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Parallel transient time of one-dimensional sand pile

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Sand dripping in one-dimensional Sand Pile Model is first studied. Patterns and signals appear. Their behaviors and interactions are explained and asymptotic approximations are made. The total collapsing time of a single stack of sand is linear in function of the number of grains.

Key words: Sand Pile Model, dynamic systems, stabilization, parallelism

1 Introduction

A one-dimensional sand pile consists of an infinite sequence of stacks. Each stack holds a finite number of grains. Sand Piles Model (SPM) and Chip Firing Games (CFG) are dynamic systems based on local balancing. The total number of grains never changes. They are both used to model flows in systems, like load-balancing in a processor network in computer science [6,7] and granular flows in physics [5]. In SPM, if a stack has at least 2 more grains than the next stack, then a grain "tumbles down" from the first stack to the second. In CFG, a stack gives a chip to each of its neighbors if it has enough chips to do so.

Goles and Kiwi [2,4] studied one-dimensional SPM and the related CFG. They detailed the dynamics and proved the convergence for various sequential cases. The problem studied here is the parallel evolution of a single non-empty stack, as illustrated by Fig. 2.

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This note is organized as follows. Definitions, notations and the equivalence of SPM and CFG are given in Sect. 2. In Sect. 3, we show that the dynamics are divided in two phases. During the first phase, the original stack has more grains than any other and always gives a grain to the second one. During the second phase, the pile stabilizes.

In Sect. 3, we study the first phase by considering an empty configuration which receives a grain in its first stack at each iteration. It can also be thought of as water dripping from a tap or sand in an hourglass. Each configuration, encoded in height differences, is partitioned in four portions of different patterns: 22, 1313, 0202 and 11. The frontiers between them act like signals.

In Sect. 4, we give the shapes of the configurations and make asymptotic approximations. The shape increases proportionally to the square root of the number of iterations. It is made of 2 sections of slopes 1 and 2 and relative length $\sqrt{2}$.

We go back to the original problem in Sect. 5. The focus is laid on the second phase: the stabilization after the height of first stack reaches the height of the second. New signals appear. The parallel collapsing time of a unique stack is linear in function of the number of grains. Compared to the sequential case, the speedup is proportional to the number of non-empty (active) stacks.

2 Definitions

We use the notation of Goles and Kiwi [4]. The only difference is that our model is parallel. The one-dimensional sand pile is modeled by a sequence of stacks. Each stack holds a finite number of grains. This number is called the height of the stack. Configurations are denoted with square brackets, *i.e.* $\nu = [[\,\nu_0\, \nu_1\, \dots\, \nu_k\,]]$. We call the difference between 2 stacks (or its average if more stacks are considered) *slope*. If a stack has at least 2 more grains than the next one, then 1 grain tumbles down. This is illustrated by the movement of the grains `a`, `b` and `c` in Fig. 1. The number of grains in the pile is finite and constant.

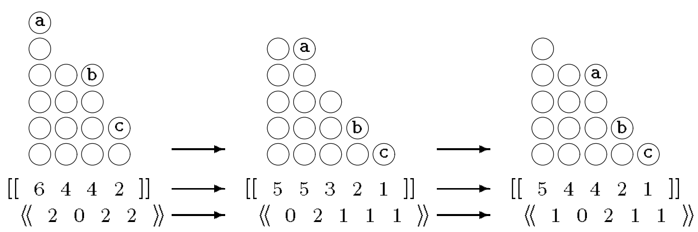

Fig. 1. Example of iterations.

Definition 1 *Let* $\mathbb{1}(n)$ *be the following threshold function:* $\forall n \in \mathbb{Z}$, $\mathbb{1}(n) = 1$ *if* $0 \leq n$, *otherwise* 0. *Let* ν *be a configuration. The* SPM *dynamics are driven by the following transition function* F *:*

$$F(\nu)_0 = \nu_0 - \mathbb{1}(\, \nu_0 - \nu_1 - 2\,) \ ,$$
$$0 < i,\ F(\nu)_i = \nu_i - \mathbb{1}(\, \nu_i - \nu_{i+1} - 2\,) + \mathbb{1}(\, \nu_{i-1} - \nu_i - 2\,) \ .$$

The negatives terms correspond to the possibility of giving a grain to the next stack, while the positive terms correspond to the possibility of receiving one. All of the stacks are updated at the same time, in parallel.

In the initial configuration all the grains are in the first stack (number 0). Since grains only move to smaller stacks, a direct induction shows that only non-increasing sequences are generated from the initial configuration. This ensures that height differences are all positive.

Any configuration can also be encoded by the list of its height differences $x = \langle\!\langle\, (\nu_0 - \nu_1)\, (\nu_1 - \nu_2)\, (\nu_2 - \nu_3)\ .$ With this encoding, the dynamics become:

$$\Theta(x)_0 = x_0 - 2\,\mathbb{1}(\, x_0 - 2\,) + \mathbb{1}(\, x_{i+1} - 2\,) \ ,$$
$$\forall i,\ 0 < i,\ \Theta(x)_i = x_i + \mathbb{1}(\, x_{i-1} - 2\,) - 2\,\mathbb{1}(\, x_i - 2\,) + \mathbb{1}(\, x_{i+1} - 2\,) \ .$$

We call these differences of grains *chips*. The above rule can be stated as: if a site has more than 2 chips, it "fires" 1 chip to both of its neighbors. This is the *chip firing game* (CFG). SPM and CFG are equivalent in a one-dimensional lattice.

2.1 Studied problem

We study the parallel collapsing of a stack of N grains located at the original stack (number 0), *i.e.* the evolution from $[[N]]$. Goles and Kiwi [4] have shown that the final configuration (or fixed point) is straightforwardly defined from the initial configuration, independently from the updating (parallel, sequential or mixed). The final configuration is a triangle with all slopes equals to 1 except, maybe for a unique 0. The sequential collapsing time is of order $N^{3/2}$. Figure 2 shows the parallel evolution in the case $N = 40$. We distinguish two phases. Before iteration 30, each time a grain falls onto the second stack (number 1). After, the pile balances and reaches stability.

During this first phase, stacks $1, 2, 3, \ldots$ have a special behavior: starting from nothing, they are balancing while a grain falls onto stack 1 every time. The new grain, like the other falling grains, arrives at the end of the iteration. Section 3 and 4 are devoted to this problem: $\mu^0 = [[0]]$ and the following dynamics: $\mu^{t+1} = [[\,(F(\mu^t)_0{+}1)\ F(\mu^t)_1\ F(\mu^t)_2\ F(\mu^t)_3 \ldots]]$. The lower part of Fig. 2 shows the first steps of this dynamics. The lengths and heights, as well as the slopes, exhibit regularity.

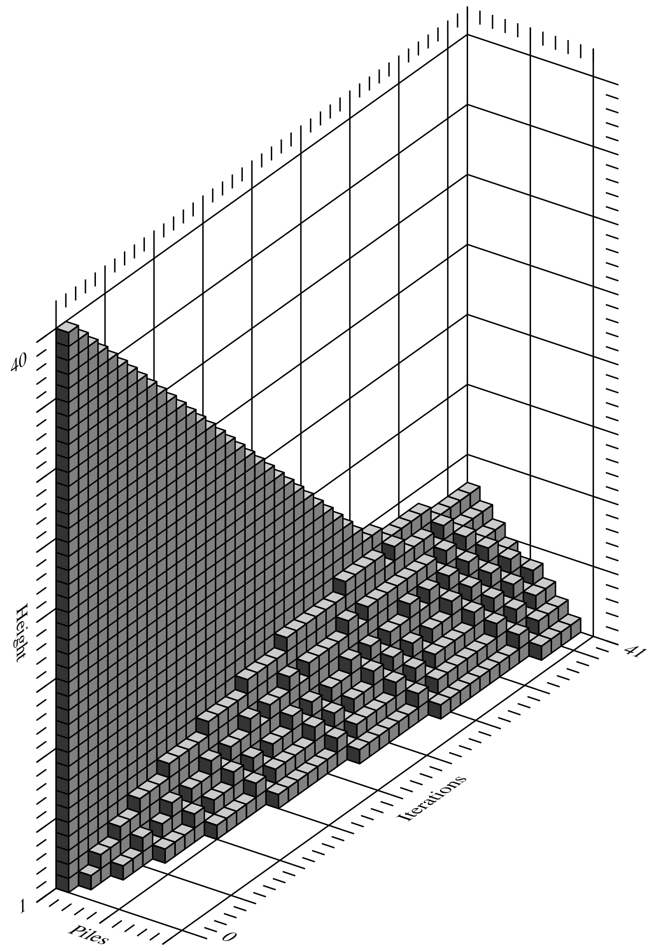

Fig. 2. Collapsing with $N = 40$.

3 Triangles and signals

Stacks $1, 2, 3, \ldots$ are encoded by height difference on Fig. 3 (steps 1 to 120). Triangles appear with patterns 22, 1313, 0202 and 11. These patterns are stable. It should be noted that for the second and third patterns, digits are alternating, like in a chessboard. Each configuration seems to be the concatenation of four portions with the following patterns: 22, 1313, 0202 and 11 respectively. We call the limit between 2 patterns and *border* the limits of the configurations *frontier*. We denote L (left), M (middle) and R (right) the frontiers between, respectively, first and second, second and third, third and fourth patterns. Geometric definitions are given in Fig. 4. In Fig. 3, L and R behave like signals moving on both sides of M.

Proposition 2 *All configurations are of the form:*

$$2^* \;(\varepsilon\,|\,3)\;(1\,3)^*\;(\varepsilon\,|\,1\,2)\;(0\,2)^*\;(\varepsilon\,|\,0)\;1^* \quad .$$

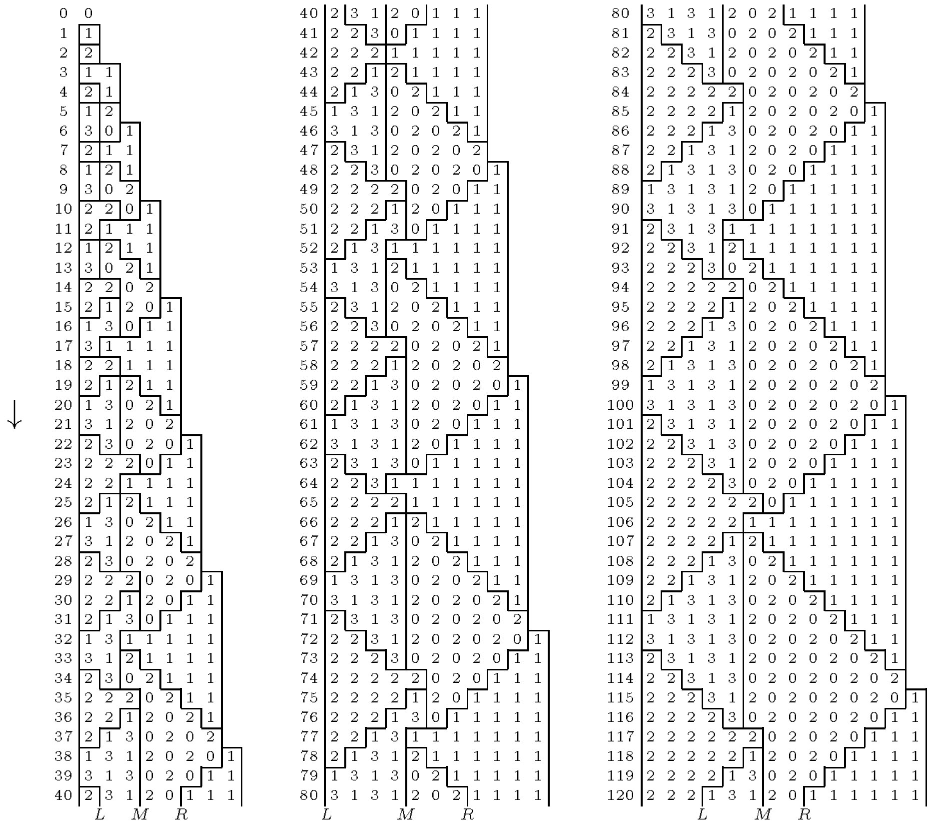

Fig. 3. Representation with height differences.

PROOF. We prove the proposition by induction. It is true for the first 120 iterations as it can be seen on Fig. 3. Interaction only depends on the 2 closest neighbors. Thus it is enough to look locally at the interactions of the frontiers on Fig. 3. Let us first investigate each signal alone, from left to right: L is going to the left (right) if it is equal to 2|1 (2|3) (lines 107 to 117); M is not moving (lines 96 to 102); and R is going to the left (right) if it is equal to 0|1 (2|1) (lines 94 to 104). While the proposition is true, only the following encounters are possible, from left to right: on the left border, L bounces (lines 59 to 65); when L meets M, L bounces and M is moved 1 step to the right (lines 81 to 87); when R meets M, R bounces and M is moved 1 step to the left (lines 50 to 57). The order is kept, and the only possible encounter with more than 2 frontiers is L-M-R. The meeting can be exactly synchronous (lines 40 to 44) or not (lines 62 to 67 and 103 to 109). In all cases the order is respected and no other case arises. □

The dynamics of the signals, L and R, are plain and simple, except when one of the signals reaches one of its limits. When L reaches the left border, it bounces back. When L or R reaches M, M is pushed one step and the signal propagates back. When R reaches the right border, it bounces back and pushes the border outwards in one position; the total length is increased by 1. When R comes back to the center, we known that the total length was incremented by 1.

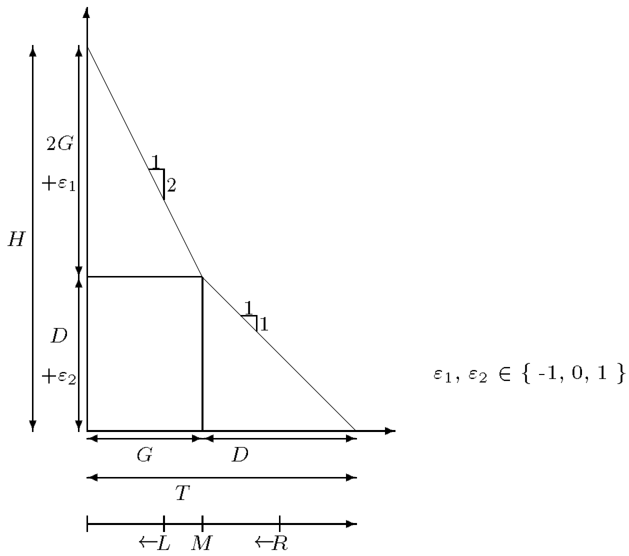

Fig. 4. Geometric definitions of G, D, T, H, L and M.

4 Asymptotic behavior

Partitions are made of two sections. The left section, amounting for patterns 22 and 1313, is of slope 2. The right section, amounting for patterns 0202 and 11, is of slope 1. We denote G and D the lengths of these sections. In this Section, we investigate the evolution of the ratio D/G.

Let G_k and D_k be the values of G and D at the time of the k^{th} return of R to the middle border M. Between 2 returns of R, the total length is increased by 1. The right section D is increased by 1 on each return of R and decreased by 1 on each return of L. It is the opposite for the left section G. Let γ_k be the number of returns of L to M between the k^{th} and $k{+}1^{th}$ return of R to M. The following relations hold: $G_{k+1} = G_k - 1 + \gamma_k$ and $D_{k+1} = D_k + 1 - \gamma_k + 1$.

Lemma 3 *Each time that R goes back to the center, either G_k or D_k is incremented by* 1 *and the other is not changed and $G_{k+1} \leq D_{k+1} \leq 2G_{k+1}$ and $1 \leq \gamma_k \leq 2$.*

PROOF. If $D_k \leq G_k$ then there is at most 1 return of L to M ($0 \leq \gamma_k \leq 1$), only the right section D increases. If $2G_k \leq D_k$ then there are more than 2 returns of L ($2 \leq \gamma_k$) and only the left section G increases. In both cases, the inequalities are changed in a finite number of iterations.

If $G_k < D_k < 2G_k$ then there are 1 or 2 returns of L to M ($1 \leq \gamma_k \leq 2$) and G_k and D_k only vary by 1. In the next collision, nothing more than equality can happen ($G_{k+1} \leq D_{k+1} \leq 2G_{k+1}$). In the case of equality, G_k and D_k can only go back to inequality as explained above. It can be seen geometrically in Fig. 3 that the inequality is verified. The Lemma follows by induction. □

Theorem 4 *The ratio D/G converges to $\sqrt{2}$.*

PROOF. The proof is only sketched; all details can be found in [3]. Let us consider 2 integers p and q such that the following relation is true:

$$1 \leq \frac{p}{q} \leq \frac{D_k}{G_k} \leq \frac{p+1}{q} \leq 2 . \quad (1)$$

with the following hypothesis over the integers p and q: $1 < q^2 \leq p^2 < G_k < D_k$ and $(2q^2+q)/2G_k \leq 1$. Since $1 \leq D_k/G_k \leq 2$, such p and q exist. Since D_k and G_k tends to infinity, with k large enough p and q are arbitrarily large.

The round trip delay for a signal is twice the length of its corresponding section (plus 1 if the signals are not synchronized in the center). Let Δ_t be the time for R to go back q times to the center. From Lemma 3, $D_k \leq D_{k+i} \leq D_k + q$ for $0 \leq i \leq q$, so that: $q2D_k \leq \Delta_t \leq q(2(D_k+q)+1)$.

Equally, $G_k \leq G_{k+i} \leq G_k + q$ for $0 \leq i \leq q$. Let α be the number of times that L reaches the center during q loops of R, the following statement holds: $2qD_k/(2(G_k+q)+1) \leq \Delta_t/(2(G_k+q)+1) \leq \alpha \leq \Delta_t/2G_k + 1 \leq q(2(D_k+q)+1)/2G_k + 1$. Enlarging these bounds, we found that: $p-1 \leq \alpha \leq p+3$. After q loops of R: $G_{k+q} = G_k + \alpha - q$ and $D_{k+q} = D_k - \alpha + 2q$ (the last $+q$ comes from the right border).

Relation (1) can also be written: $pG_k \leq qD_k \leq (p+1)G_k$. With the new values: $(p-1)G_{k+q} \leq q(D_k - \alpha + 2q) = qD_{k+q}$. And for the right section: $qD_{k+q} = q(D_k - \alpha + 2q) \leq (p+1)G_{k+q} + 2q^2 + 2q + 1 - p^2$. To get the previous two equations, we use the hypothesis made over p and q in (1). Gathering both bounds, we get:

$$\frac{p-1}{q} \leq \frac{D_{k+q}}{G_{k+q}} \leq \frac{p+2}{q} . \quad (2)$$

This means that the ratio does not change by more than $2/q$. We investigate the evolution of the inverse ratio: $D_{k+q}/G_{k+q} - D_k/G_k = (2q - \alpha - (\alpha - q)D_k/G_k)/(G_k + (\alpha - q))$. Since p and q (thus α) are much smaller than G_k, and D_k and G_k are positive: $\mathrm{sgn}\,(D_{k+q}/G_{k+q} - D_k/G_k) = \mathrm{sgn}\,(2q - \alpha - (\alpha - q)D_k/G_k)$. Since G_k only increases, $0 \leq \alpha - q$ and $2q^2 - (p+4)^2 \leq q\,(2q - \alpha - D_k/G_k(\alpha - q)) \leq 2q^2 - (p-2)^2$. Remember that $2 \leq q \leq p \leq 2q$ (from Lemma 3 and (1)). Let $A = 2q - \alpha - (\alpha - q)G_k/D_k$. If $(p+4)/q < \sqrt{2}$ then $0 < 2q^2 - (p+4)^2$, $0 < 1/q \leq A$ and D_k/G_k is increasing. If $\sqrt{2} < (p-2)/q$ then $2q^2 - (p+4)^2 < 0$, $A \leq -1/q < 0$ and D_k/G_k is decreasing. Finally, the ratio does not change by more than $2/q$. It goes toward $\sqrt{2}$ if it is more than $4/q$ away from it, in this case: $1/qG_k(1 + \alpha - q/G_k) \leq |D_{k+q}/G_{k+q} - D_k/G_k|$. Since G_k is at most linearly increasing (in k) and q and α are bounded, the sum of above terms diverges. This ensures that the ratio goes back to somewhere less than $4/q$ away from $\sqrt{2}$. From this, after some time, D_k/G_k does not differ from $\sqrt{2}$ by more than $6/q$.

When n tends to infinity, so do k, G_k and D_k (for geometric reasons), so do the possible p and q for (1) and $1/q$ tends towards zero. The ratio D_k/G_k converges to $\sqrt{2}$. Since $D_k \to \infty$ when k increases and G (D) differs by at most 1 from the next G_k (D_k). $\square$

Let H and T be, respectively, the *maximum height* (height of the first stack) and the *total length* (number of non-empty stacks) of the configuration. Theorem 4 and the fact that all quantities go to infinity allow us to relate them to the number of fallen grains n which is also the total area of Fig. 4, *i.e.*, of the 2 triangles and of the rectangle:

$$
\begin{aligned}
n &\approx \frac{D^2}{2} + G.D + G^2 \approx (2+\sqrt{2})\,G^2 \ , \\
G &\approx \sqrt{\frac{n}{2+\sqrt{2}}} \ , \qquad D \approx \sqrt{2}\,G \approx \sqrt{\frac{n}{\sqrt{2}+1}} \ , \\
H &\approx \sqrt{(2+\sqrt{2})\,n} \ , \qquad T \approx (1+\sqrt{2})G \approx \sqrt{\frac{2+\sqrt{2}}{2}n} \ .
\end{aligned}
\tag{3}
$$

It should be noted that both triangles of Fig. 4 have almost the same area, G^2. The first phase of the original problem ends when the difference of the heights of the first and second stacks is less than 2. Let $T_c(N)$ be the number of iterations before the phase changes, $N - T_c(N) \approx \sqrt{(2+\sqrt{2})\,T_c(N)}$. Since $\sqrt{N} \ll N$, $N \approx T_c(N)$ thus:

Theorem 5 *The duration of the first phase is*

$$T_c(N) \ = \ N \ - \ \sqrt{(2+\sqrt{2})\,N} \ + \ o(\sqrt{N}) \quad .$$

5 Second phase of the collapsing

We consider that the L signal is away from the left border (original stack). The beginning of the configuration is 22... The evolution is like in the first diagram of Fig. 5. Three new signals appear, from left to right: a new left signal L' pushing right a new middle frontier M' and a signal E (end of first phase) going to the right. The last two diagrams of Fig. 5 show what happens when L is present at the beginning of the stabilizing phase. The signal L is destroyed, the end signal E does not appear and neither does the static border B.

The local updating function is symmetric: changing x by $-x$, it remains the same. We use this property to restrain the cases because L' and R, and M' and M, behave symmetrically. Each time, only one case is considered.

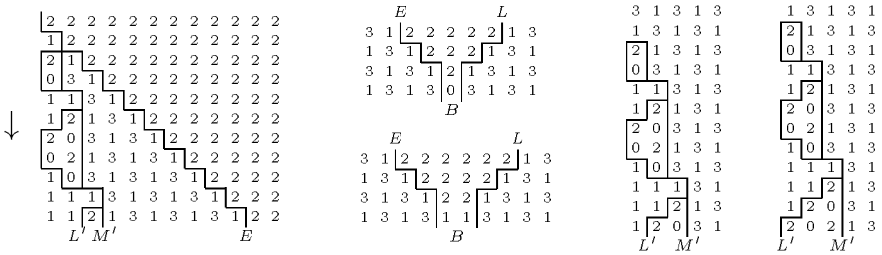

Fig. 5. Beginning of the second phase of the collapsing and generation of B.

The end signal E goes to the right until it encounters the old left signal L as shown in Fig. 5. The result is a static border B which can be 1 or 2 stacks wide depending on the parity of the distance between signals E and L. After this, the new frontier M', or the old middle frontier M, pushed by, respectively, L' and R, reaches B as shown in Fig. 6. The right column of Fig. 6 shows what happens when borders M' and M meet after the static border B has disappeared.

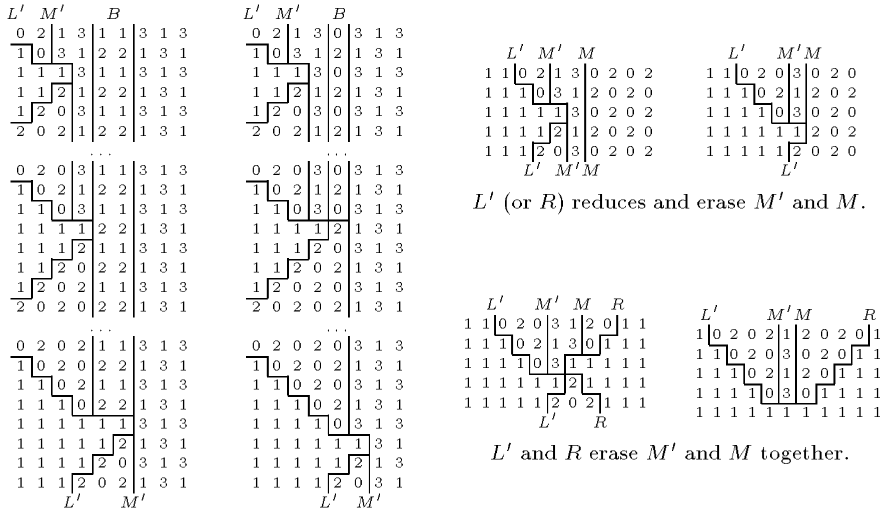

Fig. 6. Border M' absorbs B and case with no B generated.

Figure 7 shows what happens when both M and M' reach the static border B exactly at the same time. After the second case of Fig. 7 (L' and R reach synchronously the thick static border B, they remain but M' and M disappear), B can be either destroyed by 1 signal or by both L' and R synchronously. These are the last three cases of Fig. 7. The pile reaches stability.

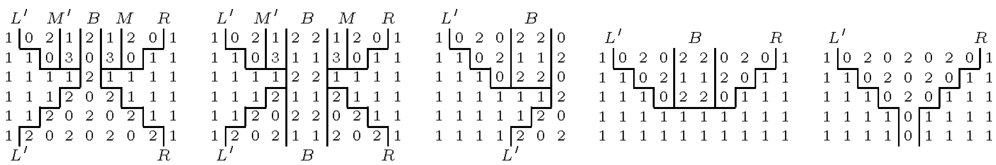

Fig. 7. Both signals L' and R reaching the static border B at the same time and corresponding ends.

5.1 Asymptotic Time

We summarize the interactions of the second phase in Fig. 8. Special cases studied above are not indicated and can always be considered as gains of time.

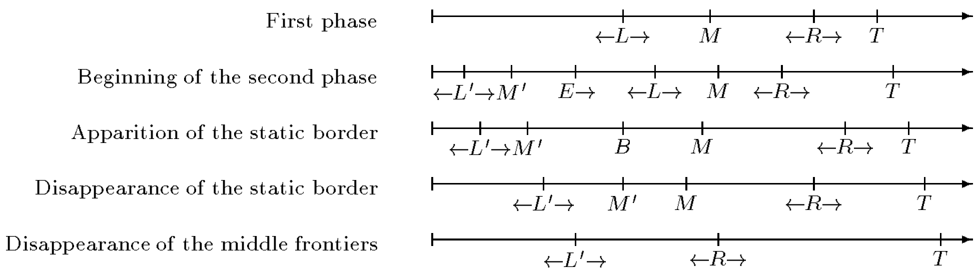

Fig. 8. Steps of the collapsing.

The static border B is not important to the dynamics, it only helps one of the 2 borders, M' or M, to advance faster. Border M' (M) is only pushed to the right (left) by L' (R). To approximate, we neglect the fact that M is going towards M' (M' is faster because L' has a shorter way to come and go). Let d_0 be the distance that M' has to cross to reach M. It is up-bounded by M_0, the position of M plus 1 (L might move M before disappearing). M' moves 1 stack to the right each time L' comes back. To reach M_0, it needs $\sum_{i=0}^{M_0} 2.i = M_0(M_0+1)$. From (3), $M_0 = \sqrt{N/(2+\sqrt{2})} + o(\sqrt{N})$. At most $N/(2+\sqrt{2}) + o(N)$ iterations are needed. Signals L' and R need at most $2\sqrt{N}$ iterations to join. Together with the time of the first phase given in Theorem 5 becomes:

Theorem 6 *The collapse time of a unique stack in the one-dimensional* SPM*,* $T_{\text{par}}(N)$*, is linear in the number of grains. It is bounded by:* $N + o(N) < T_{\text{par}}(N) < N(1 + 1/(2+\sqrt{2})) + o(N)$*.*

Let us recall the last result of [4, Part 3]: $\Omega(N) \leq T_{\text{par}}(N) \leq \mathrm{O}(n^{3/2})$. We have found that the time is linearly bounded from above. It cannot be less since $\mathrm{O}(n^{1/2})$ stacks (processors) are used to make exactly the same things as in sequential (parallel speedup limit).

6 Conclusion

The parallel collapsing time of a single stack in one-dimensional SPM is linear in function of the number of grains N. In the sequential case, Goles and Kiwi [4] have shown that the stabilization time was of order $N^{3/2}$. In comparison, the speedup is $\sqrt{N}$ which is the number of nonempty stacks. This is a real parallel process.

The dynamics are decomposed in two phases: dripping then stabilizing. During the dripping process, configurations are made of two different sections of slopes 2 and 1. The ratio of their relative lengths tends to $\sqrt{2}$. During the second phase, there are three sections of slopes 1, 2 and 1. We found asymptotic approximation for the different parameters of the configurations. The signal encoding techniques developed here can be used to study dynamic systems as in [1].

If the original stack is in the middle of the pile, then the dripping is symmetrical on both side. During the second phase, left signals L meet, bringing some disturbances, but all in all the process is still in linear time.

With respect to the CFG, we have a different result than Anderson *et al.* in [1]. This comes first because they do not bound the number of grains which can tumble from a stack to the next one; in our study, it is at most 1. The other reason is that their starting configuration is $[\![\ldots N\,N\,N\,0\,0\,0\,\ldots]\!]$ while ours is $[\![N\,0\,0\,0\,\ldots]\!]$ as already stated by Goles and Kiwi [4].

We believe that the time bound for the total collapsing time of any finite configuration is also bounded by the number of grains.

Thanks

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