Solving Analytic Differential Equations in Polynomial Time over Unbounded Domains

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May 24, 2011

Outline

The case of integers

Many models:

- Recursive functions
- Turing machines
- λ-calculus
- circuits
- $\ldots$

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Church Thesis

All reasonable discrete models of computation are equivalent.

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- BSS model (Blum Shub Smale)
- Computable analysis
- GPAC (General Purpose Analog Computer)
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- Church Thesis for analog computers ? $\Rightarrow$ No (GPAC $\subsetneq$ BSS)
- Comparison with digital models of computation ? $\Rightarrow$ How ?
- What is a reasonable model ?

GPAC

General Purpose Analog Computer

- by Claude Shanon (1941)
- idealization of an analog computer: Differential Analyzer
- circuit by from:

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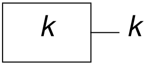
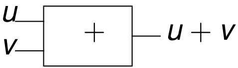
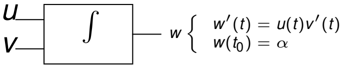


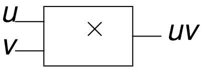
A constant unit



An adder unit



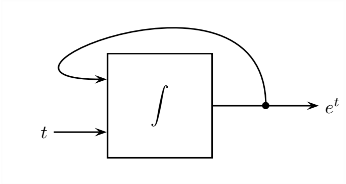
An integrator unit



A multiplier unit

GPAC: examples

Example (Exponential)



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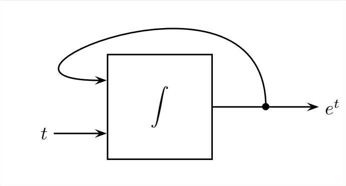

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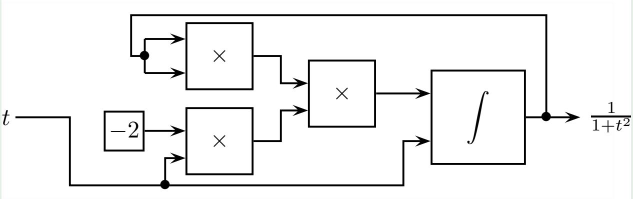
Example (Nonlinear)



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GPAC: beyond the circuit approach

Theorem

y is generated by a GPAC iff it is a component of the solution $y = (y_1, \ldots, y_d)$ of the ordinary differential equation (ODE):

$$\begin{cases} \dot{y} &= p(y) \\ y(t_0) &= y_0 \end{cases}$$

where p is a vector or polynomials.

Example (Counter-example)

$$\begin{cases} \dot{y} &= \frac{1}{y} \\ y(0) &= 1 \end{cases}$$

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Definition

A real $r \in \mathbb{R}$ is computable is one can compute an arbitrary close approximation for a given precision:

$$\text{Given } p \in \mathbb{N}, \text{ compute } r_p \text{ s.t. } |r - r_p| \leqslant 2^{-p}$$

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Counter-Example

$$r = \sum_{n=0}^{\infty} d_n 2^{-n}$$

where

$$d_n = 1 \Leftrightarrow \text{the } n^{th} \text{ Turing Machine halts on input } n$$

Computable function

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is computable if there exist a Turing Machine M s.t. for any $x \in \mathbb{R}$ and oracle $\mathcal{O}$ computing x, $M^{\mathcal{O}}$ computes $f(x)$.

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Counter-Example

$$f(x) = \lceil x \rceil$$

Church Thesis

We have:

- TM: $\mathbb{N} \to \mathbb{N}$
- GPAC: $\mathbb{R} \to \mathbb{R}$, analytic ($\Rightarrow C^\infty$)
- CA: $\mathbb{R} \to \mathbb{R}$, continuous

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Church Thesis

Turing Machines, GPAC and Computable analysis are equivalent models of computations.

Really ?

▷ GPAC $\subseteq$ CA: computing the solution of an ODE:
"Of course, the resulting algorithms are highly inefficient in practice"

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Effective Church Thesis ?

Are all (sufficiently powerful) "reasonable" models of computations with "reasonable" measure of time polynomially equivalent ?

Computational complexity

- TM: well known and understood
- CA:
  - A few technical differences
  - Relatively clear
- GPAC: unclear

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$$\text{TM} \subseteq \text{GPAC}$$

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- Preserves time: state at step $n \Leftrightarrow$ function value at time n
- Satisfying

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- Computing the solution of an ODE quickly
- Lots of algorithms…
- Few theoretical results
- Not satisfying

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Problem statement & facts

We want to solve:

$$\begin{cases} \dot{y} & = p(y) \\ y(t_0) & = y_0 \end{cases}$$

Solve ?

▷Compute $y_i(t)$ with arbitrary precision

Properties & hypothesis:

- Assume y defined over $\mathbb{R}$: no loss of generality
- y is analytical over $\mathbb{R}$:

$$y(t+\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n$$

- Problem: local Taylor series, difficult to use
- Idea: stronger assumption: y analytical over $\mathbb{C}$
- Consequence: Taylor series valid over $\mathbb{C}$

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New problems

Two problems:

- Compute Taylor series from function and vice-versa
- Compute Taylor series of the solution to an ODE

Poly-boundedness

Definition

$f : \mathbb{R} \to \mathbb{R}$ poly-bounded if $|f(t)| \leqslant e^{p(\log t)}$.

Theorem

$f : \mathbb{R} \to \mathbb{R}$ computable in polynomial time $\Rightarrow$ f poly-bounded

Theorem

If $f : \mathbb{C} \to \mathbb{C}$ analytical over $\mathbb{C}$: $f(t) = \sum_{n=0}^{\infty} a_n t^n$ and poly-bounded then:

$$f \text{ polynomial time computable}$$

$$\Updownarrow$$

$$\{a_n\} \text{ polynomial time computable}$$

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Result

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If y is poly-bounded, analytical over $\mathbb{C}$ and satisfies

$$\begin{cases} \dot{y} &= p(y) \\ y(t_0) &= y_0 \end{cases}$$

Then y is polynomial time computable.

- Poly-bounded: optimal, easy to check
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Limitations and workaround

Consider:

$$\begin{cases} \dot{y} &= -2ty^2 \\ y(0) &= 1 \end{cases} \Rightarrow \quad y(t) = \frac{1}{1+t^2}$$

Problem:

- two poles over $\mathbb{C}$: i and $-i$
- Workaround possible !
- But there is a real limitation here

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Future work

- Develop method specific to $\mathbb{R}$
- Understand what complexity means for the GPAC

Questions ?

- Do you have any questions ?