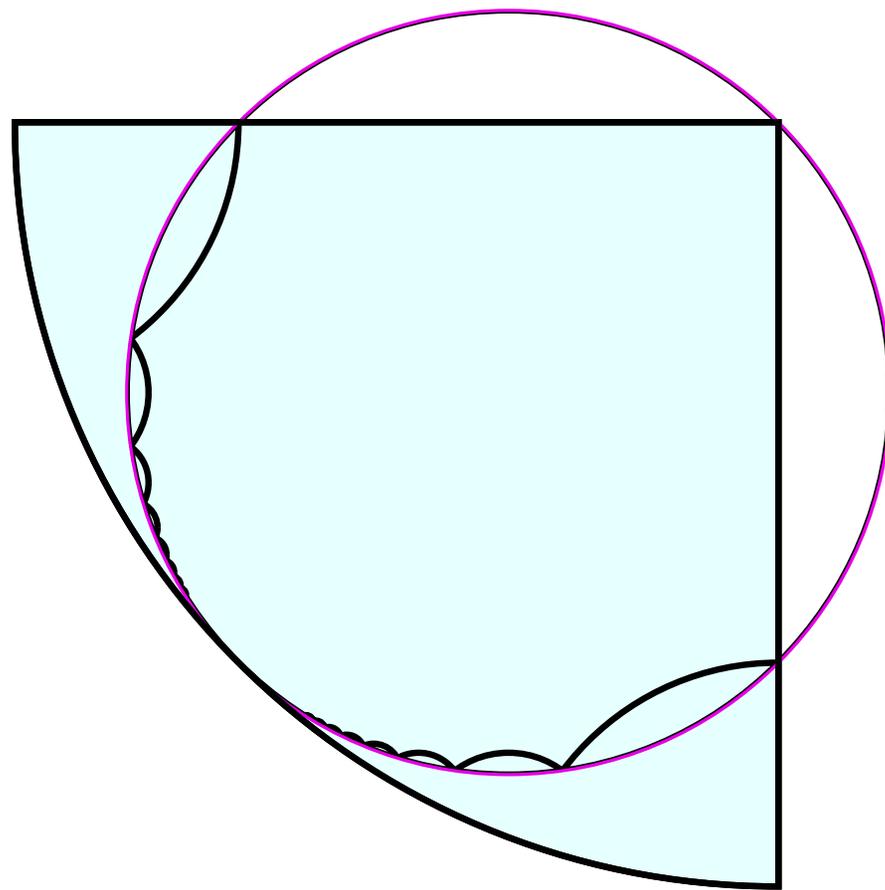
infinigons and infinigrids

in the hyperbolic plane, consider **all regular rectangular polygons under the same view**

their limit
 \Rightarrow an infinigon:



the visual limit:



basic construction

let $x_n x_{n+1}$, $n \in \mathbb{Z}$,

be the sequence of segments such that:

- $\forall n : x_n x_{n-1}, x_n x_{n+1} = x_{n+1} x_n, x_{n+1} x_{n+2}$
- $\forall n : \|x_n x_{n+1}\|_h = \|x_{n+1} x_{n+2}\|_h$

result:

the x_n 's are on an e -cercle Γ

if $x_0 = 0$ and $\|x_n x_{n+1}\|_e = x$, $x \in]0, 1[$

then, diameter of $\Gamma = \frac{x}{\cos(\frac{\alpha}{2})}$

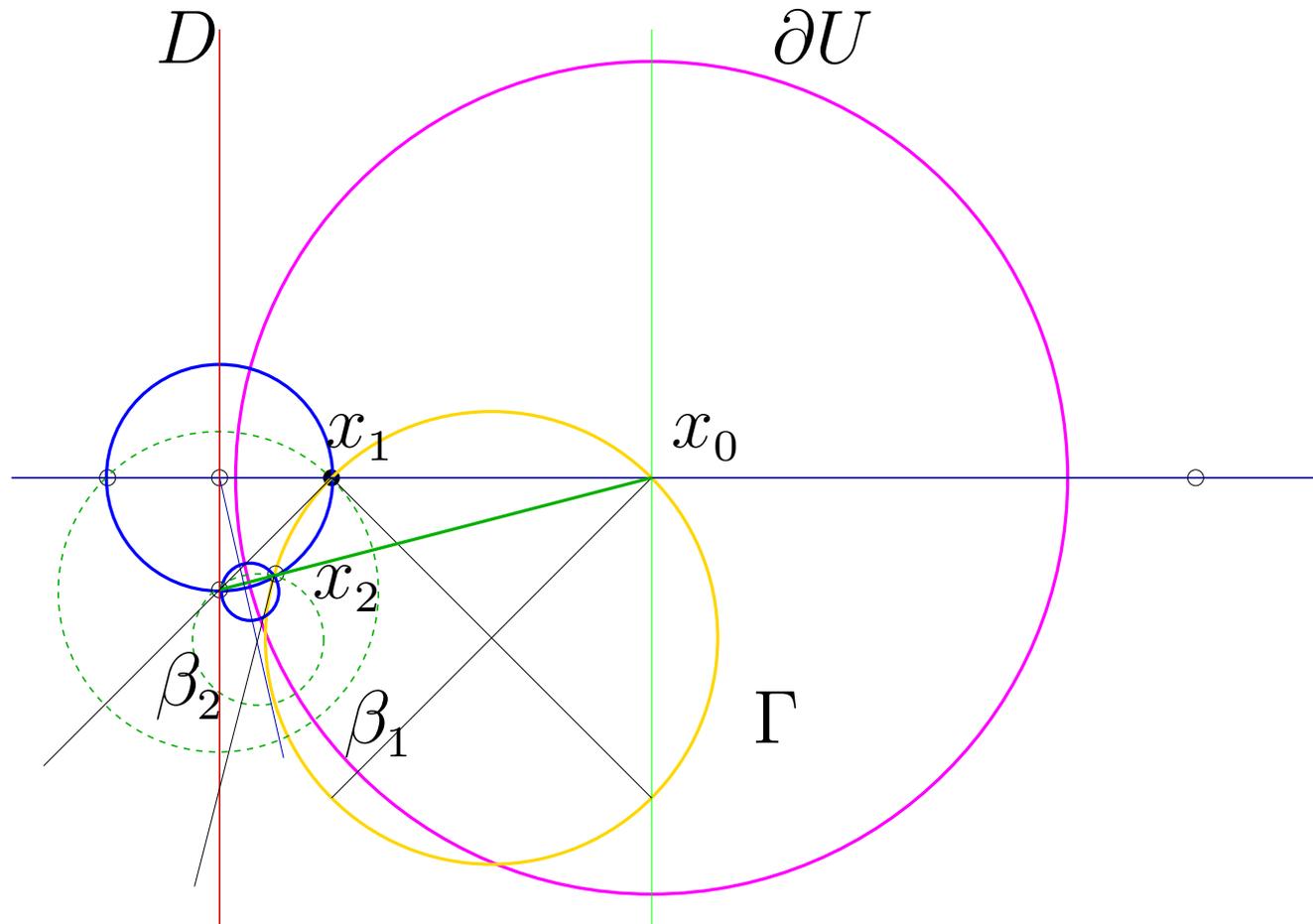
let U be the open unit disc ;

- if $\Gamma \subset U$, the x_n 's describe: either a regular polygon, or a dense subset in an annulus
- if $\Gamma \subset \bar{U}$ and $\Gamma \not\subset U$,
then Γ is a *horocycle* and the x_n 's are on a *closed infinigon*
- if $\Gamma \not\subset \bar{U}$, then Γ is an *equidistant curve*
and the x_n 's are on an *open infinigon*

points at infinity of an infinigon:

- closed infinigon: a single point
- open infinigon: closed interval of ∂U

basic construction in the disc model:



infinigrids:

tessellation:

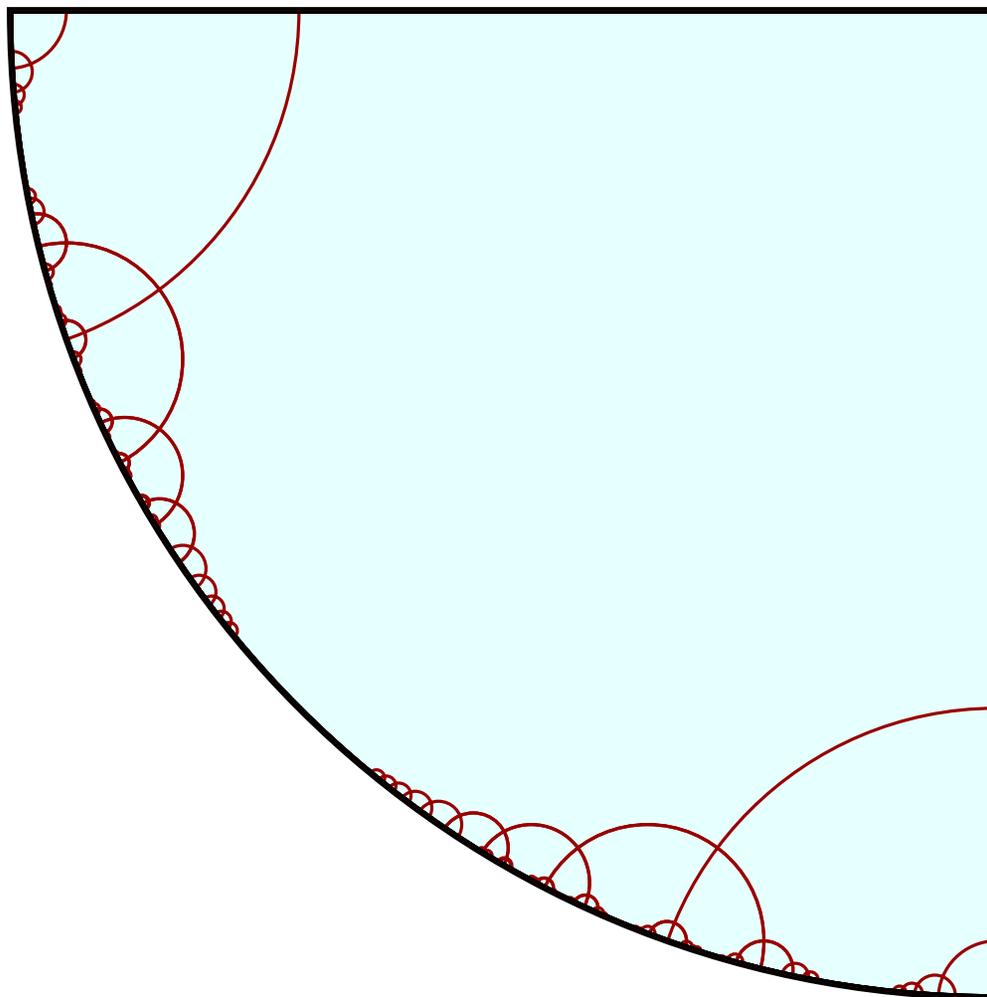
fix an infinigon ; replicate it by reflection in its sides and recursively repeat with the images

theorem 1 (Coxeter/Rozenfeld/Margenstern)

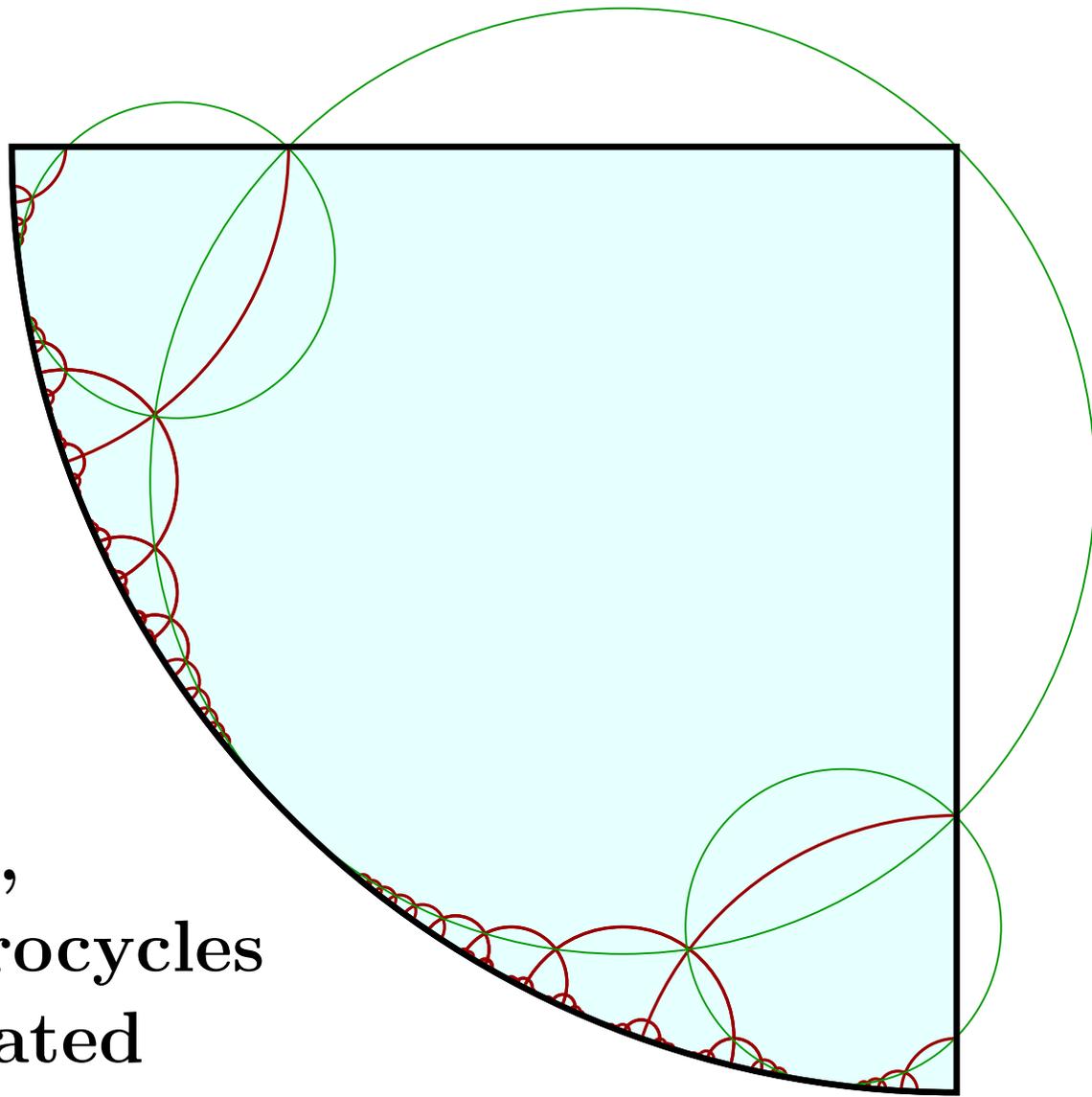
– *an infinigon generates a tiling by tessellation if and only if its interior angle is $\frac{2\pi}{k}$, $k \geq 3$*

infinigon: either closed or open

disc model: the rectangular infinigrd



the same,
when horocycles
are indicated



extension of the splitting method:

theorem 2 (Margenstern) – *the tiling generated by an infinigon is in bijection with an infinite tree with infinite branching at every node*

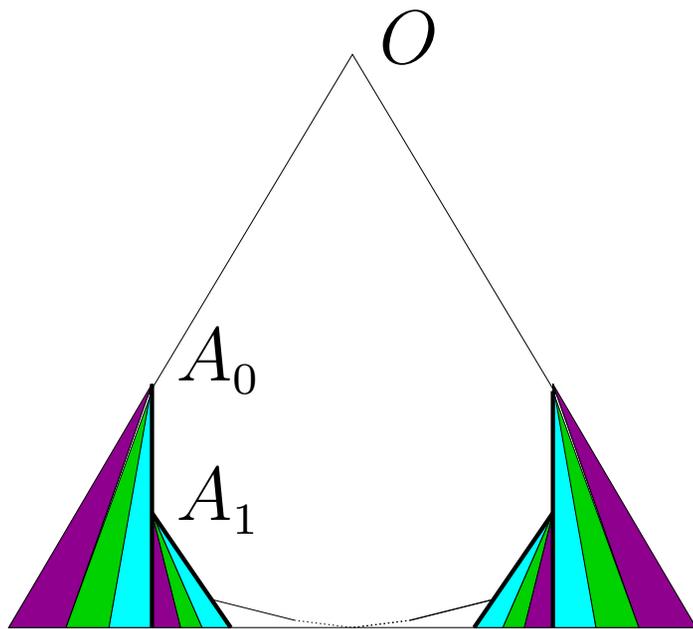
proof based on a recursive splitting:

regions of the splitting

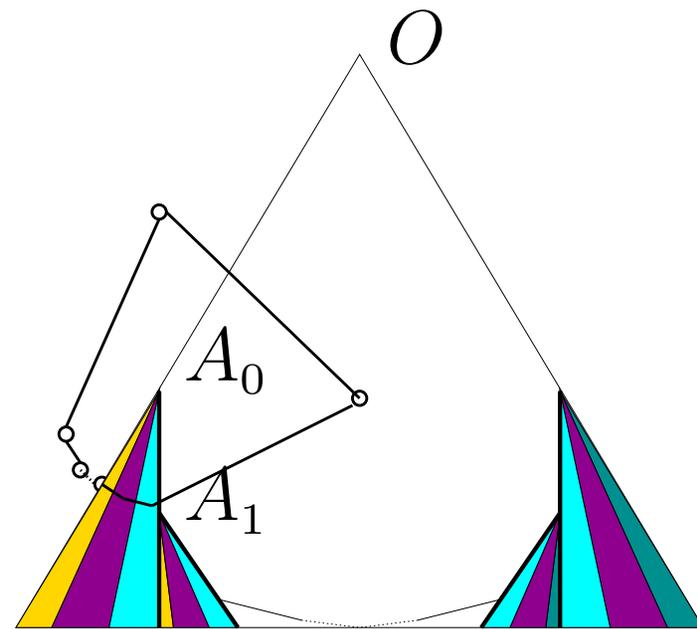
= copies of initial regions

by induction, generates a spanning tree of the dual graph

splitting of the hyperbolic plane:



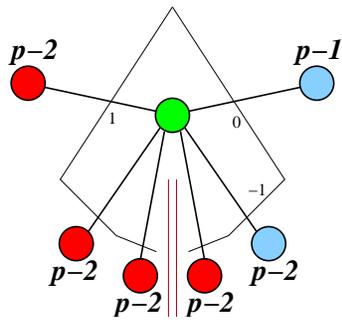
even case



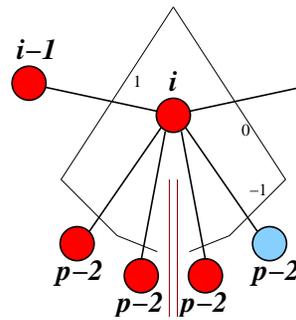
odd case

algorithms of construction:

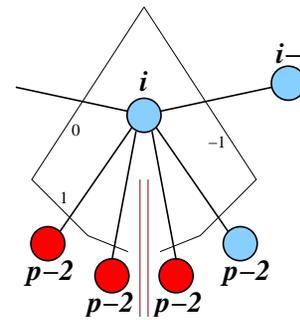
(i)



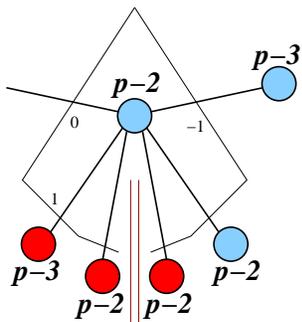
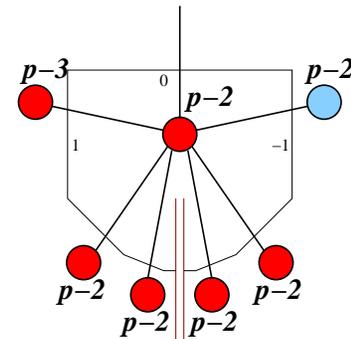
(ii)



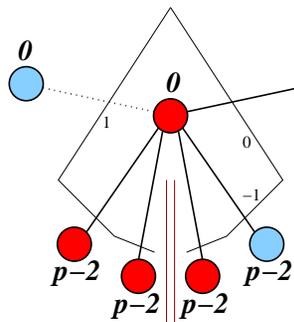
(iii)



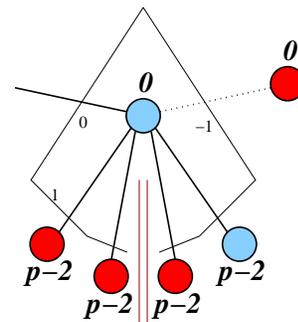
(iv)



(v)



(vi)



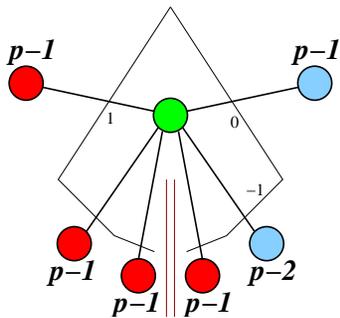
(vii)

in (iii):
 $i < p-2$

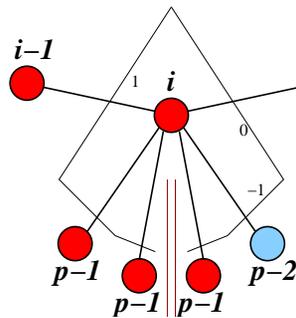
even case

algorithms of construction:

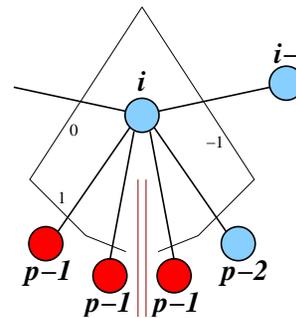
(i)



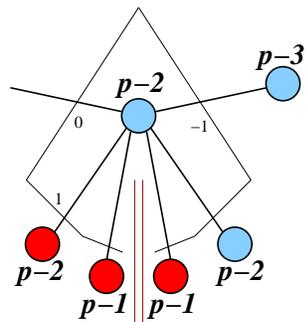
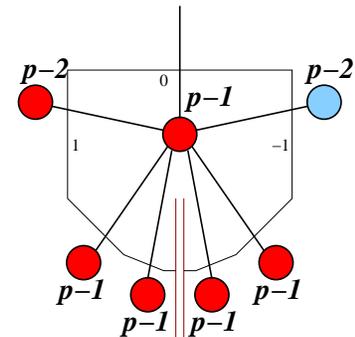
(ii)



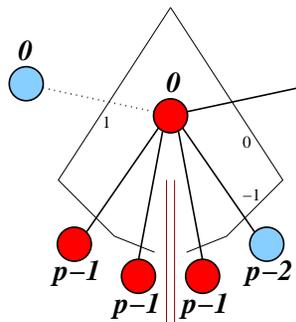
(iii)



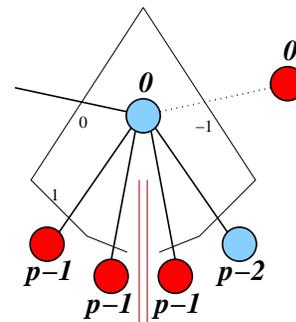
(iv)



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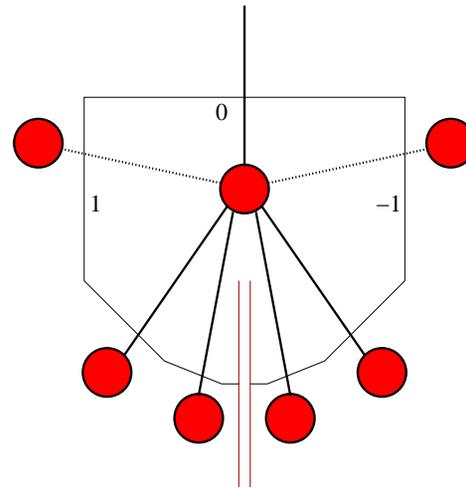
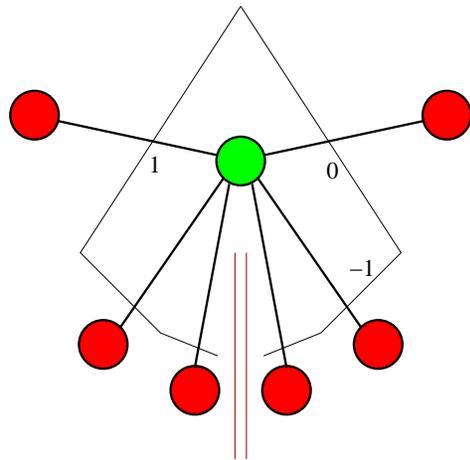
(vii)

in (iii):
 $i < p-2$

odd case

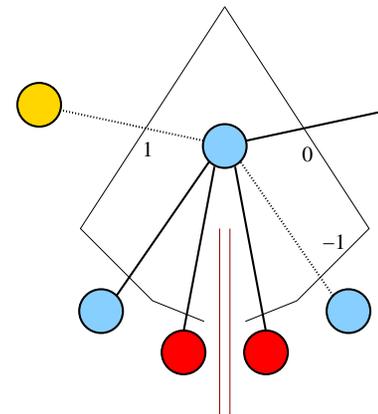
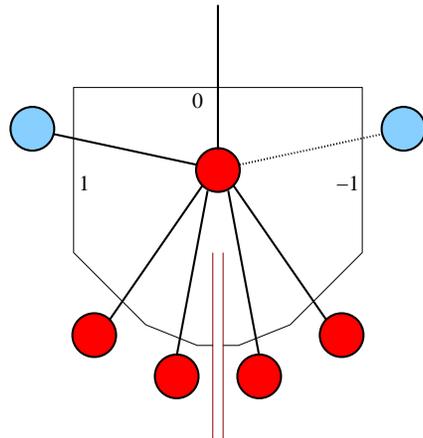
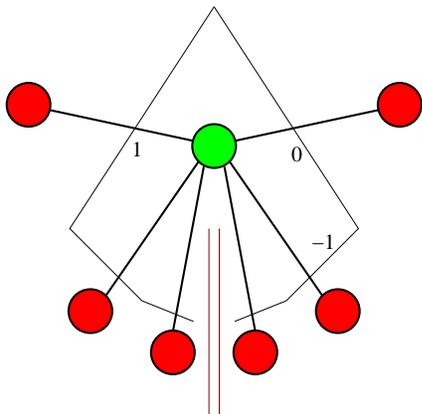
particular cases:

$$k = 3$$



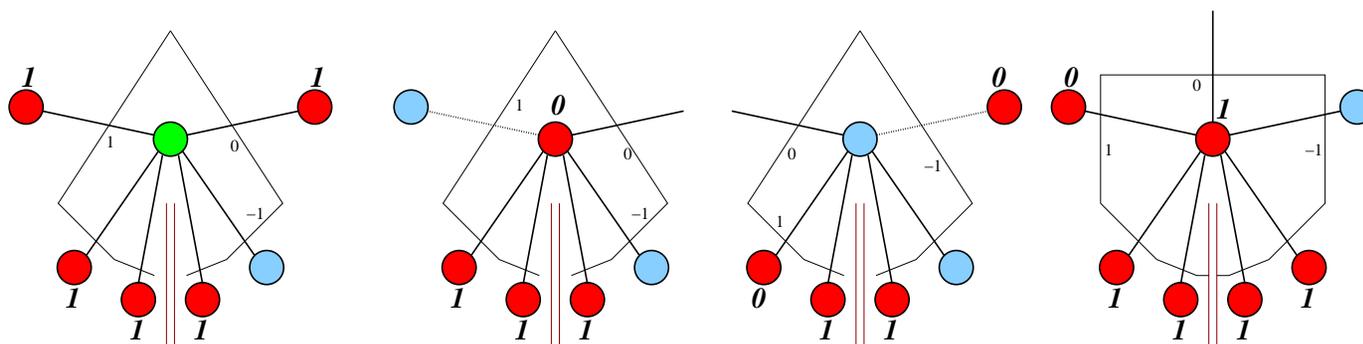
particular cases:

$$k = 4$$



particular cases:

$$k = 5$$



infinigrind: a new class of CA's

definition 1 – (Grigorieff-Margenstern, SCI 2002 ; *FI*, **61**, 2004)

a CA on an infinigrind is adapted if its transition function δ satisfies:

$$\delta : Q \times \{0, 1\}^{|Q|} \mapsto Q$$

with

$$\langle s, t+1 \rangle = \delta(\langle s, t \rangle, z_1(s, t), \dots, z_{|Q|}(s, t))$$

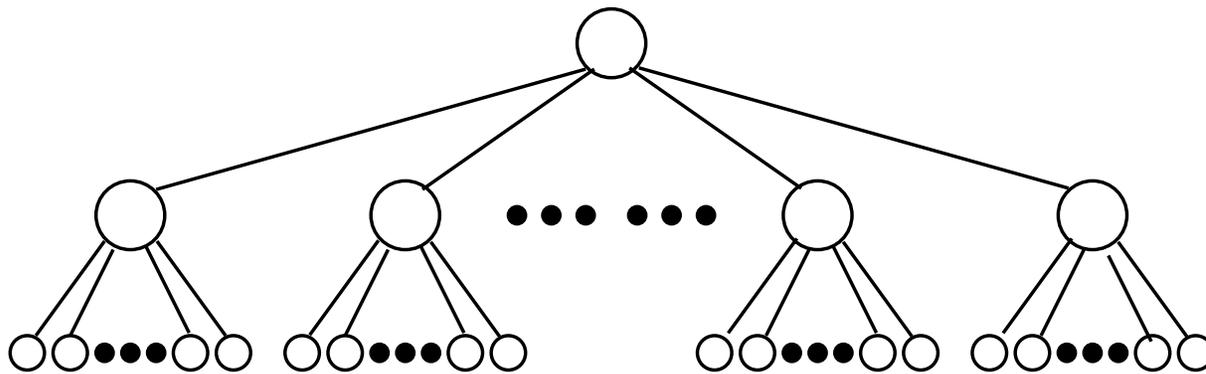
where $1, \dots, k$ are the states of the automaton

and:

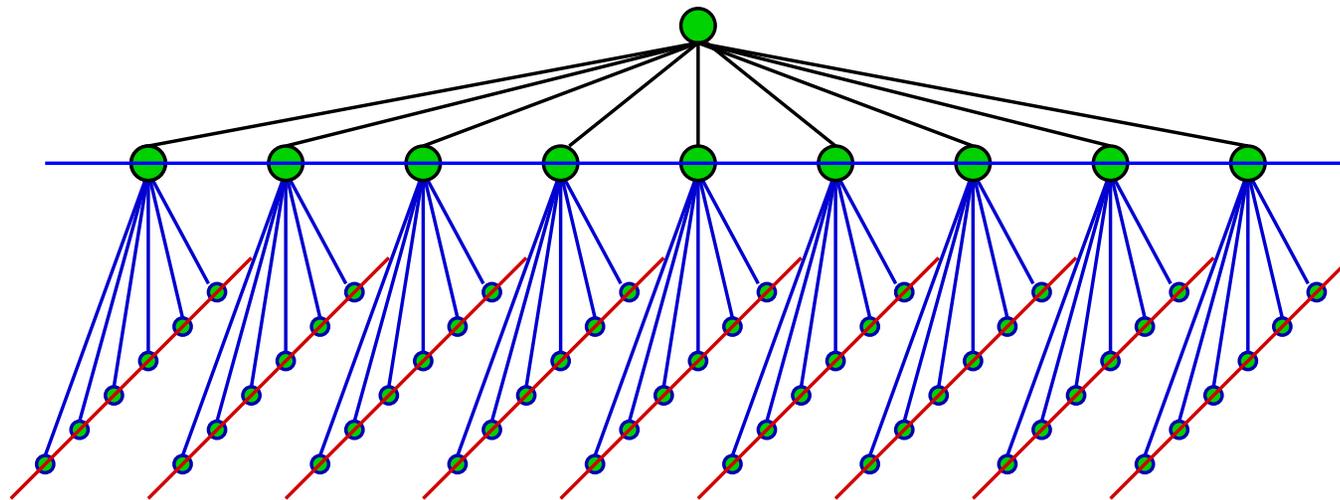
$$z_i(s, t) = \begin{cases} 1 & \text{if there is a neighbour of the} \\ & \text{cell under the state } i \text{ at time } t \\ 0 & \text{otherwise} \end{cases}$$

addresses of the cells: (a_1, \dots, a_n) , $a_i \in \mathbb{N}$

infinigrids: tree representation



infinigrids: another representation



theorem 1 – (G.-M., quoted papers)
*there is an adapted CA on the infinigrd such
that for any arithmetic formula F from Σ_n^0 or
from Π_n^0 , U recognizes whether F is true or not*

proof

we may assume F to be closed

let $F = \exists x_1 \forall x_2 \dots \xi x_n G(x_1, \dots, x_n)$

where G is prim. rec. with values in $\{0, 1\}$

initialization of F : put $G(a_1, \dots, a_n)$ in the
cell (a_1, \dots, a_n)

for all the z 's,

(a_1, \dots, a_n, z) is seen by (a_1, \dots, a_n)



infinigrd: another type of CA

definition 2 – (G.-M., quoted papers)
a CA on the infinigrd is said to be with registers, if it is adapted and if:

- **accept** and **reject** are two states
- each cell has two **registers** : **a** and **x**
 - a** read only: address of the cell
 - x** i/o: to compute integers

*accepted operations: copy **a**, +, −, /, *,
mod, sg, \overline{sg} , $\{(n)_i\}_{i=1}^{|n|}$, all in 1 step*

data in unary through the root (init.^t not in 1)
halt: root in **accept** or **reject**

theorem 2 – (G.-M., quoted papers)

CA's with registers on the infinigrd: they are able to decide the truth of any Σ_n ; the operation is linear in the size of the formula

proof

basically, same idea as in theorem 1 but:

from Matiyasevich, we find a polynomial P such that for all e, n and y , $\varphi_e(n) \simeq y$ if and only if there is x_1, \dots, x_9 such that $P(e, n, y, x_1, \dots, x_9) = 0$

$P(e, n, y, x_1, \dots, x_9)$ is computed by the cell $(e, n, y, x_1, \dots, x_9)$; same time for all the cells

constant time to report the result to the root: in fact, 12 levels ■