On the legacy of Euler, Fourier, Poincaré and Einstein about Computation

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Computation

	Mechanical	Mathematical
Algorithm	1930s	Ancient Greece Archimedes π Hero $\sqrt{2}$
Incompleteness	gödel/ Turing	Pythagoras $\sqrt{2}$ / $\mathbb{Q}$ Cardano $\sqrt{-1}$ / $\mathbb{R}$
Complexity	algorithmic 1960s complex/simple	algebraic 16th Cent. complex/real
Hypercomputation	beyond Turing 1970s	beyond $\mathbb{C}$ 19th Century 1843

Computation

$\Downarrow$ Machine

Computer Science

fixed rules of inference

$\Downarrow$

Technology

algorithmic randomness = tautologic

$\neq$

$\Downarrow$ Mind

Mathematics

$\Downarrow$

paradoxes $\Leftrightarrow$ logic evolves

$\sqrt{2}$, $\sqrt{-1}$, ...

$\Downarrow$

Cognition

clash linear/nonlinear
alg/transcend.

$\Downarrow$

deterministic chaos

Mathematical Computation over $\mathbb{R}$ or $\mathbb{C}$ i.e. comm. fields
= classical calculus

Beyond commutative fields

- quaternions 1843 raging battle

Hamilton non commut

- octonions 1843 ignored ...

Graves non-assoc.

4 division algebras over $\mathbb{R}$

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ (with euclidean rings of integers

dim 1, 2, 4, 8

beyond: ∃ zerodivisors
Dickson algebras

I Evolution of computation through complexification by X over $\mathbb{R}$

A_k dim 2^k

$\mathrm{Zer}(A_4) \cong \mathrm{Aut}(\mathbb{G})$

14D manifold — group of automorphisms

[less algebraic rules = more geometric options

multiplication map : $x \overset{L_a}{\longmapsto} a \times x$

SVD of $L_a = \sqrt{\text{eigenvalues}}\,(L_a L_a^T)$

$0 \leq k \leq 3$ $\qquad \|a\| =$ euclidean norm

$k \geq 4 \quad a \neq 0 \Rightarrow$ up to $2^{k-2} \neq$ sv

with X^{ty} $4p$, $p \geq 1$, $\|a\|$ included

nonassociativity $\Rightarrow$ paradoxes
$k \geq 4$

the result of SVD depends
on computational route

$k \geq 5$ $\quad u \times v \neq 0$ but
$\|u \times v\| = 0$ possible

$\Rightarrow$ no backward analysis
based on $\|\cdot\|_2$ can be reliable!

These are but two examples.
$\Rightarrow$ flexible geometry depending
on context: $A_k \subset A_{k+1}$

$$A_{k+1} = A_k \oplus A_k \times \tilde{\imath}$$

$\tilde{\imath}$ complex unit $\notin A_k$

algebra $\mathcal{D}er(A_k)$, $k \geq 0$
= Lie algebra based on $[\,,\,]$
= derivation algebra

$\mathcal{D}er(\mathbb{R}) = \mathcal{D}er(\mathbb{C}) = \{0\}$

$D(x \times y) = (Dx) \times y + x \times (Dy)$
Leibniz

$\mathcal{D}er(\mathbb{H}) = \{x \mapsto [a, x],\ a \in \mathbb{H}\}$

$\mathcal{D}er(\mathbb{O})$ = Lie algebra of G_2

Lie group $G_2 \cong Aut(\mathbb{O})$ E. Cartan 1914

nonlinear core in A_k:

$$K(A_k) = \bigcap_{D \in \mathcal{D}er(A_k)} \operatorname{Ker} D$$

irreducible by linear derivation
= measure of algebraic complexity

$K(A_k)$ = "residue after derivation"
= out of reach of causality by
linear derivation

$\dim_{\mathbb{R}} K(A_k) = d_a(k)$ = algebraic depth (over $\mathbb{R}$)

k	0	1	2	3	$k \geq 4$
$d_a(k)$	1	$\begin{cases}1\\2\end{cases}$	1	1	2^{k-3}

$d_a(k) = 1$ $\quad k = 0, 2, 3$
$\quad\geq 2$ $\quad k = 1$ and ≥ 4

linear / complex causality
$\mathbb{R}$ / $\mathbb{C}$ or A_{k-3}, $k \geq 4$
$d_a = 1$ for nonlinear core $\quad d_a \geq 2$

II Relative Computation through nonstandard $\oplus$

Einstein's view

Part I standard $+$
complexified $\times$

Part II non standard $+$ $\oplus$

i.e. Weak {commutativity + associat.}
ruled by a relator

Origin = abstract computation form of Special Relativity

Einstein 1905 c, $B_c = \{x, \|x\| < c\} \subset \mathbb{R}^3$

B_c = relativistically admissible velocities

$$\gamma_x = \left(1 - \frac{1}{c^2}\|x\|^2\right)^{-1/2} = \frac{1}{\text{Lorentz contract.}}$$

$$x \oplus y = \frac{1}{1+\frac{\langle x,y\rangle}{c^2}}\left(x + \frac{1}{\gamma_x} y + \frac{1}{c^2}\frac{\gamma_x}{1+\gamma_x}\langle x,y\rangle\, x\right)$$

deduced from $y \oplus x$ by a <u>rotation</u> in the plane (x,y) about O.

Borel 1913

in physics : Thomas rotation (1926)

Lorentz transformation in $\mathbb{H}$

Poincaré 1905

physical interpretation Minkowski 1907

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no follow-up on $\oplus$ until 1988

$\boxed{\text{Ungar}}$ algebraic theory based on "gyro"-addition

$\Rightarrow$ new view of <u>non</u>-euclidean geometry

gyrator, gyrovector space
$\Rightarrow$ natural setting for
SR in mathematical physics

A.A. Ungar 2008 Analytic hyperbolic geometry and A Einstein's special theory of relativity, World Sc.

$\Rightarrow$ export to mathematical computation

relator : $a, b \to \mathrm{rel}(a,b)$
$\in \mathrm{Aut}(S, \oplus)$

S = groupoid with binary $\oplus$

A_1 $\quad a \oplus b = \mathrm{rel}(a,b)(b \oplus a)$

A_2 $\quad a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{rel}(a,b)c$

1) organ$(G, \oplus) = (S, \oplus, \mathrm{rel})$
satisfying A_1, A_2

$$L_a x = a \oplus x = b \Rightarrow$$
$$x = \ominus a \oplus b$$

$L_a \neq R_a$

$$R_a y = y \oplus a = b \Rightarrow$$
$$y = b \ominus \mathrm{rel}(a,b)\, a = b \,\hat{-}\, a$$

induced addition $\hat{+}$ commutative

$$\mathrm{Aut}(G, \oplus) = \mathrm{Aut}(G, \hat{+})$$

2) V- framed metric cloth W

V linear vector space over $\mathbb{R}$, dim $n \geq 2$
inner product $\langle , \rangle$ norm $\|\cdot\|$

$$B_\lambda = \{x \in V ; \|x\| < \lambda\}, \quad 0 < \lambda < \infty$$

$G = B_\lambda$ or V with scalar mult. $\cancel{\otimes}$
relator $\in$ group of isometries

Ex: Einstein's organ B_c
$\Rightarrow$ cloth W_E framed in $\mathbb{R}^3$
with $\|\cdot\|_2$

$$\oplus \Rightarrow \text{metric } \overset{\circ}{d}(x,y) = \|-x \oplus y\| = \|y \ominus x\|$$

$$\hat{+} \Rightarrow \quad \hat{d}(x,y) = \|y \,\hat{-}\, x\|$$

Weaving Information Processing by cloth geometry and lines $\rightarrow$ $L\oplus$, $R\oplus$

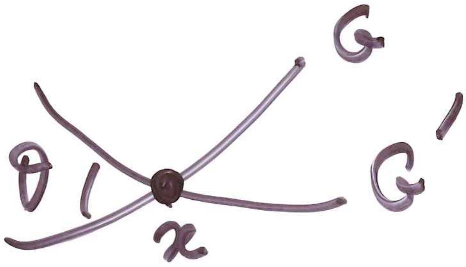
$\hat{+}$

Cerfacs TR/PA/11/27

$\oplus \Rightarrow$ 2 geodesics $\begin{cases} L & \text{metric} \\ R & \text{homotopic} \end{cases}$

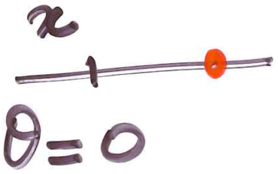
$\hat{+} \Rightarrow$ line $\neq$ geodesic

linear causality $\Leftrightarrow$ a ~~and~~ b linearly dependent



angle θ between geodesics $\mathring{G}$ and $\mathring{G}'$

$\mathring{d}$

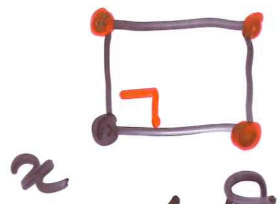


$$\boxed{0 \le \theta \le \frac{\pi}{2}}$$

$\{$	line	$v = 2$
	triangles	3
	squares	4

$0<\theta\le\frac{\pi}{3}$

$v=3$: 2 couches



$\frac{\pi}{3}<\theta\le\frac{\pi}{2}$

$v=4$: 3 couches

$\frac{\pi}{2}<\theta\le\pi$?

θ obtuse ?

III Revisit Poincaré's view with hypercomputation and logistic

Poincaré 1905 (Palermo) CRAS

evolution of $q=\alpha+X$ $\alpha\in\mathbb{R}$, $X\in \mathrm{Im}\,\mathbb{H}$

in SR: $\mathrm{Re}\,q^2=f\in\mathbb{R}$ invariant

$$q^2=(\underbrace{\alpha^2-\|X\|^2}_{f=\mathrm{Re}\,q^2=\alpha^2+X^2})+2\alpha X$$

additional constraint ?

- in Physics Poincaré Palermo
- in Computation (i) $|\mathcal{R}_e q| = \|\mathcal{I}m\, q^2\|$

or (ii) $(\mathcal{R}_e q)^2 = \|\mathcal{I}m\, q^2\|^2$.

$$\Updownarrow$$

quadratic equation in x, f

$x = \frac{\alpha^2}{f}$ $\boxed{x = -4f\, x(1-x)}$

logistic

$\alpha^2 = x f \in \mathbb{R}$

$\|X\|^2 = -X^2 = -4f x(1-x)$ with

$$\begin{cases} X \in \mathcal{J}\mathbb{H} \text{ iff } \|X\|^2 > 0 \\ X \in \mathbb{R} \text{ iff } X^2 > 0 \end{cases}$$

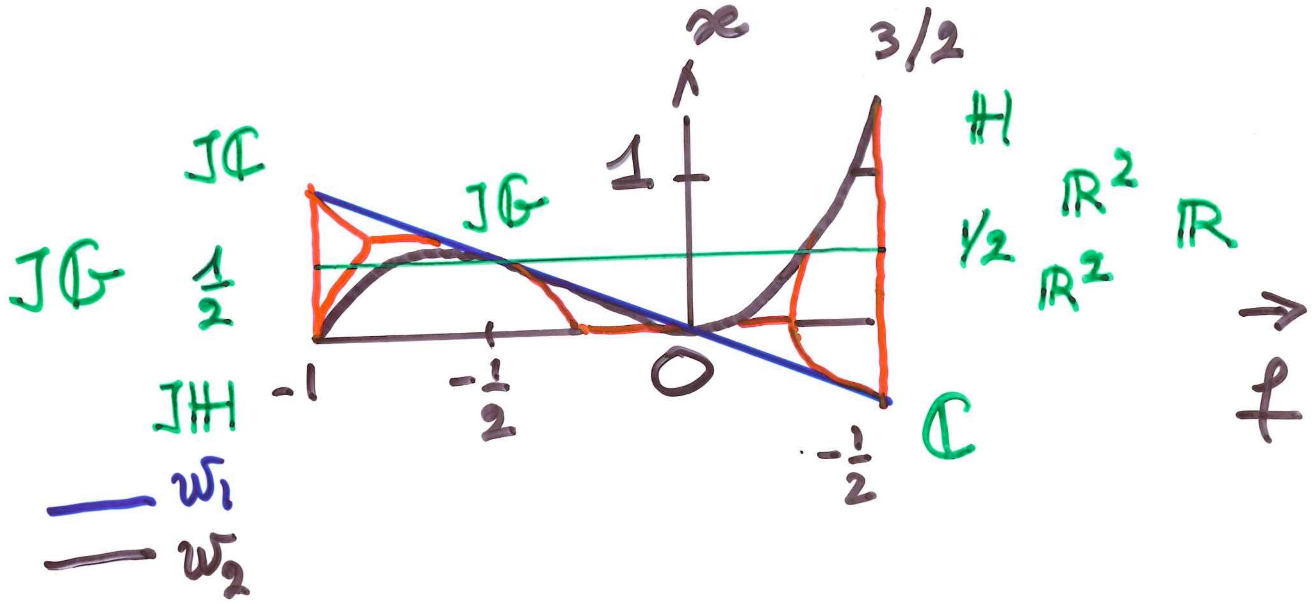
$\Rightarrow$ evolutive context:

$\mathcal{J}\mathbb{C}, \mathcal{J}\mathbb{H}, \mathcal{J}\mathbb{G}$ for $f < 0$

$\mathbb{R}$ $\mathbb{C}$ $\mathbb{H}$ for $f > 0$

Successive iterations   Picard

$$x_{n+1} = -4f\, x_n (1 - x_n)$$



- no divergence to ∞ iff $f \in [-1, \frac{1}{2}]$
- closed form:

* $f = -1$ $\begin{cases} 1 - 2x = \cos 2\psi, & \psi \in [0, \frac{\pi}{2}] \\ x = \sin^2 \psi \end{cases}$

$f = -\frac{1}{2}$ $|1 - 2x| = e^{-y}$, $y \in \mathbb{R}^-$

$f = 0$ arbitrary

* $f = \frac{1}{2}$ $\frac{1}{2} - x = \cos\theta$, $\theta \in [0, \pi]$

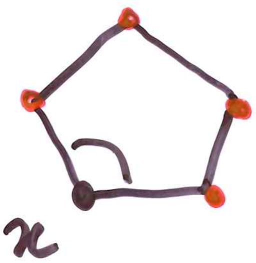
* interpretation of organic causality
$v > 4$

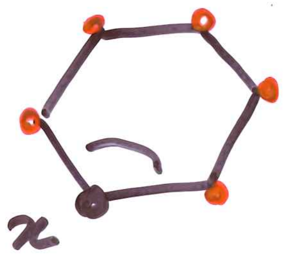
v-polygon in hyperbolic geometry

$\Rightarrow$ organic causality

$3 \leq v < \infty \Rightarrow v-1 \geq 2$ causes of organic origin

$\theta > \frac{\pi}{2} \Leftrightarrow v > 4$
discrete & complex organic causality



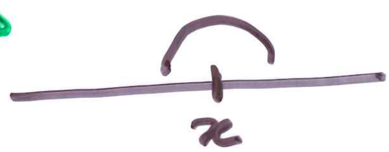


$\frac{\pi}{2} < \theta \leq \frac{2\pi}{5}$
$v = 5$

$\frac{2\pi}{5} < \theta \leq \frac{2\pi}{3}$
$v = 6$

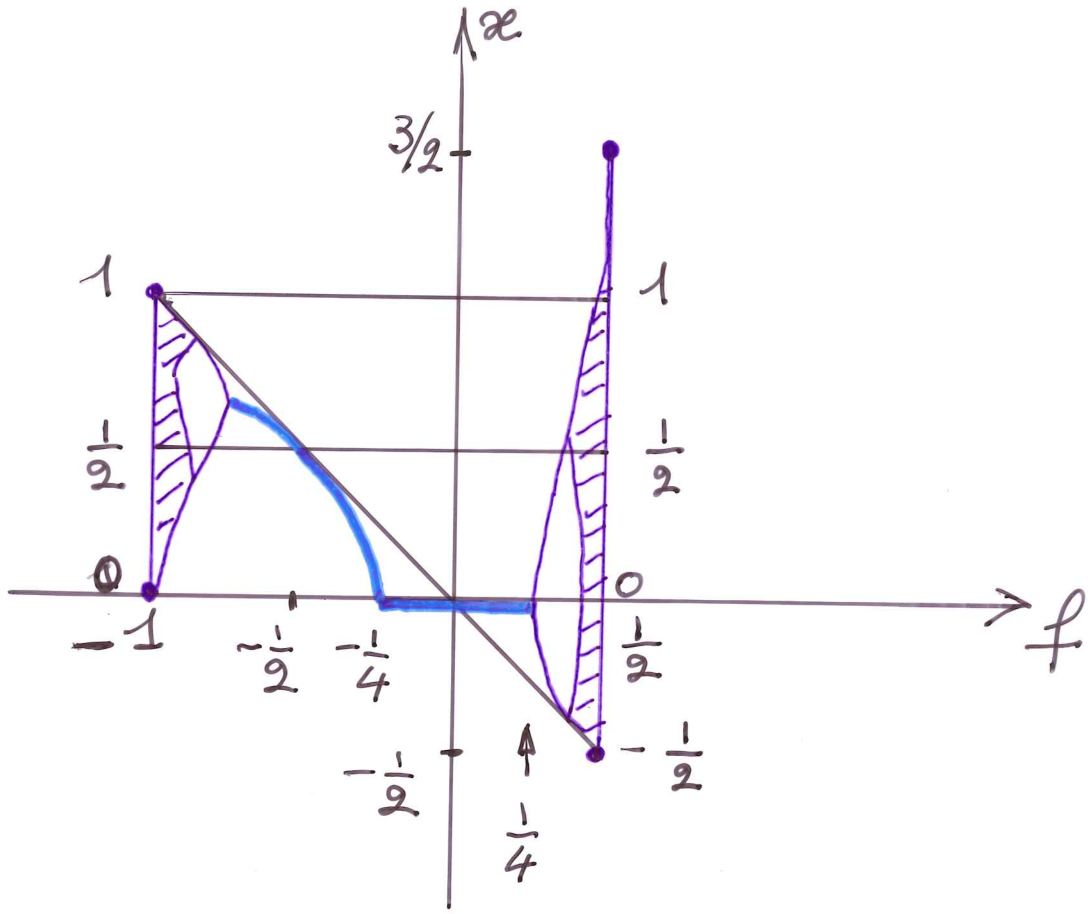
$$0 \leq \varphi(v-1) < \theta \leq \varphi(v) = \pi \frac{v-2}{v} < \pi$$

2 limit cases



$v = 2$, $\theta = 0$
1 cause

$v = \infty$, $\theta = \pi$
transcendent

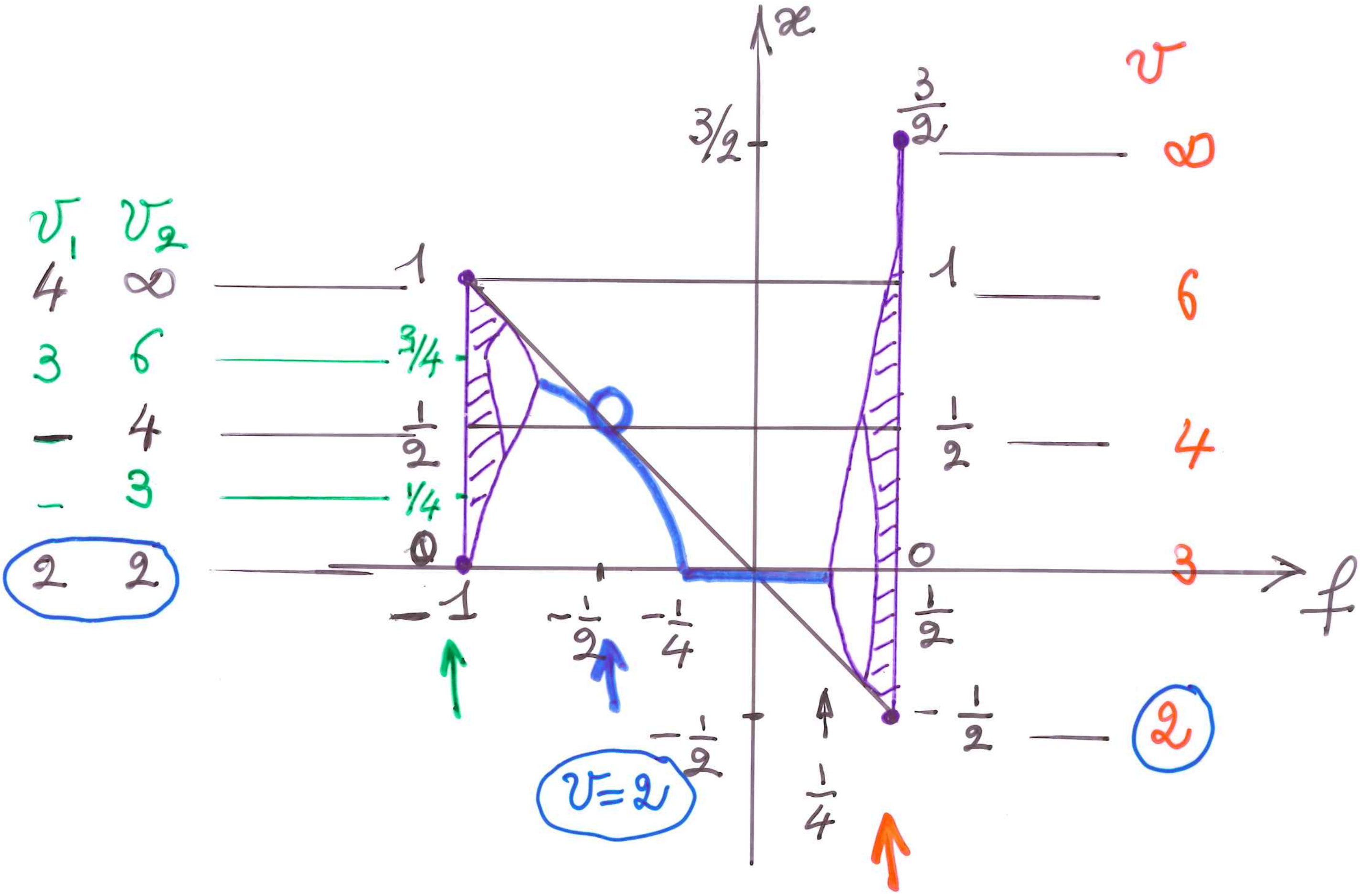


convergence

divergence confinée | controlled divergence confined

Organic causality

Poincaré + Einstein $f = -1, -\frac{1}{2}, \frac{1}{2}$



$v = 2 \iff$ linear causality

$3 \leq v < \infty \iff$ finite complex

$v = \infty \iff$ transcendent

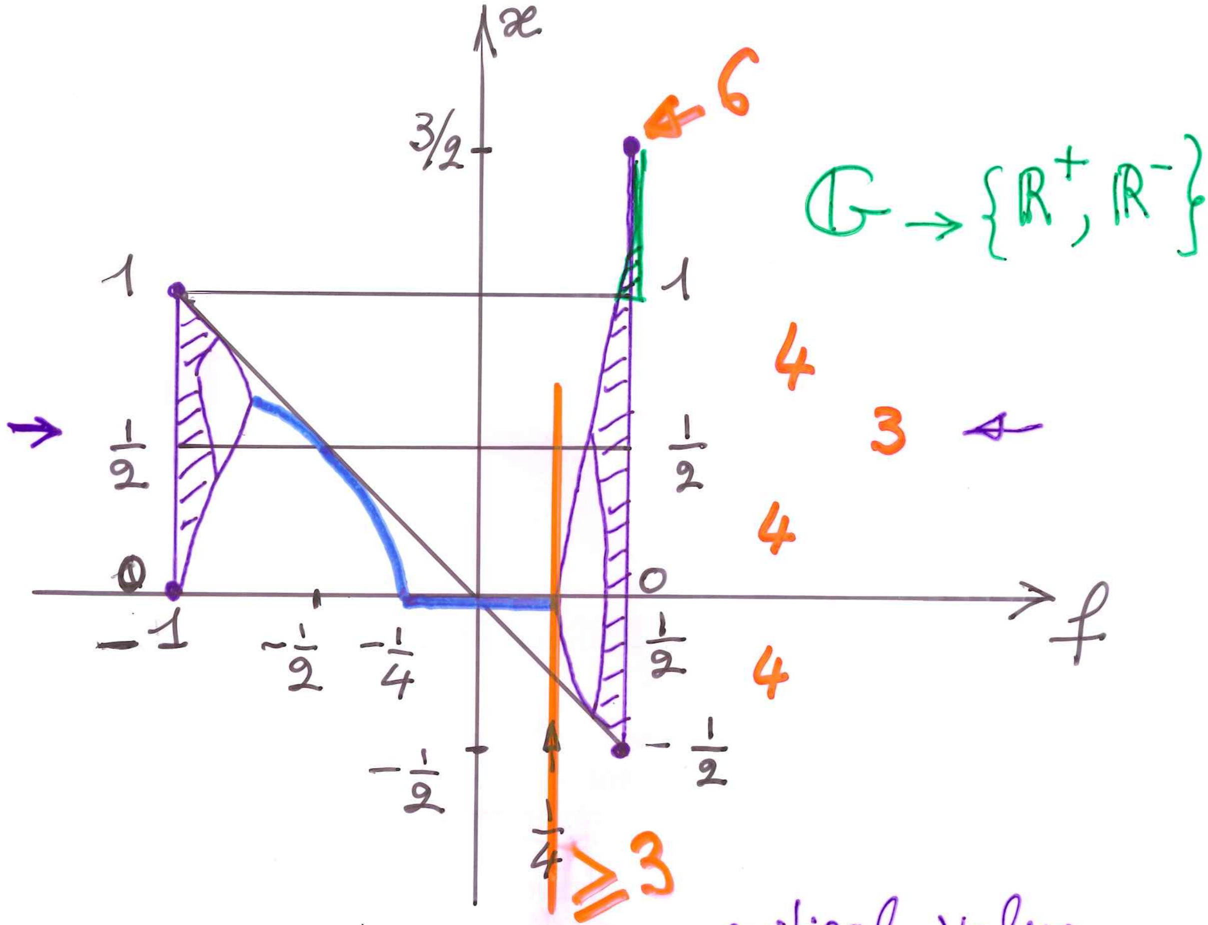
convergence

divergence confinée | controlled divergence confined

Algebraic complexity

= degree of polynomial

Simplexity $c < f \leq \frac{1}{2}$



critical value

$x = \frac{1}{2}$

$\Downarrow$

2 if $-1 \leq f \leq \frac{1}{4}$

$\{3, 4, 6\}$ if $\frac{1}{4} < f \leq \frac{1}{2}$

$c < f \leq \frac{1}{2}$ $\left\{ \begin{array}{l} g = (q, -\bar{q}) \\ \Downarrow \\ g = 2\alpha = \{\pm\sqrt{2f}\} \end{array} \right.$

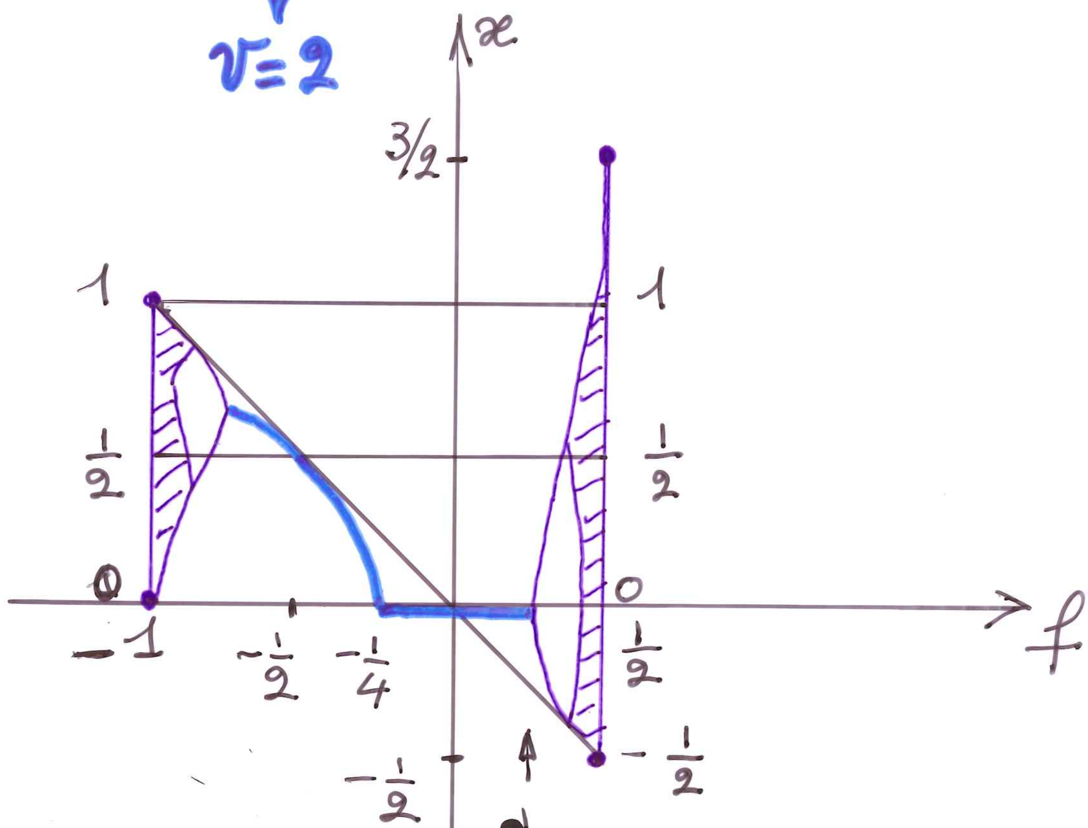
convergence

divergence confinée et structurée

- $f \in [-1, -\frac{1}{2}]$: zerodivisors in A_4
- $f \in [\frac{\sqrt{5}-1}{4}, \frac{1}{2}]$: $\{0, \pm\sqrt{2f}\}$ controlled divergence confined

Real dynamics = linear causality [17]

$v = 2$



$= -\frac{1}{2} \|f\|^2$

Fourier $\sigma(f) = \langle f', tf \rangle$ min $\Leftrightarrow f = e^{-\frac{t^2}{4}}$

Poincaré + Einstein $|1 - 2x| = e^{-y}$ $y = \frac{t^2}{4}$

parabola

convergence

divergence confinée | controlled divergence confined

IV Euler + Fourier

f arbitrary $\quad \partial f = f'$

$\varpi f = t f$

$\varpi(\partial f) = t f' \quad \partial(\varpi f) = t f' + f$

$(\varpi\partial - \partial\varpi)(f) = -f \quad \neq 0 \Leftrightarrow f \neq 0$

<u>But</u> $\quad \sigma(f)$ min $\Leftrightarrow \partial f = -\frac{1}{2}\varpi f$

ϖ and ∂ commutent

$f' = -\frac{1}{2} t f \Leftrightarrow f = C e^{-\frac{t^2}{4}}$

$\lim_{|t|\to\infty} \sqrt{|t|}\, f = 0$

Paradox in real analysis $\left\{ \begin{array}{l} \text{transcendent} \\ \text{algebraic} \end{array} \right.$

clash : linear / nonlinear
$\partial \qquad \overline{\varpi}$

exposed by exponential

V Moral of talk

Mathematical computation ⇒ paradoxes

2 unavoidable unless one puts artificial limits

ex. Turing computability for mechanical comp.

} good news because Life is paradoxical

+ non commutative

X non associative   X = repeated +

and exponentiation = repeated X

$\underbrace{a \times \dots \times a}_{a^n}$ $n \in \mathbb{Z} \to \mathbb{Q} \to \mathbb{R}$

a^x for $a > 0$, $x \in \mathbb{R}$

Paradoxes ⇒ Mathematics is NOT tautologic

0, ∞