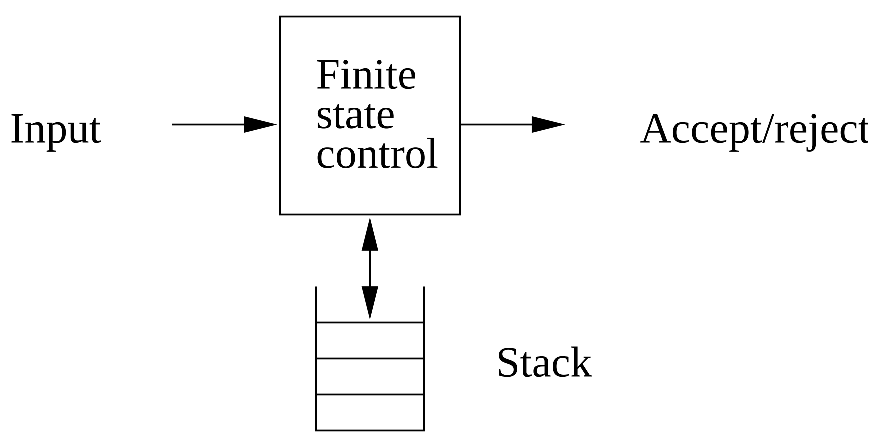
Pushdown Automata

A pushdown automata (PDA) is essentially an ϵ-NFA with a stack.

On a transition the PDA:

1. Consumes an input symbol.
2. Goes to a new state (or stays in the old).
3. Replaces the top of the stack by any string (does nothing, pops the stack, or pushes a string onto the stack)



Example: Let's consider

$$L_{wwr} = \{ww^R : w \in \{0,1\}^*\},$$

with "grammar" $P \to 0P0$, $P \to 1P1$, $P \to \epsilon$. A PDA for L_{wwr} has tree states, and operates as follows:

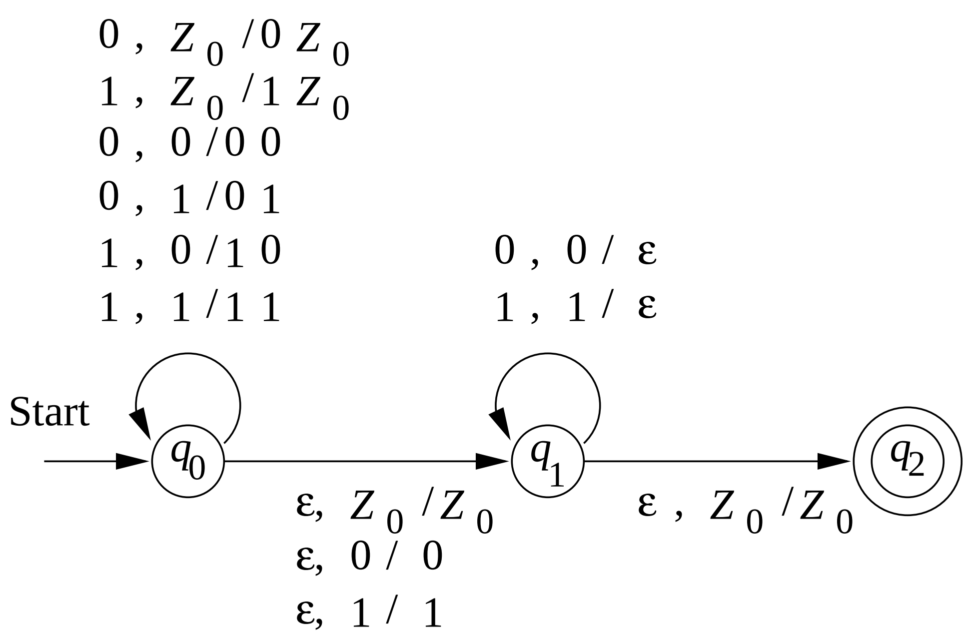
1. Guess that you are reading w. Stay in state 0, and push the input symbol onto the stack.

2. Guess that you're in the middle of ww^R. Go spontanteously to state 1.

3. You're now reading the head of w^R. Compare it to the top of the stack. If they match, pop the stack, and remain in state 1. If they don't match, go to sleep.

4. If the stack is empty, go to state 2 and accept.

The PDA for L_{wwr} as a transition diagram:



PDA formally

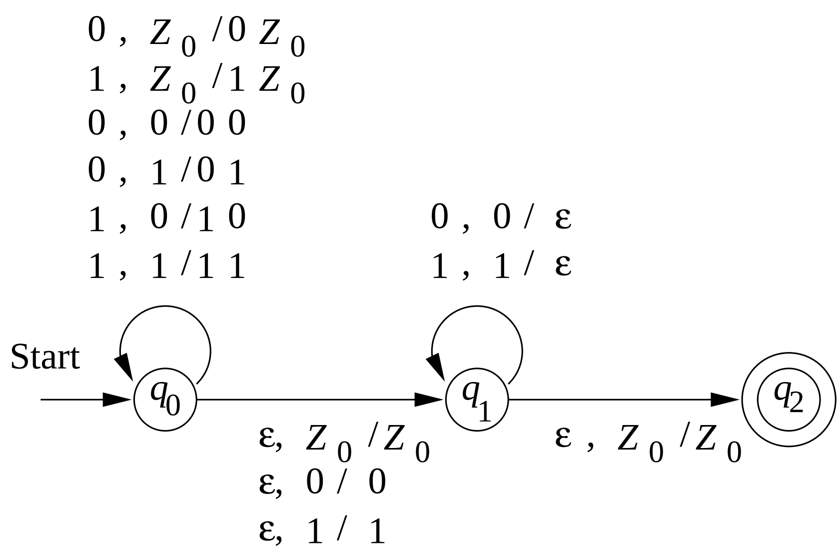
A PDA is a seven-tuple:

$$P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F),$$

where

- Q is a finite set of states,
- Σ is a finite *input alphabet,*
- Γ is a finite *stack alphabet,*
- $\delta : Q \times \Sigma \cup \{\epsilon\} \times \Gamma \rightarrow 2^{Q \times \Gamma^*}$ is the *transition function,*
- q_0 is the *start state,*
- $Z_0 \in \Gamma$ is the *start symbol* for the stack, and
- $F \subseteq Q$ is the set of *accepting states.*

Example: The PDA



is actually the seven-tuple

$$P = (\{q_0, q_1, q_2\}, \{0, 1\}, \{0, 1, Z_0\}, \delta, q_0, Z_0, \{q_2\}),$$

where δ is given by the following table (set brackets missing):

	$0, Z_0$	$1, Z_0$	0,0	0,1	1,0	1,1	ϵ, Z_0	$\epsilon, 0$	$\epsilon, 1$
$\rightarrow q_0$	$q_0, 0Z_0$	$q_0, 1Z_0$	$q_0, 00$	$q_0, 01$	$q_0, 10$	$q_0, 11$	q_1, Z_0	$q_1, 0$	$q_1, 1$
q_1			q_1, ϵ			q_1, ϵ	q_2, Z_0		
$\star q_2$									

Instantaneous Descriptions

A PDA goes from configuration to configuration when consuming input.

To reason about PDA computation, we use *instantaneous descriptions* of the PDA. An ID is a triple

$$(q, w, \gamma)$$

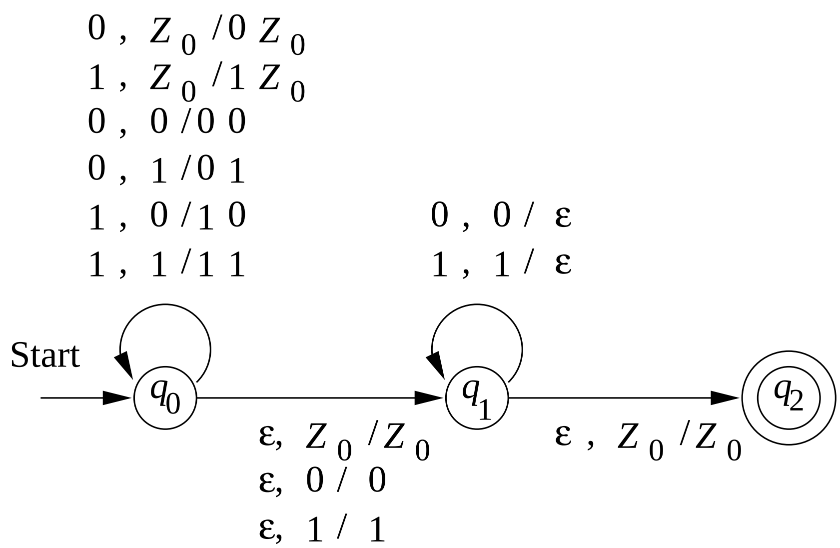
where q is the state, w the remaining input, and γ the stack contents.

Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA. Then $\forall w \in \Sigma^*, \beta \in \Gamma^*$:

$$(p, \alpha) \in \delta(q, a, X) \Rightarrow (q, aw, X\beta) \vdash (p, w, \alpha\beta).$$

We define $\overset{*}{\vdash}$ to be the reflexive-transitive closure of $\vdash$.

Example: On input 1111 the PDA



has the following computation sequences:

$$
\begin{tikzcd}
(q_0, 1111, Z_0) \arrow[d] \arrow[dr] & & \\
(q_0, 111, 1Z_0) \arrow[d] \arrow[dr] & (q_1, 1111, Z_0) \arrow[r] & (q_2, 1111, Z_0) \\
(q_0, 11, 11Z_0) \arrow[d] \arrow[dr] & (q_1, 111, 1Z_0) \arrow[r] & (q_1, 11, Z_0) \arrow[d] \\
(q_0, 1, 111Z_0) \arrow[d] \arrow[dr] & (q_1, 11, 11Z_0) \arrow[dr] & (q_2, 11, Z_0) \\
(q_0, \varepsilon, 1111Z_0) \arrow[d] & (q_1, 1, 111Z_0) \arrow[d] & (q_1, 1, 1Z_0) \arrow[d] \\
(q_1, \varepsilon, 1111Z_0) & (q_1, \varepsilon, 11Z_0) & (q_1, \varepsilon, Z_0) \arrow[d] \\
 & & (q_2, \varepsilon, Z_0)
\end{tikzcd}
$$