The physical Church-Turing thesis and the principles of quantum theory

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Abstract

Notoriously, quantum computation shatters complexity theory, but is innocuous to computability theory [5]. Yet several works have shown how quantum theory as it stands could breach the physical Church-Turing thesis [10, 11]. We draw a clear line as to when this is the case, in a way that is inspired by Gandy [7]. Gandy formulates postulates about physics, such as homogeneity of space and time, bounded density and velocity of information — and proves that the physical Church-Turing thesis is a consequence of these postulates. We provide a quantum version of the theorem. Thus this approach exhibits a formal non-trivial interplay between theoretical physics symmetries and computability assumptions.

The Church-Turing thesis. The physical Church-Turing thesis states that any function that can be computed by a physical system can be computed by a Turing Machine. There are many mathematical functions that cannot be computed on a Turing Machine (the halting function $h : \mathbb{N} \to \{0, 1\}$ that decides whether the i^{th} Turing Machine halts, the function that decides whether a multivariate polynomial has integer solutions, etc.). Therefore, the physical Church-Turing thesis is a strong statement of belief about the limits of both physics and computation.

The quantum computing challenge. The shift from classical to quantum computers challenges the notion of complexity: some functions can be computed faster on a quantum computer than on a classical one. But, as noticed by Deutsch [5], it does not challenge the physical Church-Turing thesis itself: a quantum computer can always be (very inefficiently) simulated by pen and paper, through matrix multiplications. Therefore, what they compute can be computed classically.

The quantum theoretical challenge. Yet several researchers [8, 10, 11] have pointed out that Quantum theory does not forbid, in principle, that some evolutions would break the physical Church-Turing thesis. Indeed, if one follows the postulates by the book, the only limitation upon evolutions is that they be unitary operators. Then, according to Nielsen's argument [11], it suffices to consider the unitary operator $U = \sum |i, h(i) \oplus b\rangle\langle i, b|$, with i over integers and b over $\{0, 1\}$, to have a counter-example.

The paradox between Deutsch's argument and Nielsen's argument is only an apparent one; both arguments are valid; the former applies specifically to Quantum Turing Machines, the latter applies to full-blown quantum theory. Nevertheless, this leaves us in a unsatisfactory situation: if the point about the Quantum Turing Machine was to capture Quantum theory's computational power, then it falls short of this aim, and needs to be amended! Unless Quantum theory itself needs to be amended, and its computational power brought down to that of the Quantum Turing Machine?

Computable quantum theory. Quantum theory evolutions are about possibly infinite-dimensional unitary operators and not just matrices — for a good reason: even the state space of a particle on a line is infinite-dimensional. Can this fact be reconciled with the physical Church-Turing thesis, at least at the theoretical-level? Mathematically speaking, can we allow for all those unitary operators we need for good physical reasons *and at the same time* forbid the above $U = \sum |i, h(i) \oplus b\rangle\langle i, b|$, but for good physical reasons as well? These are the sort of questions raised by Nielsen [11], who calls for a programme of finding the non-ad-hoc, natural limitations that one could place upon Quantum theory in order to make it computable: we embark upon this programme of a computable Quantum theory.

Our result. The idea that physically motivated limitations lead to the physical Church-Turing thesis has, in fact, already been investigated by Gandy [7]. Although some similarities exist, Gandy's proof of the Church-Turing thesis serves different goals from those of the proof by Dershowitz and Gurevich [4], as it is based not on an axiomatic notion of algorithm, but on physical hypotheses. In Gandy's proof, one finds the important idea that causality (i.e. bounded velocity of information), together with finite density of information, could be the root cause of computability (i.e. the physical Church-Turing thesis). More generally, Gandy provides a methodology whereby hypotheses about the real world have an impact upon the physical Church-Turing thesis; an idea which can be transposed to other settings: we transpose it to Quantum theory. A formal statement and a proof of the result are available in [1].

Future work. This result clarifies when it is the case that Quantum theory evolutions could break the physical Church-Turing thesis or not; a series of examples shows that it suffices that one of the above hypotheses be dropped. This draws the line between the positive result of [5] and the negative results of [8, 10, 11]. Because these hypotheses are physically motivated, this is a step along Nielsen's programme of a computable Quantum theory. Further work could be done along this direction by introducing a notion of 'reasonable measurement' [12], or investigating the continuous-time picture as started by [13, 14]. Prior to that however this work raises deeper questions: Is the bounded density of information really compatible with modern physics? For instance, can we really divide up space into pieces under this condition, without then breaking further symmetries such as isotropy?

Bigger picture. The question of the robustness of the physical Church-Turing thesis is certainly intriguing; but it is hard to refute, and fundamentally contingent upon the underlying physical theory that one is willing to consider. For instance in the General Relativity context a series of paper explain how 'hypercomputation' might be possible in the presence of certain space-times [6, 9]. Beyond this sole question however, we find that it essential to exhibit the formal relationships that exist between the important hypotheses that we can make about our world. Even if some of these hypotheses cannot be refuted, whenever some can be related to one another at the theoretical level, then one can exclude the inconsistent scenarios.

Acknowledgments

We are particularly thankful to Vincent Nesme and Reinhard Werner, since their paper with one of the authors [2, 3] contains a rather central ingredient to our proof. We would like to thank Alexei Grimbaum and Zizhu Wang for several comments, as well as Simon Perdrix, Mehdi Mhalla and Philippe Jorrand.

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