Substitutions and Strongly Deterministic Tilesets

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1. Substitution Systems

2. Wang tiles

3. Substitutions and tilesets

4. Deterministic tilesets

5. Main Result

Substitutions

A **k-substitution** s on Σ is a map from Σ to Σ^k. It defines a **morphism** both on finite and infinite words.

$$\tau : a \mapsto ab, \quad b \mapsto ba$$

A substitution such that $s(a) = au$ with $u \in \Sigma^+$ admits a **fixpoint** $s^\infty(a) = \lim_{n\to\infty} s^n(a)$.

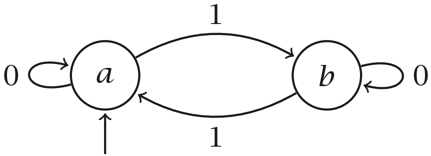
$s^n(a) = aus(u)s^2(u)\cdots s^{n-1}(u)$

$$\tau^\infty(a) = a|b|ba|baab|baababba|baababbaabbabaab|\cdots$$

N. Pytheas Fogg. Substitutions in Dynamics, Arithmetics and Combinatorics. Springer, 2002.

Cobahm's theorem

Theorem A sequence $w \in \Sigma^\omega$ is **k-automatic** if and only if it is the **coding of the fixpoint** of a k-substitution.



Remark A finite machine to **generate** the fixpoint.

J-P Allouche and J. Shallit. Automatic Sequences: Theory, Applications, Generalizations. CUP, 2003.

Dynamical systems

A **dynamical system** is a pair (X, F) where X is a compact and $F : X \to X$ is continue.

Endow $\Sigma^{\mathbb{Z}}$ with the (metric) **product topology**.

$$d(u, v) = 2^{-\min\{|k| \mid u_k \neq v_k\}} \quad \forall u, v \in \Sigma^{\mathbb{Z}}$$

The **shift map** $\sigma : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ translates its argument:

$$\sigma(u)_k = u_{k+1} \quad \forall u \in \Sigma^{\mathbb{Z}}, \forall k \in \mathbb{Z}$$

A **symbolic dynamical system** is a dynamical system (X, σ) where $X \subseteq \Sigma^{\mathbb{Z}}$ is closed and σ-invariant.

$\Sigma^{\mathbb{Z}}$ is called the **full shift** and any such X is called a **subshift**.

D. Lind and B. Marcus. An Introduction to Symbolic Dynamics and Coding. CUP, 1995.

Orbit closure and limit set

The **orbit closure** of the **fixpoint** of a substitution defines a **symbolic dynamical system** (X_s, σ)

$$X_s = \{u \,|\, \forall p \prec u, \exists n \in \mathbb{N}, p \prec \sigma^n(a)\}$$

Let $S : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ extend s on $\Sigma^{\mathbb{Z}}$:

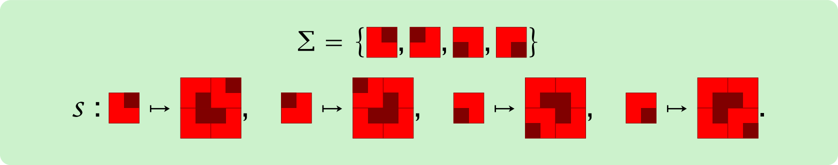
$$s(u_k) = S(u)_{2k} S(u)_{2k+1} \quad \forall u \in \Sigma^{\mathbb{Z}}, \forall k \in \mathbb{Z}$$

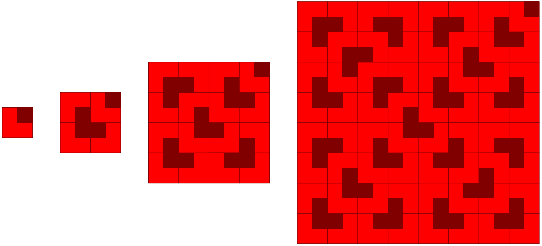
The **limit set** $\Lambda_s \subseteq \Sigma^{\mathbb{Z}^2}$ is the maximal attractor of S:

$$\Lambda_s = \bigcap_{t \in \mathbb{N}} \left\langle S^t \left(\Sigma^{\mathbb{Z}^2} \right) \right\rangle_\sigma$$

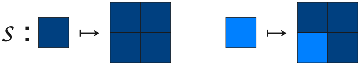
Remark $X_s \subseteq \Lambda_s$

2D Substitutions

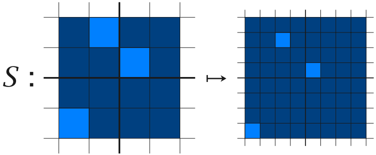




Two-by-two substitutions



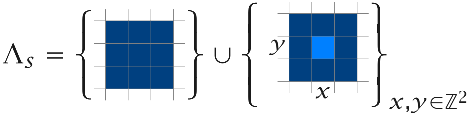
A **2x2 substitution** $s : \Sigma \to \Sigma^{\boxplus}$ maps letters to squares of letters on the same finite alphabet.



The substitution is extended as a **global map** $S : \Sigma^{\mathbb{Z}^2} \to \Sigma^{\mathbb{Z}^2}$ on colorings of the plane:

$$\forall z \in \mathbb{Z}^2, \forall k \in \boxplus, \quad S(c)(2z+k) = s(c(z))(k)$$

Limit set and history



The **limit set** $\Lambda_s \subseteq \Sigma^{\mathbb{Z}^2}$ is the maximal attractor of S:

$$\Lambda_s = \bigcap_{t\in\mathbb{N}} \left\langle S^t\left(\Sigma^{\mathbb{Z}^2}\right)\right\rangle_\sigma$$

The limit set is the set of colorings admitting an **history** $(c_i)_{i\in\mathbb{N}}$ where $c_i = \sigma_{u_i}(S(c_{i+1}))$.

Finite machinery

Question Can we devise some sort of **finite machine** to capture **limit sets**?

Limit sets can be **infinite**, even **uncountable**.

Remark As **subshifts**, limit sets are characterized by **sets of forbidden patterns**.

Let's find some **finite machinery** that captures **sets of forbidden patterns** to characterize **limit sets**.

SFT and sofic subshifts

Definition A **subshift of finite type** (SFT) is a subshift that can be defined by a **finite set** of forbidden patterns.

Definition A **sofic subshift** is the **coding** of a **SFT**.

In 1D, sofic subshifts admit **regular languages** as sets of forbidden patterns.

In 2D, sofic subshifts are captured by **coding of tilings** by finite sets of Wang tiles.

Question Can we characterize limit sets with Wang tiles?

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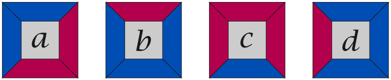
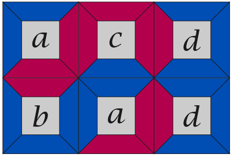
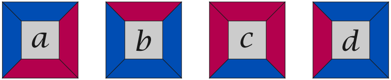
The Domino Problem (DP)

> *"Assume we are **given a finite set of square plates** of the same size with **edges colored**, each in a different manner. Suppose further there are infinitely many copies of each plate (plate type). We are **not permitted to rotate or reflect a plate**. The question is to find an effective procedure by which we can **decide**, for each given finite set of plates, **whether we can cover up the whole plane** (or, equivalently, an infinite quadrant thereof) **with copies of the plates subject to the restriction that adjoining edges must have the same color**."*
>
> *(Wang, 1961)*

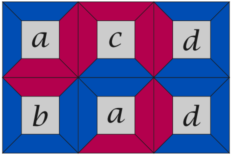




Wang tiles



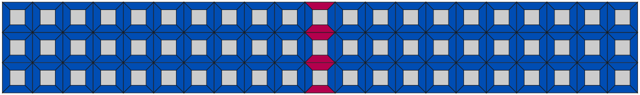
A **tileset** $\tau \subseteq \Sigma^4$ is a tile set with colored edges.



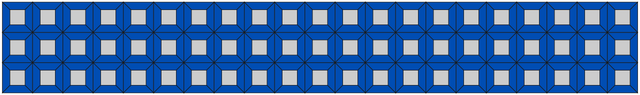
The set of **τ-tilings** $X_\tau \subseteq \tau^{\mathbb{Z}^2}$ is the set of colorings of $\mathbb{Z}^2$ by τ where colors match along edges.

Periodic Tilings

Definition A tiling is **periodic** with period p if it is invariant by a **translation** of vector p.



Lemma If a tile set admits a **periodic** tiling then it admits a **biperiodic** tiling.

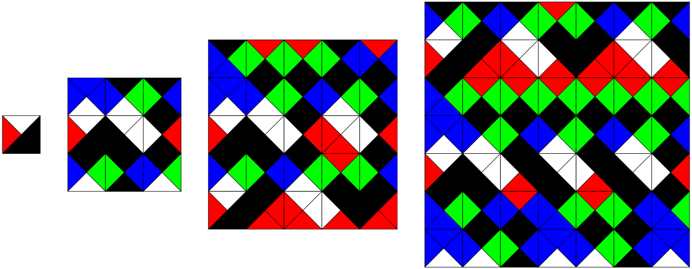


Lemma Tile sets tiling the plane biperiodically are **re** (**recursively enumerable**).

co-Tiling

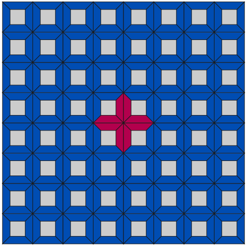
Lemma Tile sets tiling the plane are **co-re**.

Sketch of the proof Consider tilings of larger and larger square regions. If the set does not tile the plane, by compacity, there exists a size of square it cannot cover with tiles.



Aperiodicity

Definition A tiling is **aperiodic** if it admits no non-trivial period.



Definition A tile set is **aperiodic** if it admits a tiling and all its tilings are aperiodic.

Remark If there were **no aperiodic** tile set, the Domino Problem would be **decidable**.

Undecidability of DP

Theorem[Berger 1964] **DP** is **undecidable**.

Remark To prove it one needs **aperiodic** tile sets.

Seminal self-similarity based proofs *(reduction from* ***HP****)*:

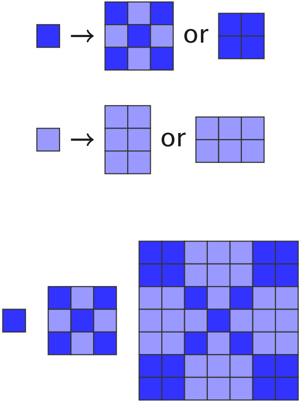
- Berger, 1964 *(20426 tiles, a full PhD thesis)*
- Robinson, 1971 *(56 tiles, 17 pages, long case analysis)*
- Durand et al, 2007 *(Kleene's fixpoint existence argument)*

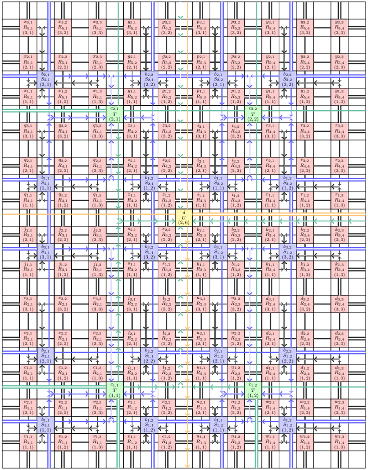
Tiling rows seen as transducer trace based proof:
Kari, 2007 *(affine maps, reduction from* ***IP****)*

And others!

- Mozes, 1990 *(non-deterministic substitutions)*
- Aanderaa and Lewis, 1980 *(1-systems and 2-systems)*

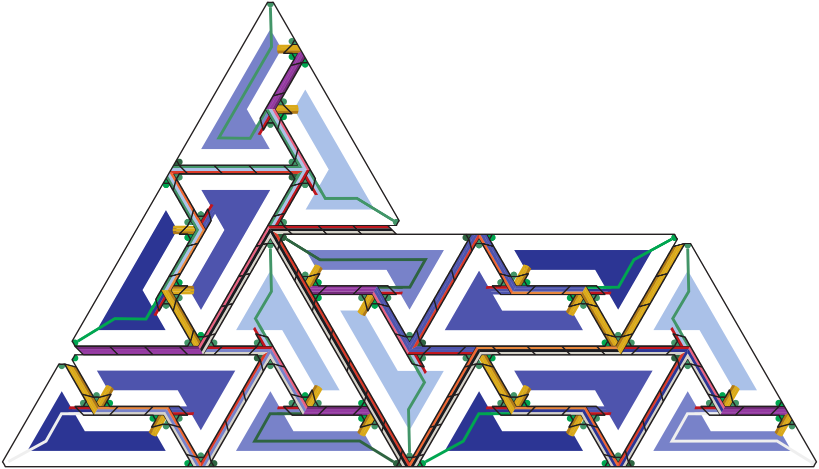
Mozes 1990





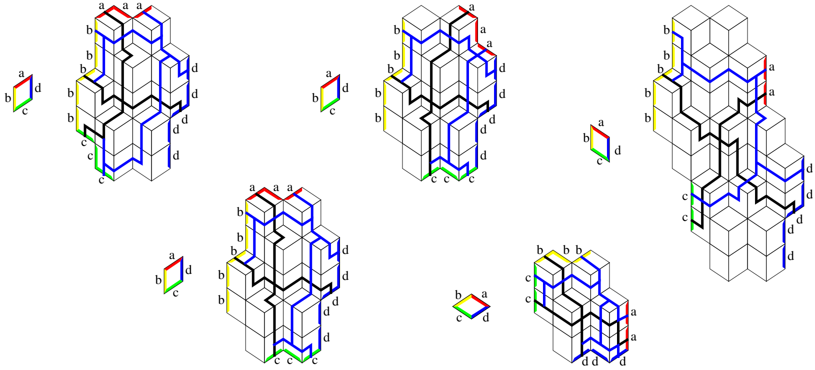
Theorem[Mozes 1990] The limit set of a **non-deterministic rectangular substitution** *(+ some hypothesis)* is sofic.

Goodman-Strauss 1998



Theorem[Goodman-Strauss 1998] The limit set of **homothetic substitution** *(+ some hypothesis)* is sofic.

Fernique-O 2010



Theorem[Fernique-O 2010] The limit set of a **combinatorial substitution** *(+ some hypothesis)* is sofic.

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In this talk

A simple original construction of an aperiodic tile set based on two-by-two substitution systems. . .

. . . and its application both to recognizing limit sets of substitutions and to an old historical construction.

This work combines tools and ideas from:

- **[Berger 64]** *The Undecidability of the Domino Problem*
- **[Robinson 71]** *Undecidability and nonperiodicity for tilings of the plane*
- **[Grünbaum Shephard 89]** *Tilings and Patterns, an introduction*
- **[Durand Levin Shen 05]** *Local rules and global order, or aperiodic tilings*

Unambiguous substitutions

A substitution is **aperiodic** if its limit set Λ_s is aperiodic.

A substitution is **unambiguous** if, for every coloring c from its limit set Λ_s, there exists a unique coloring c' and a unique translation $u \in \boxplus$ satisfying $c = \sigma_u(S(c'))$.

Proposition **Unambiguity** implies **aperiodicity**.

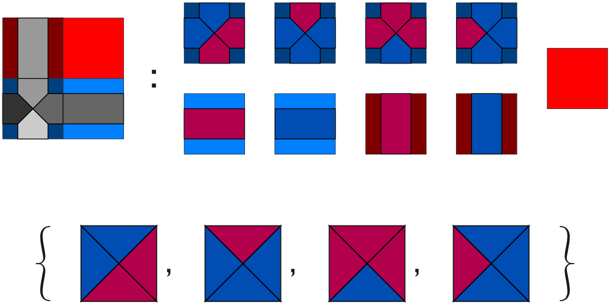
Sketch of the proof. Consider a periodic coloring with minimal period p, its preimage has period $p/2$. ◊

Idea. Construct a tile set whose tilings are in the limit set of an unambiguous substitution system.

Coding tile sets into tile sets

Definition A tile set τ' **codes** a tile set τ, according to a **coding rule** $t : \tau \to \tau'^{\boxplus}$ if t is injective and

$$X_{\tau'} = \{\sigma_u(t(c)) | c \in X_\tau, u \in \boxplus\}$$



Unambiguous self-coding

Definition A tile set τ **codes** a substitution $s : \tau \to \tau^{\boxplus}$ if it codes itself according to the coding rule s.

Proposition A tile set both admitting a tiling and **coding** an **unambiguous** substitution is **aperiodic**.

Sketch of the proof. $X_\tau \subseteq \Lambda_s$ and $X_\tau \neq \emptyset$. $\Diamond$

Idea. Construct a tile set whose tilings are in the limit set of a **locally checkable** unambiguous substitution embedding a whole history.

a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*

a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
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a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
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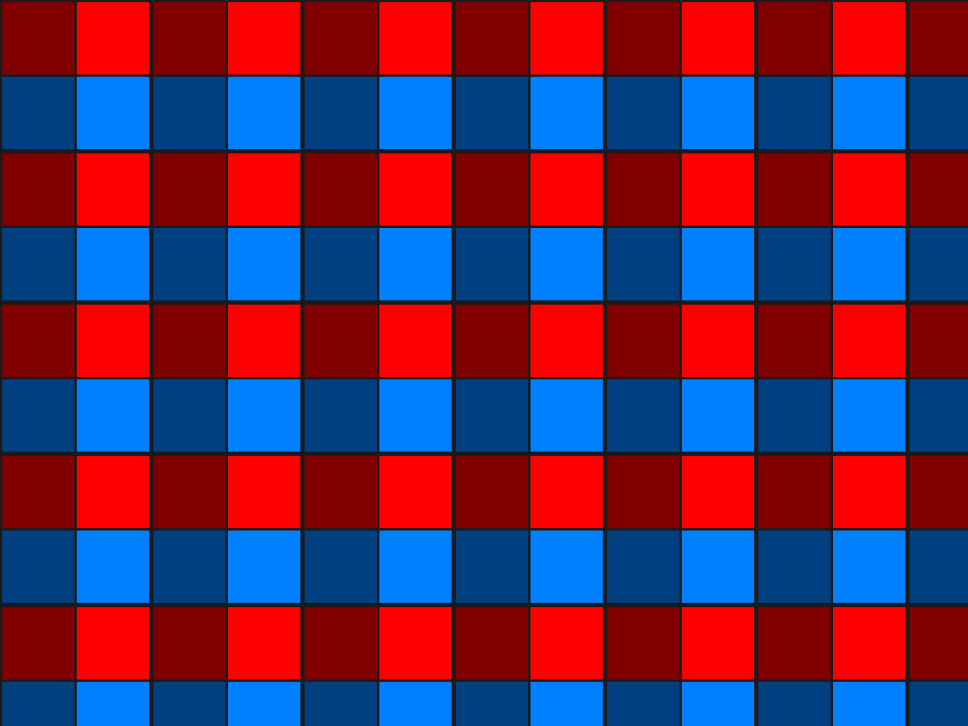

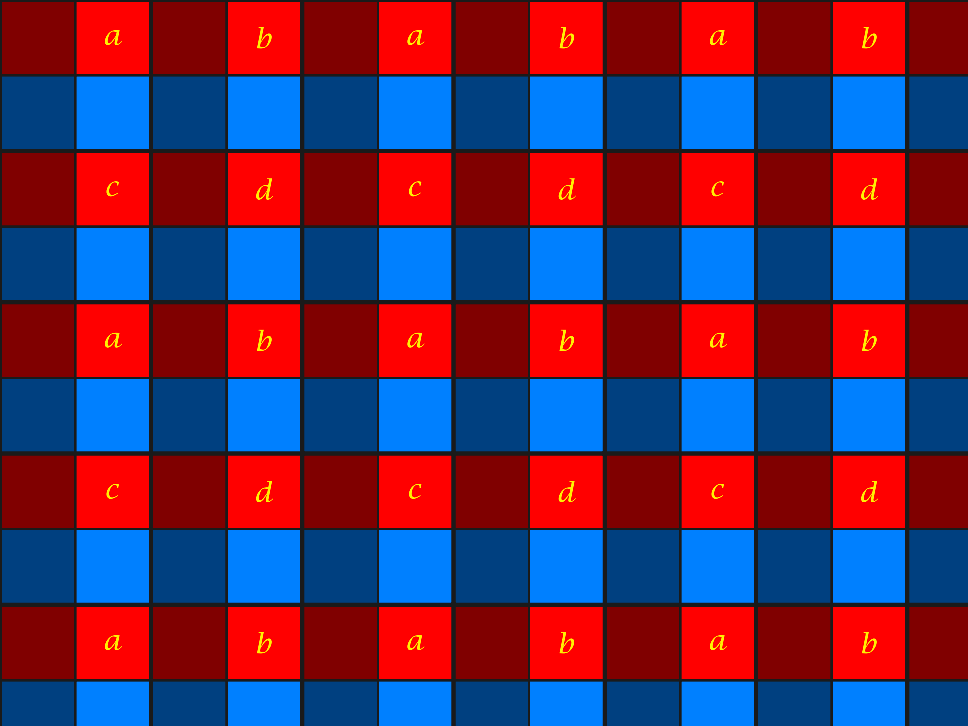
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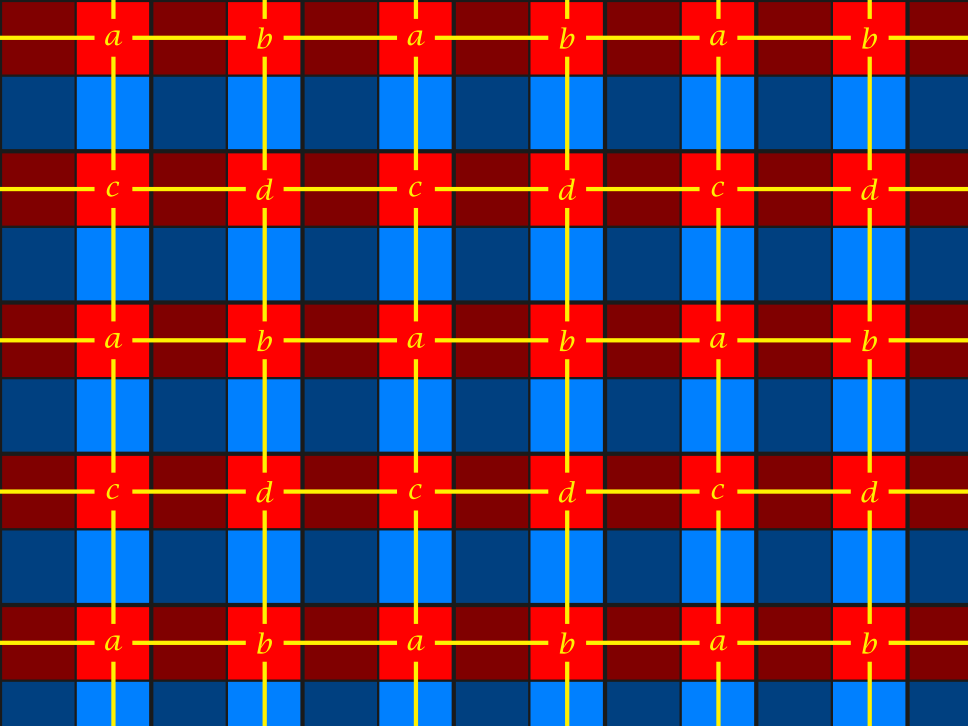
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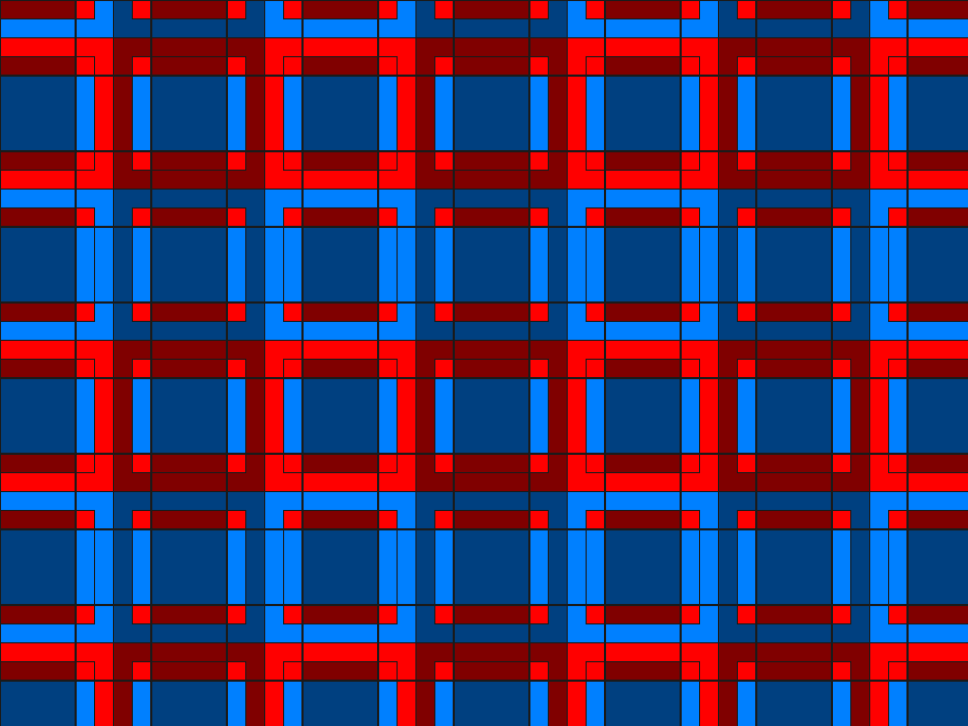

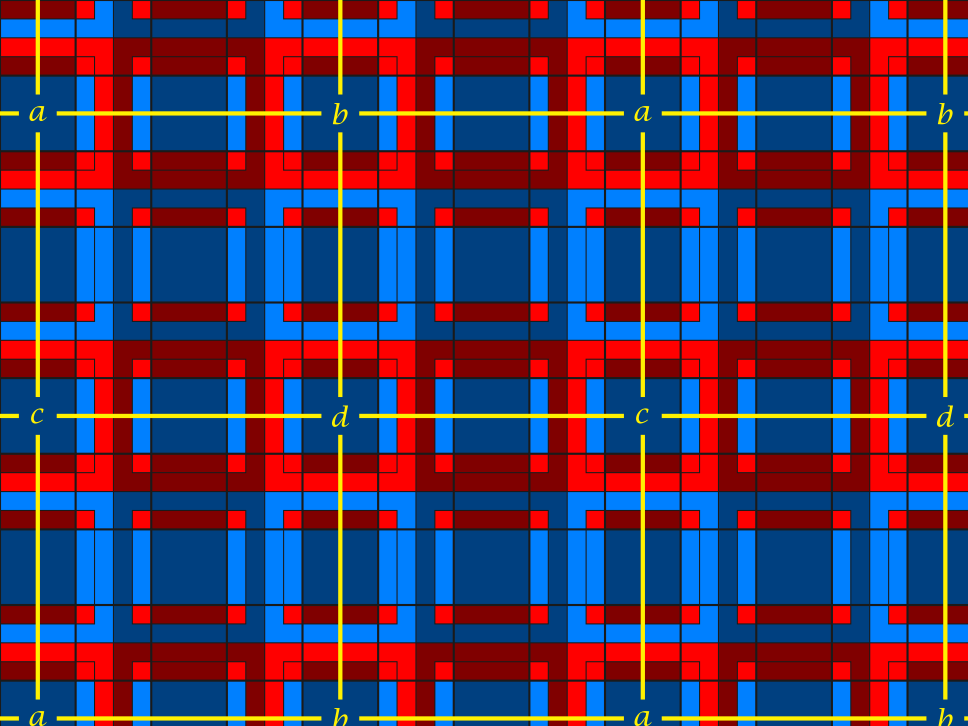
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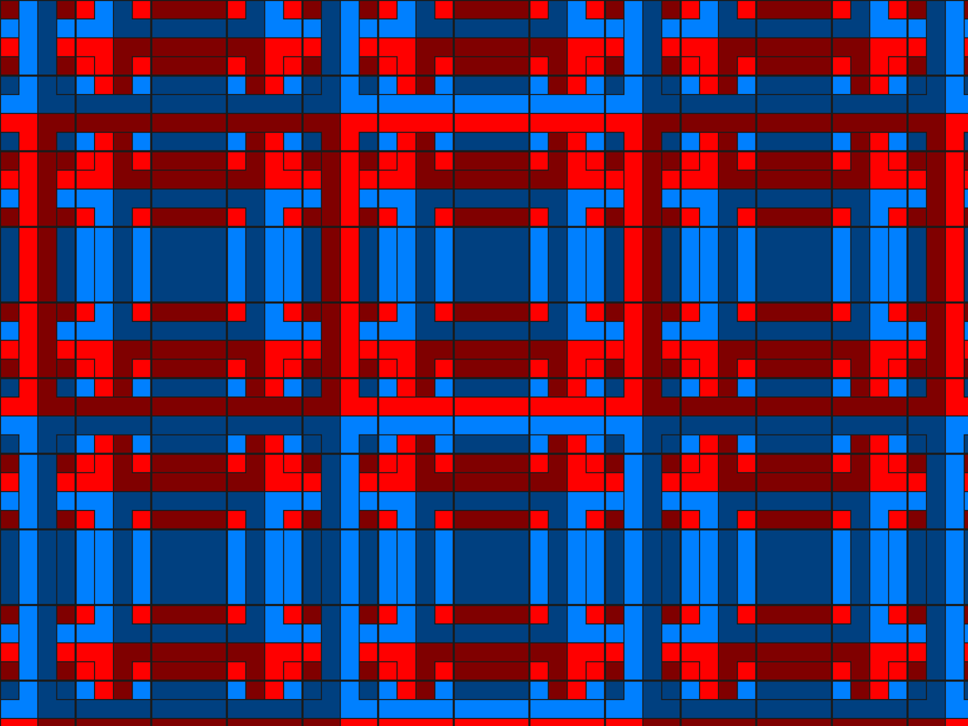
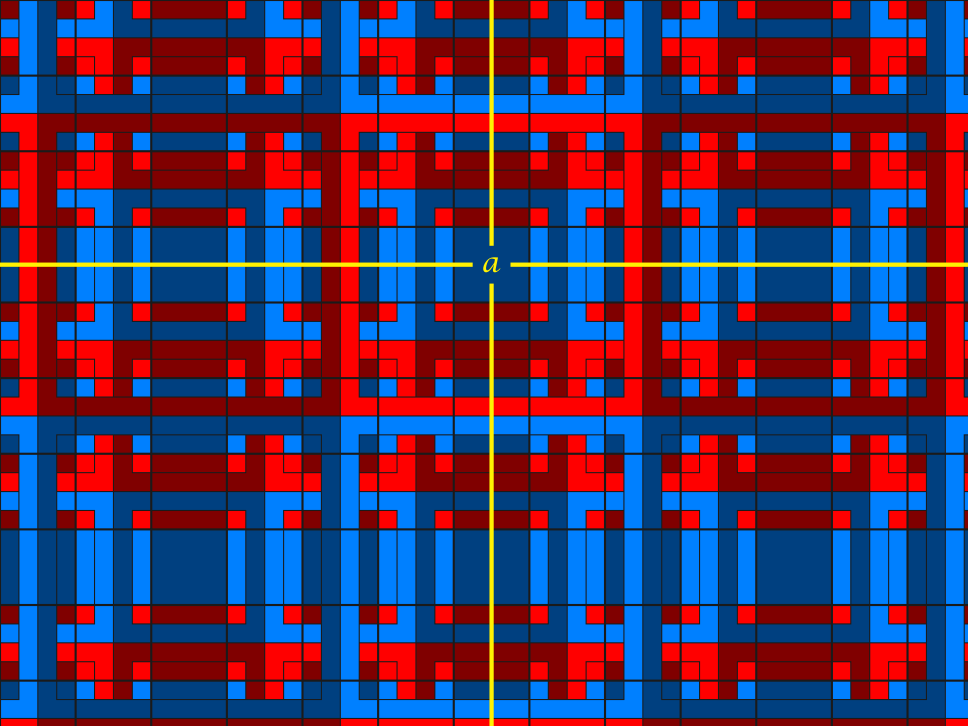
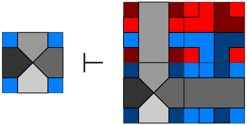




a

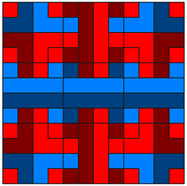
Is this self-encoding?

Iterating the coding rule one obtains 56 tiles.



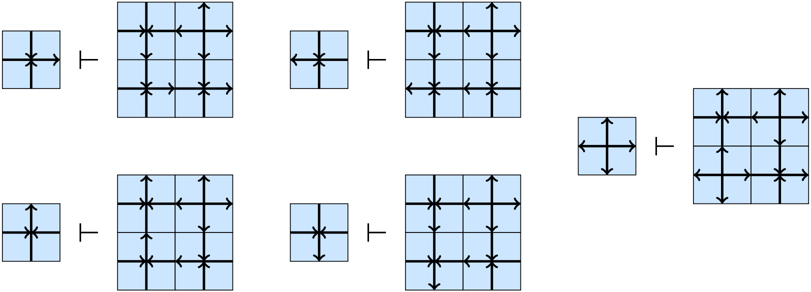
coding rule

Unfortunately, this tile set is **not self-coding**.



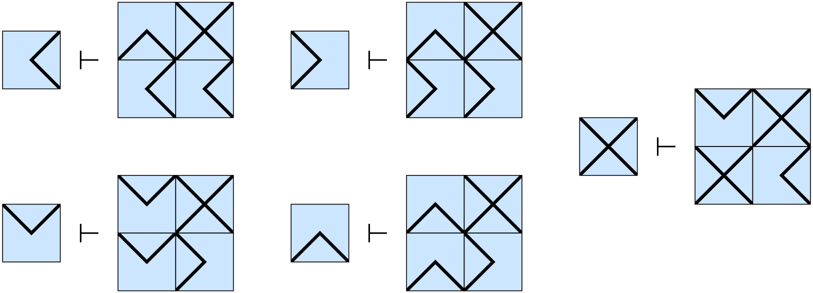
Idea Add a **synchronizing substitution** as a **third layer**.

à la Robinson



Proposition The associated tile set of **104 tiles** admits a tiling and **codes** an **unambiguous** substitution.

à la Robinson



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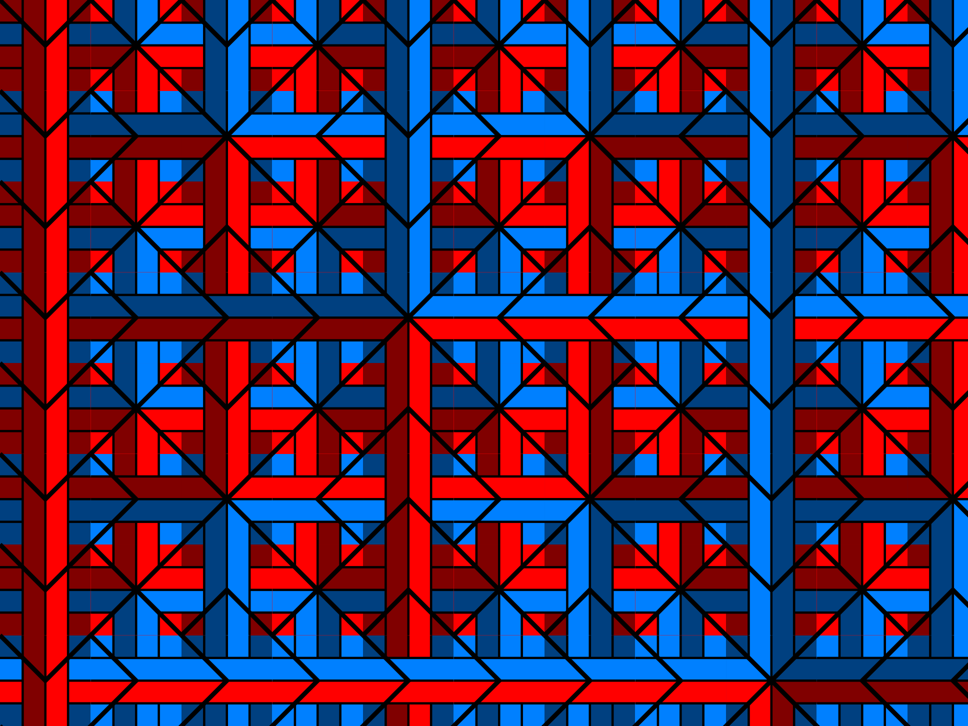


Aperiodicity: sketch of the proof

1. The tile set admits a tiling:
Generate a valid tiling by iterating the substitution rule:
$X_\tau \cap \Lambda_S \neq \emptyset$*.*

2. The substitution is unambiguous:
It is injective and the projectors have disjoined images.

3. The tile set codes the substitution:

(a) each tiling is an image of the canonical substitution
Consider any tiling, level by level, short case analysis.

(b) the preimage of a tiling is a tiling
Straightforward by construction (preimage remove constraints).

To continue…

Theorem The **limit set** of a 2x2 substitution is **sofic**.

Idea To encode Λ_s via **local matching rules**, decorate s into a **locally checkable** $s^\bullet$ embedding a whole history.

Corollary[Berger 1964] **DP** is **undecidable**.

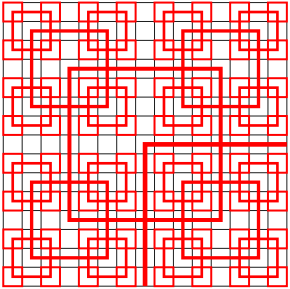
Idea Construct a **2x2 substitution** whose **limit set** contains everywhere **squares** of larger and larger size, insert **Turing computation** inside those squares.

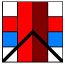
Enforcing substitutions *via* tilings

Let π map every tile of $\tau(s')$ to $s'(a)(u)$ where a and u are the letter and the value of $\boxplus$ on layer 1.

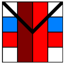
Proposition. Let s' be any substitution system. The tile set $\tau(s')$ enforces s': $\pi\left(X_{\tau(s')}\right) = \Lambda_{s'}$.

Idea. Every tiling of $\tau(s')$ codes an history of s' and every history of s' can be encoded into a tiling of $\tau(s')$.





$b = s(a)\binom{1}{1}$



$b = s(a)\binom{0}{0}$

Squares everywhere

↦

t	t
t	t

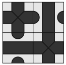
↦

h	h
h	h

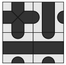
↦

v	v
v	v

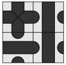
t ↦



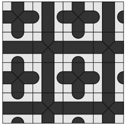
h ↦



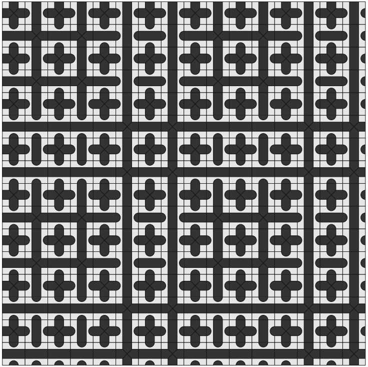
v ↦



t	t
t	t



t	t	v	v	t	t	v	v
t	t	v	v	t	t	v	v
h	h	t	t	h	h	t	t
h	h	t	t	h	h	t	t
t	t	v	v	t	t	v	v
t	t	v	v	t	t	v	v
h	h	t	t	h	h	t	t
h	h	t	t	h	h	t	t



1. Substitution Systems

2. Wang tiles

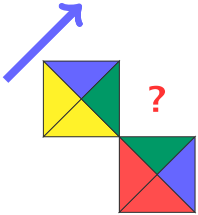
3. Substitutions and tilesets

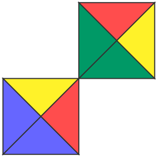
4. Deterministic tilesets

5. Main Result

Deterministic tilesets

Deterministic tileset. A tileset τ is **NE-deterministic** if for any pair of tiles $(t_W, t_S) \in \tau^2$, there exists **at most one** tile t compatible to the west with t_W and to the south with t_S.





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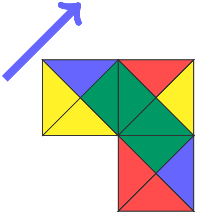

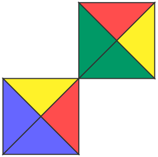

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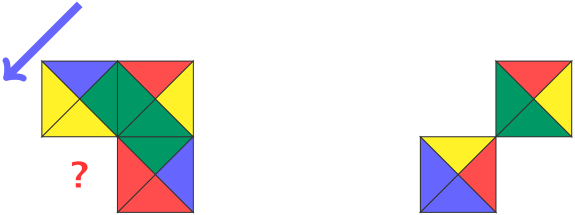

Strong determinism. A tileset is **4-way deterministic** if it is simultaneously deterministic in the **four directions** NE, NW, SW and SE.

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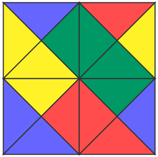

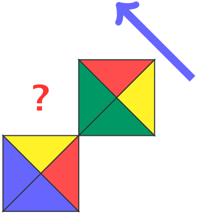

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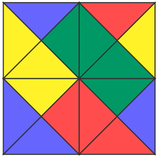

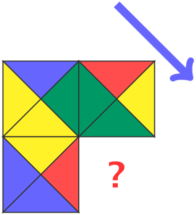

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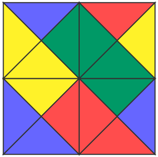

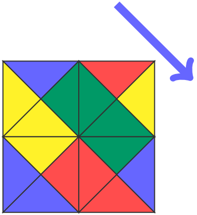

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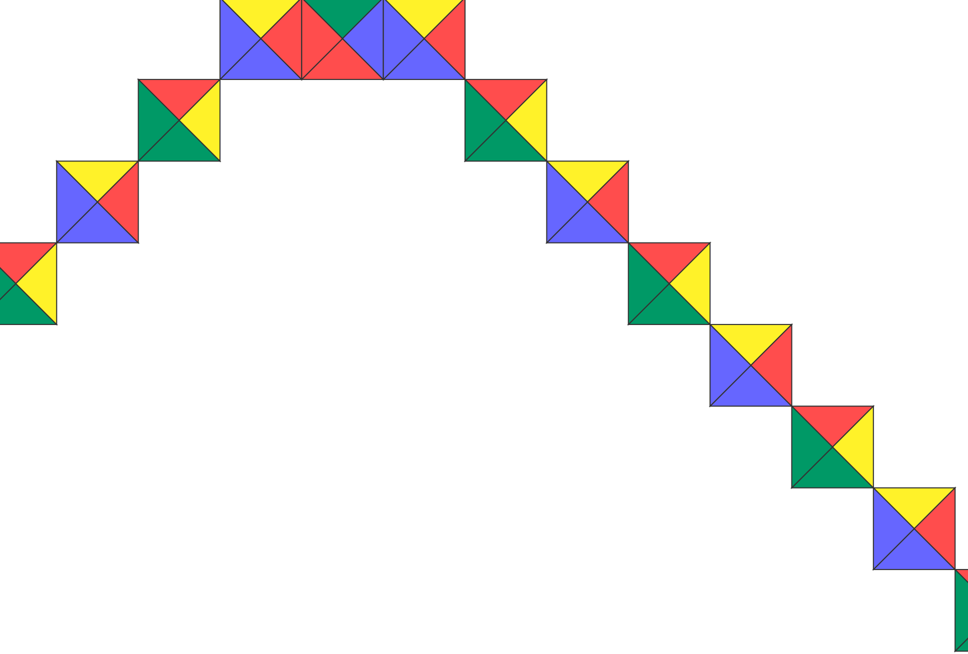

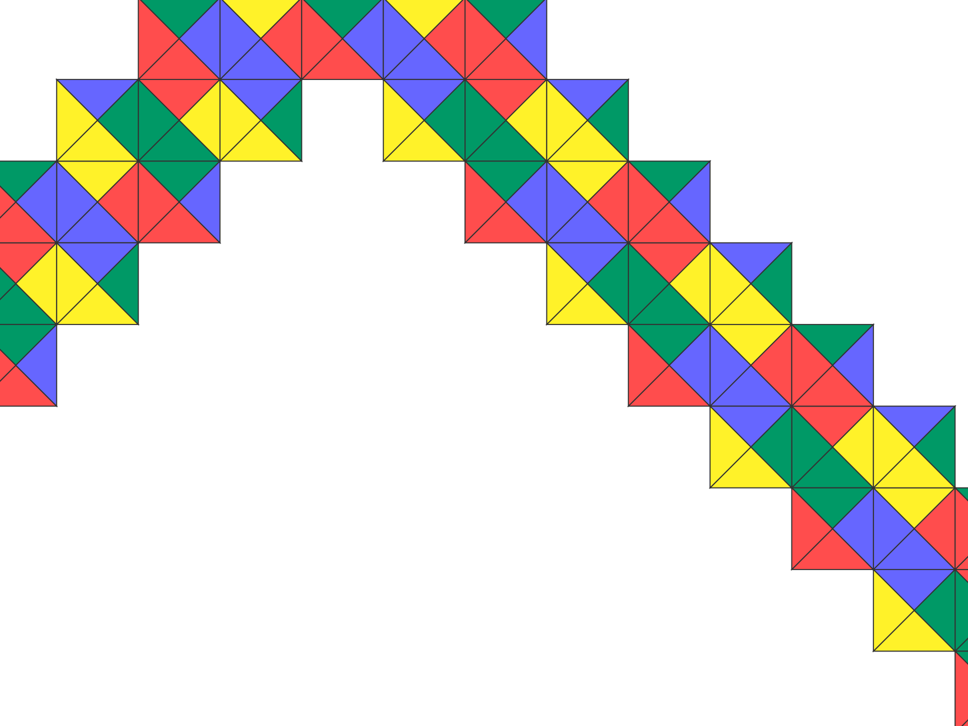

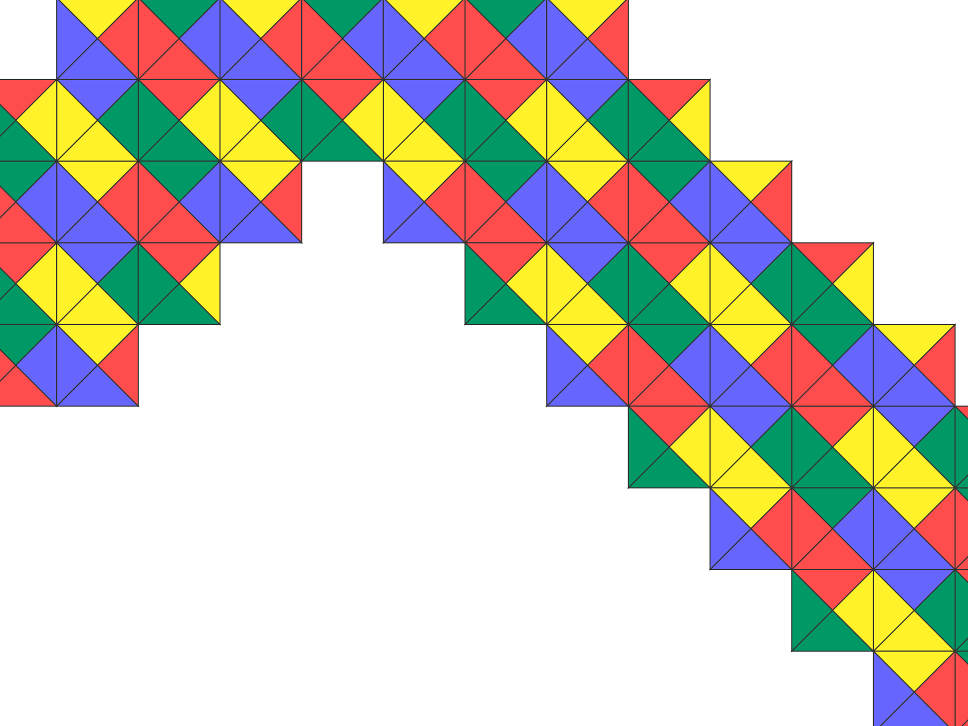

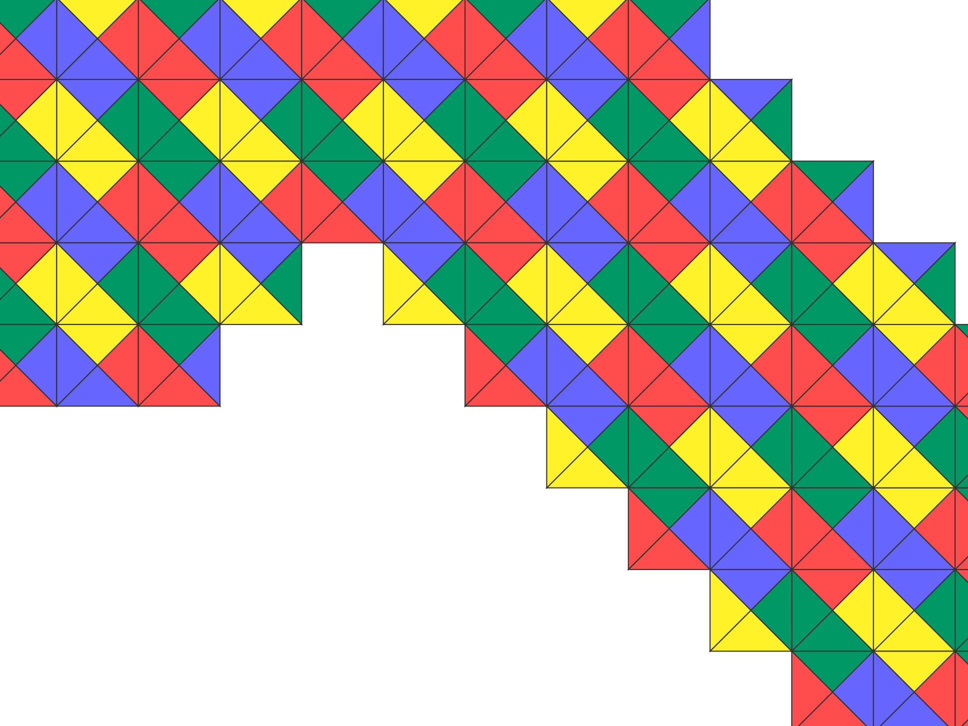

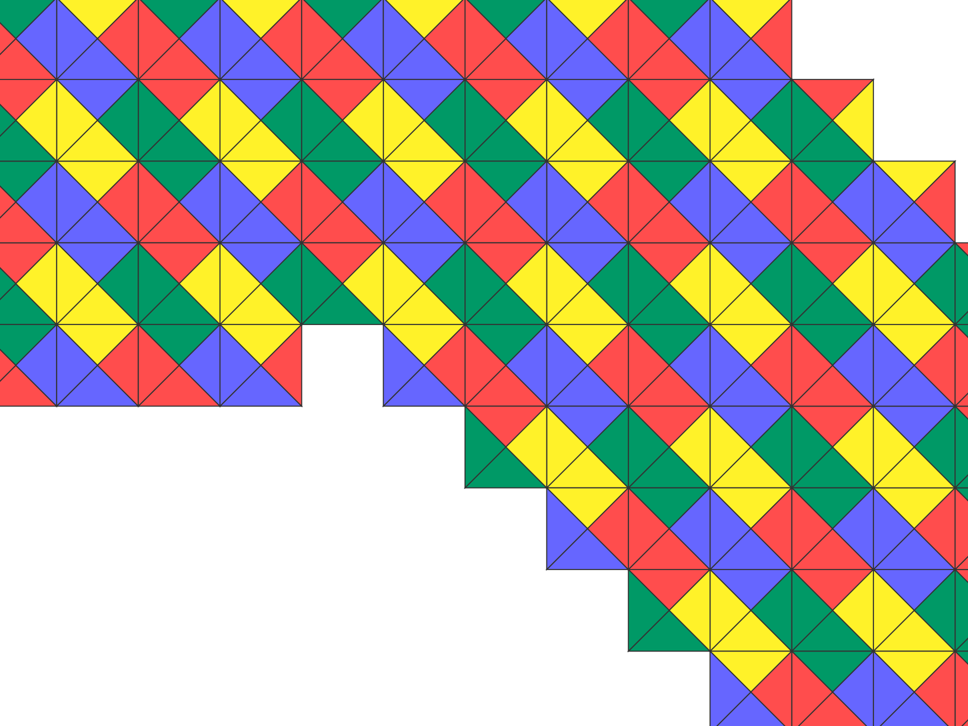

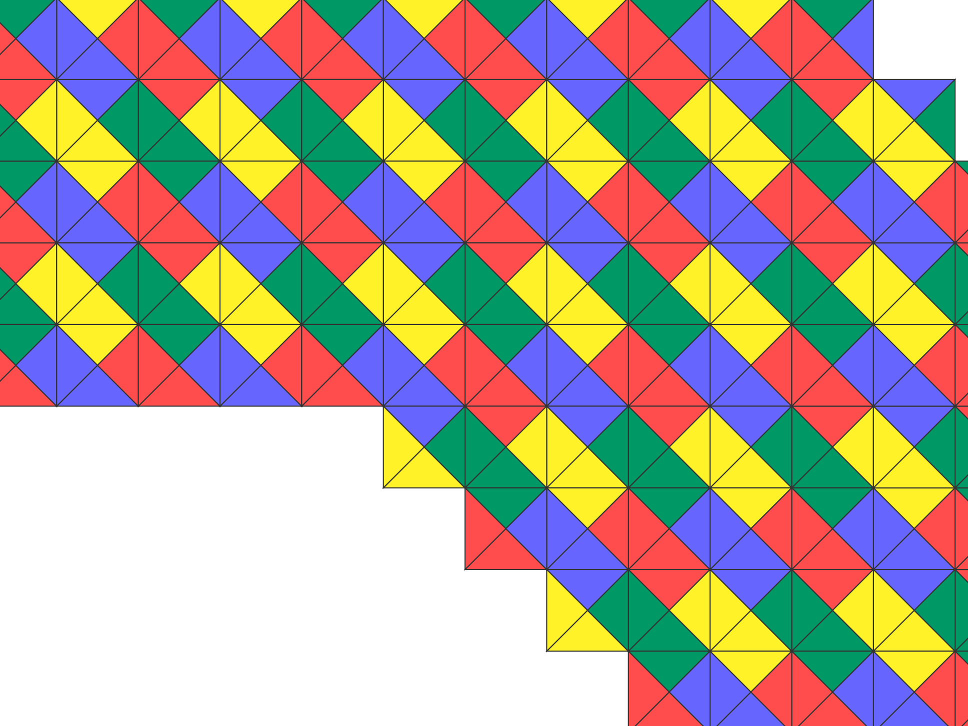

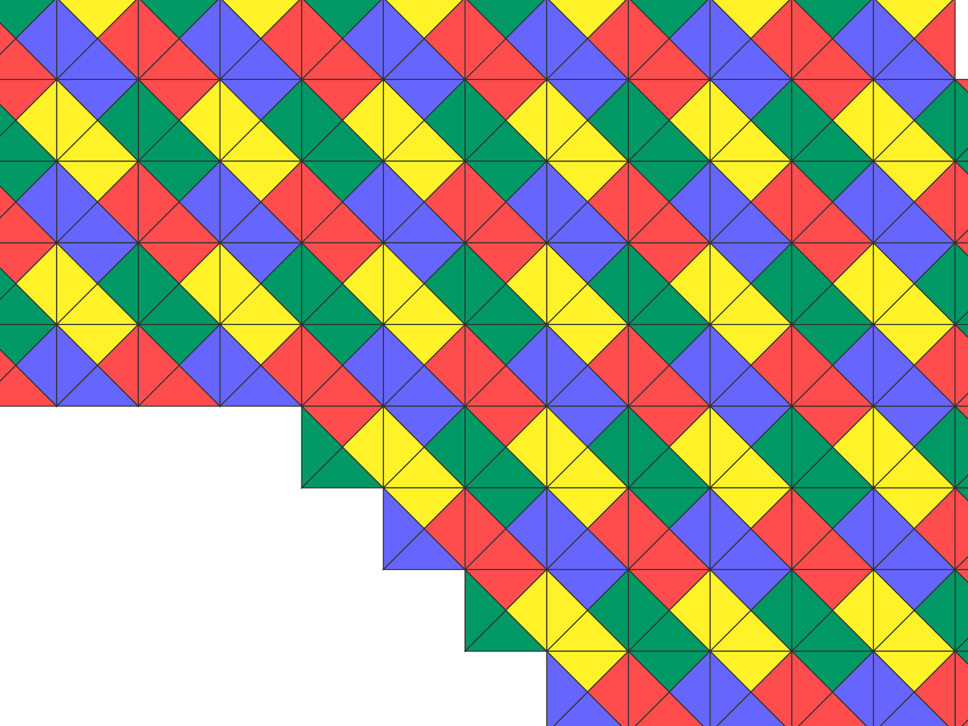

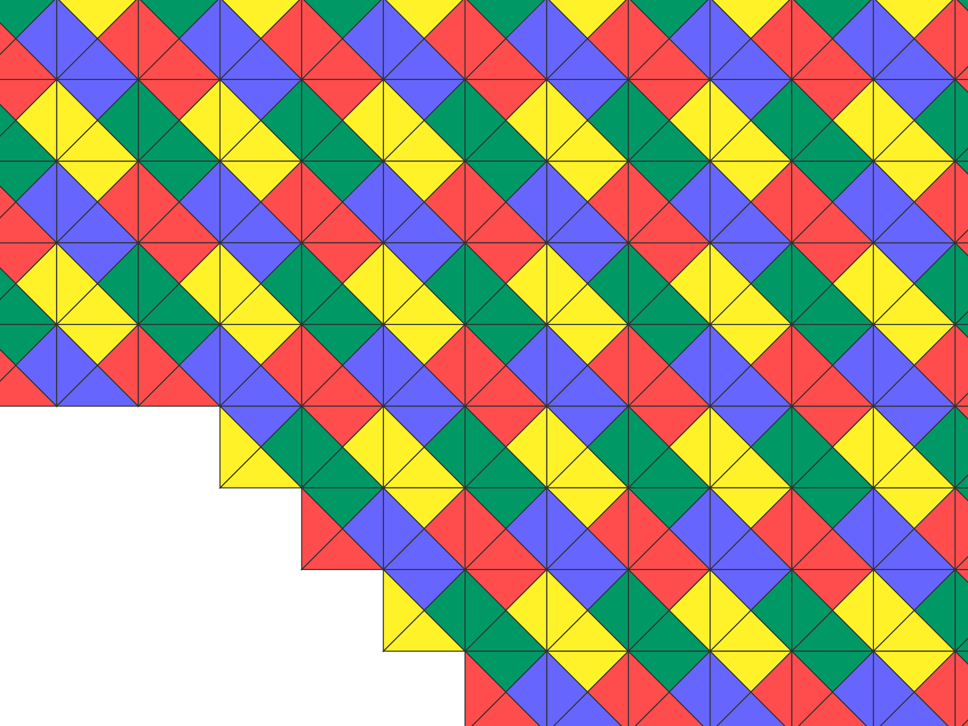

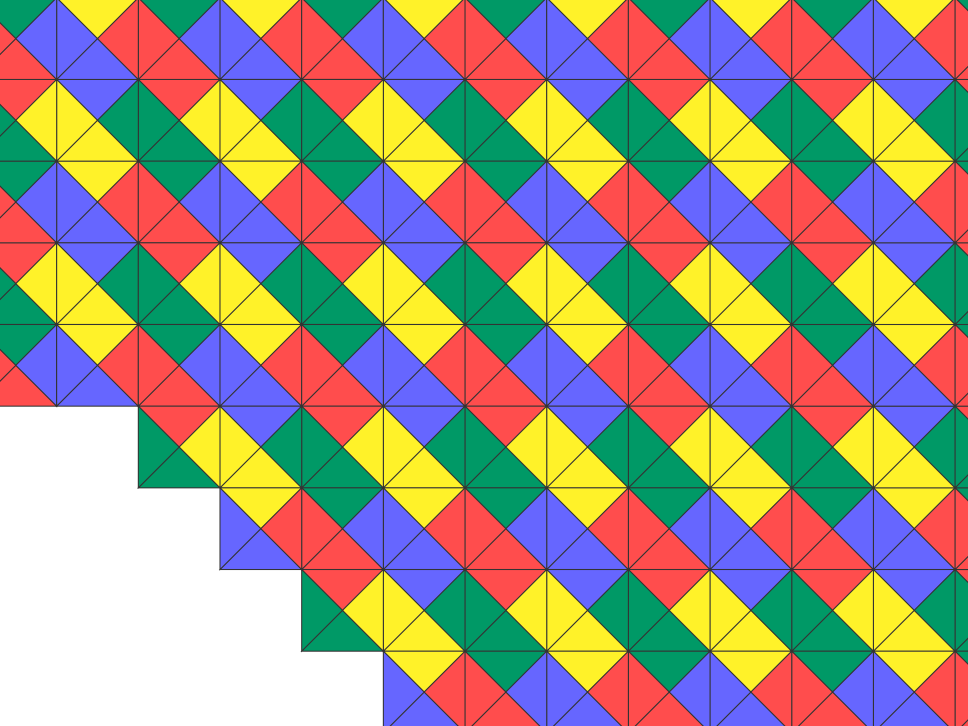

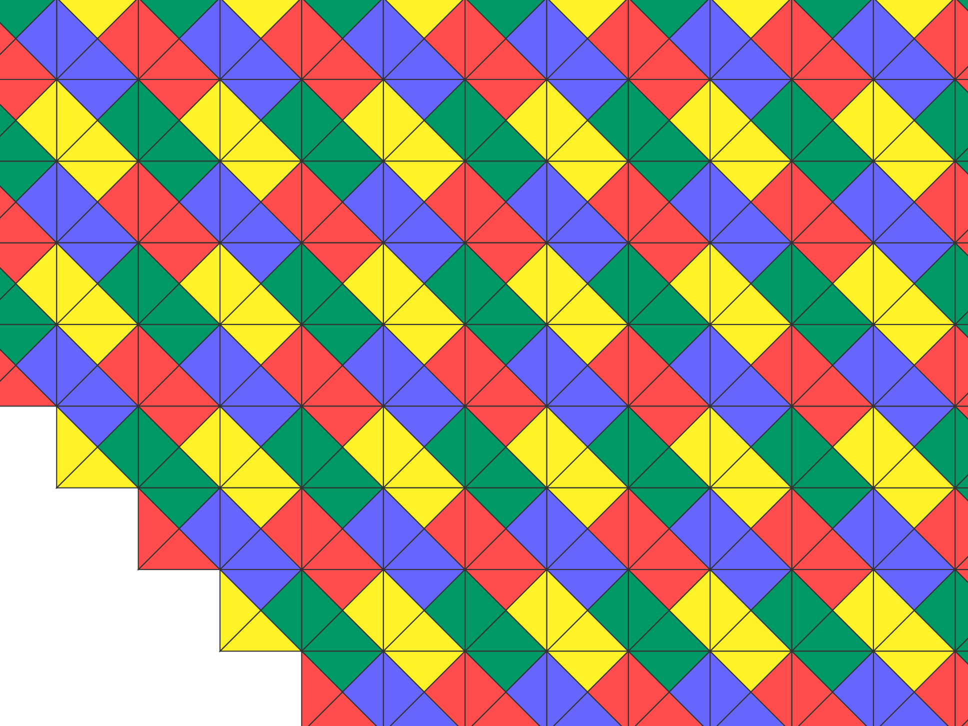

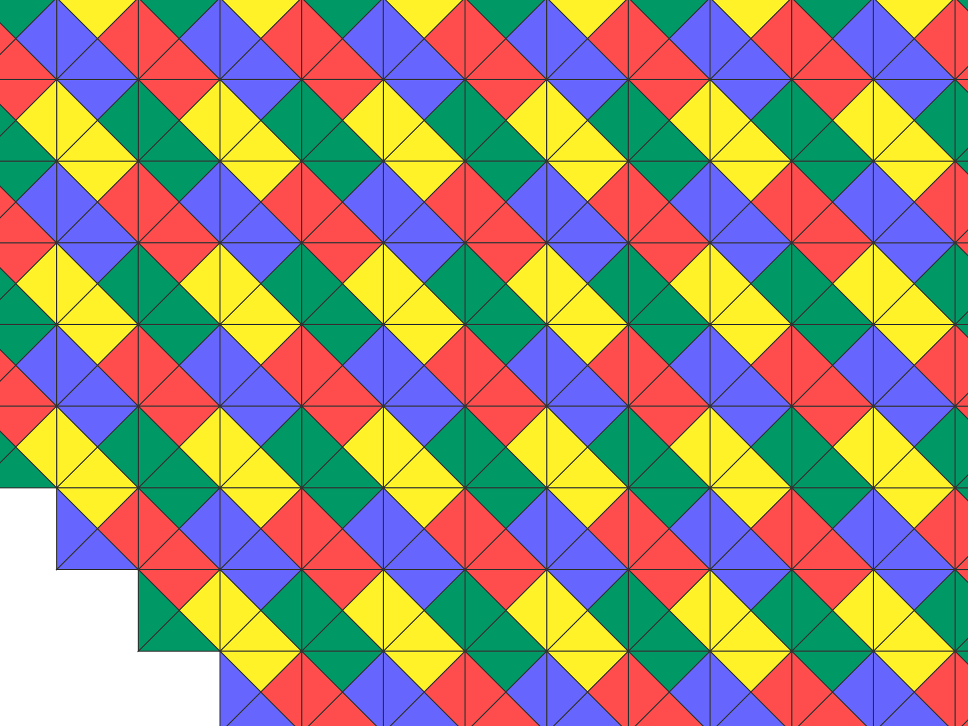

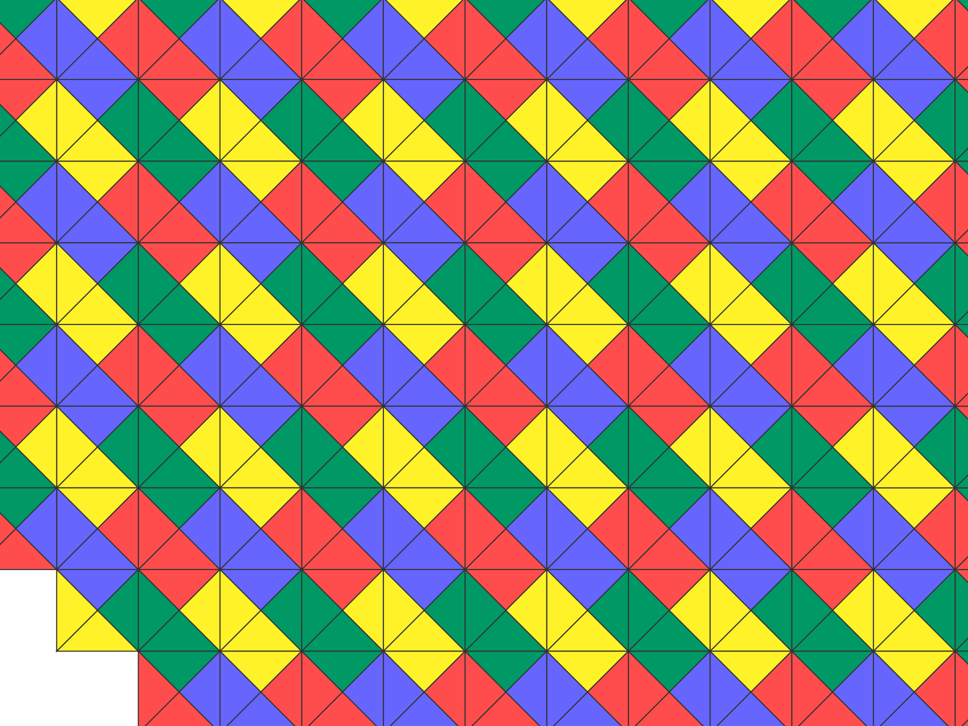

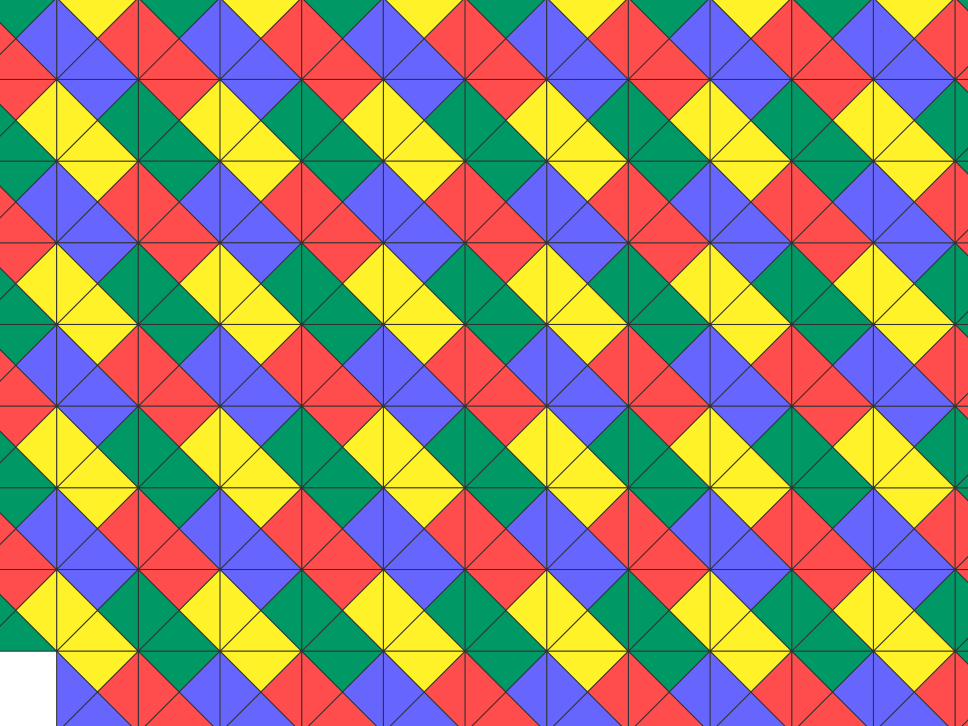

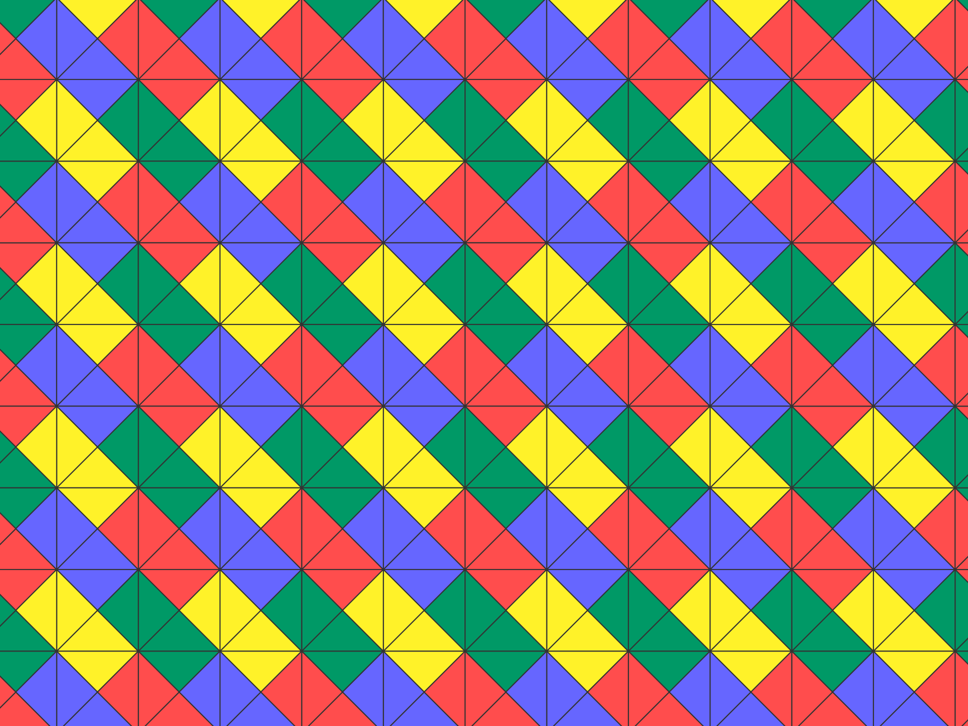

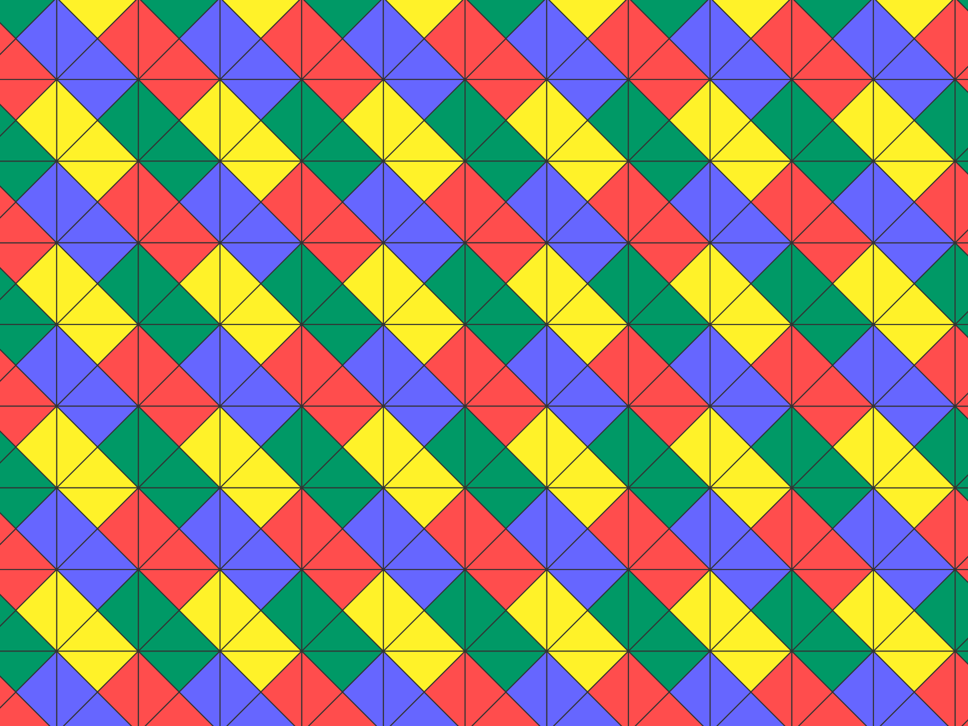

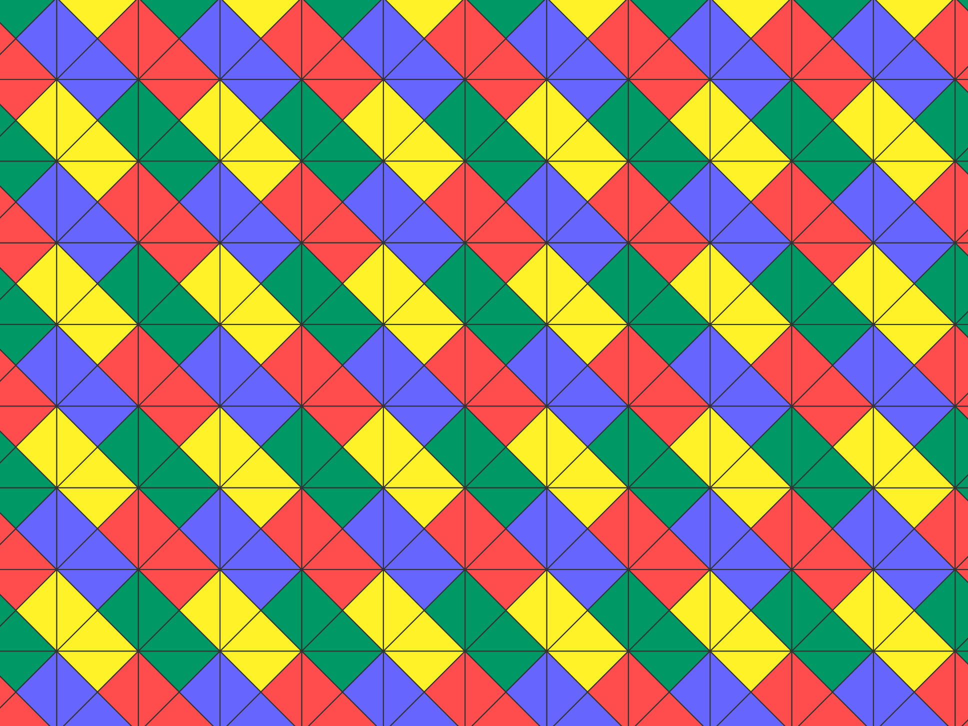

A very short history

[Kari 91] introduces the notion of deterministic tileset.

Theorem [Kari 91]. **DP** remains undecidable for **deterministic** tilesets (and there exist some **(bi)deterministic** aperiodic tilesets).

Theorem [Kari-Papasoglu 99]. There exist some **4-way deterministic** aperiodic tilesets.

Theorem [Lukkarila 09]. **DP** remains undecidable for **4-way deterministic** tilesets.

1. Substitution Systems

2. Wang tiles

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5. **Main Result**

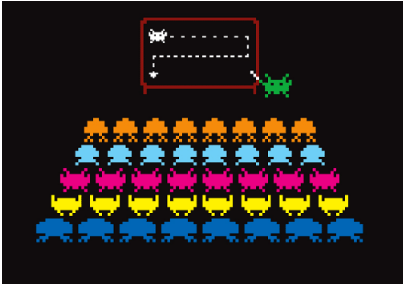
Directional soficity of Λ_s

We want to force the hierarchical structure imposed by the substitution using **4-way deterministic** local rules.

In practice. Let's adapt some (rather technical) existing constructions coding the history of colorings into tilings to make them deterministic.

Theorem. Limit sets of 2×2 substitutions are ***4-way*** **sofic**.

Directional soficity of Λ_s



Battle plan.

1. We determinize **104** in the **four directions** simultaneously.
2. We determinize **104 + substitutions** in **one** direction.
3. We **bi**determinize **104 + substitutions**.
4. We **strongly** determinize **104 + substitutions**.

104-way

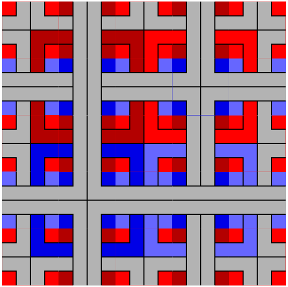
104 is not deterministic in **any** direction.

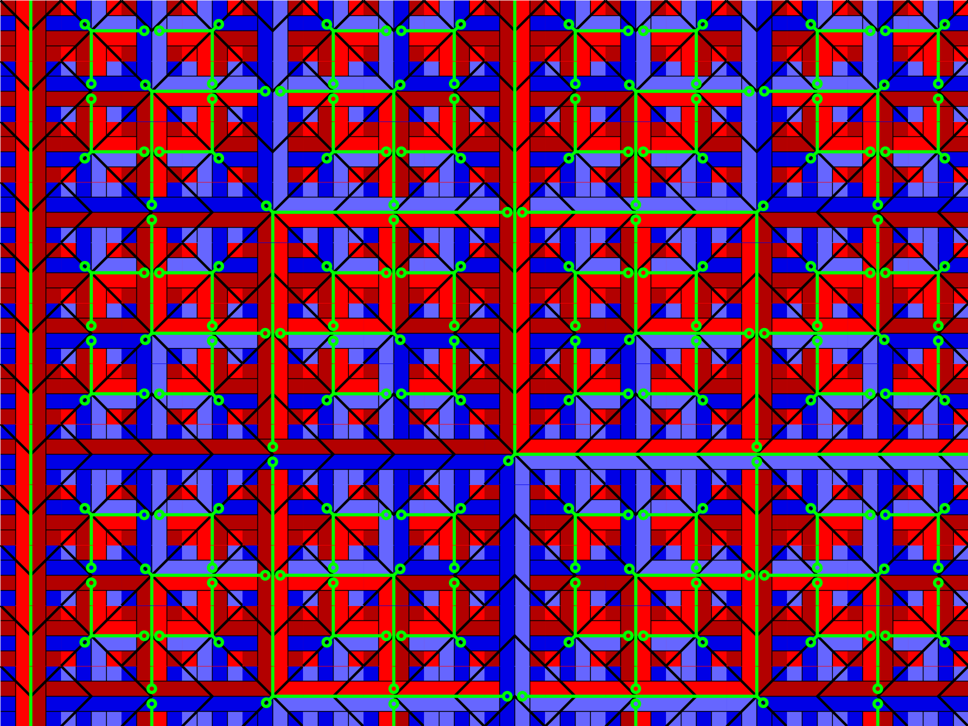
Problem. How to chose between H and V tiles in $\substack{\square\square\\\blacksquare\square}$ positions?

Idea. Go & search for information on the mother tile back in the history.

Need for:

- new constraints at radius 1 for level 0 ;
- new wires (carrying H/V/X labels) to inferior levels.





104 1-way + Substitutions

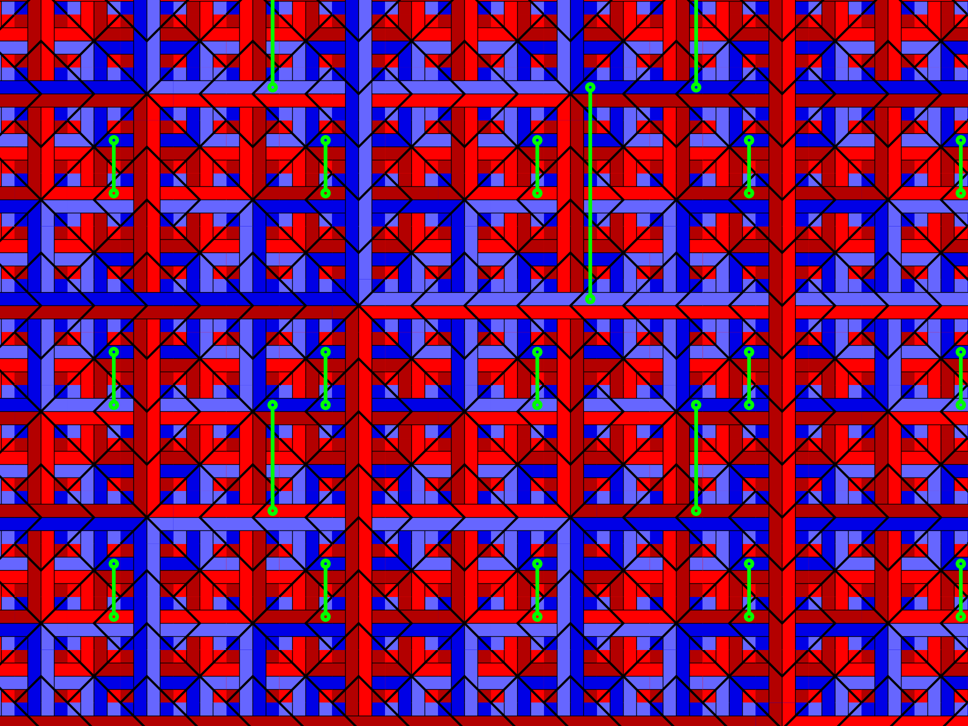
We combine the **strongly deterministic** version of 104 with the **encoding of a substitution** s' on the **quaternary tree**.

Again, the obtained tileset is not deterministic in **any** direction!

Problem. For the NE direction, we do not know how to “predict” the letter carried by cables of color ▖ in NE position on X tiles.

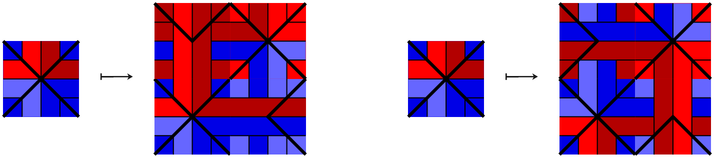
Idea. In our construction, hereditary information is **translated** in the SW direction. We could **set up some wires** to go & find it.

By Jove! We have already done something similar at the previous step.



104 2-way + Substitutions

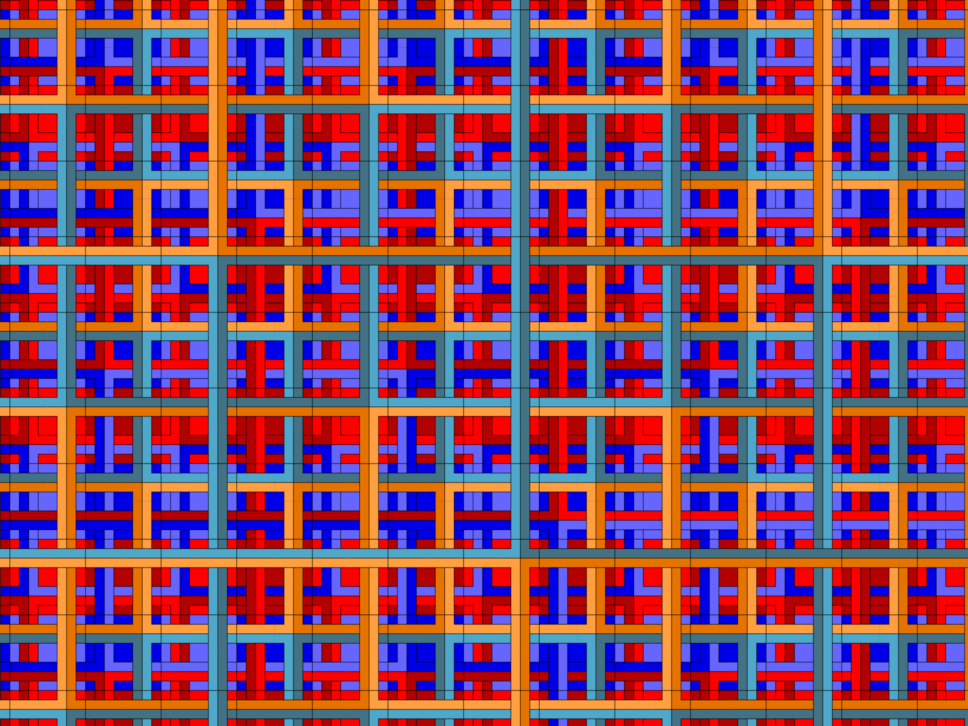
We consider the cartesian product of the two **104 1-way + substitution** tilesets obtained from the following two **symmetrical substitution schemes**.

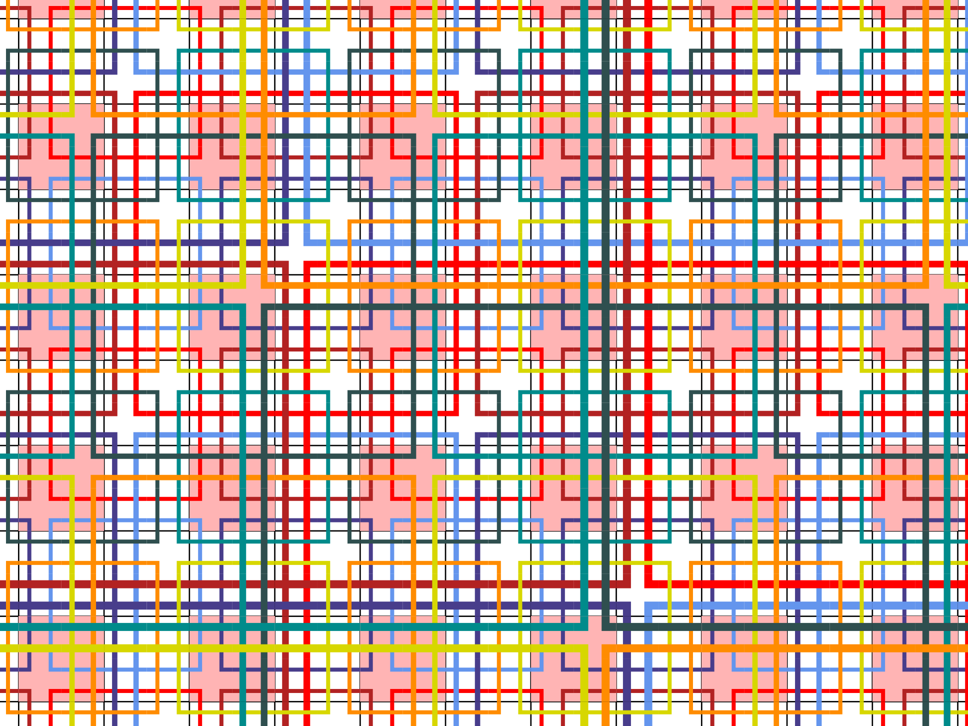


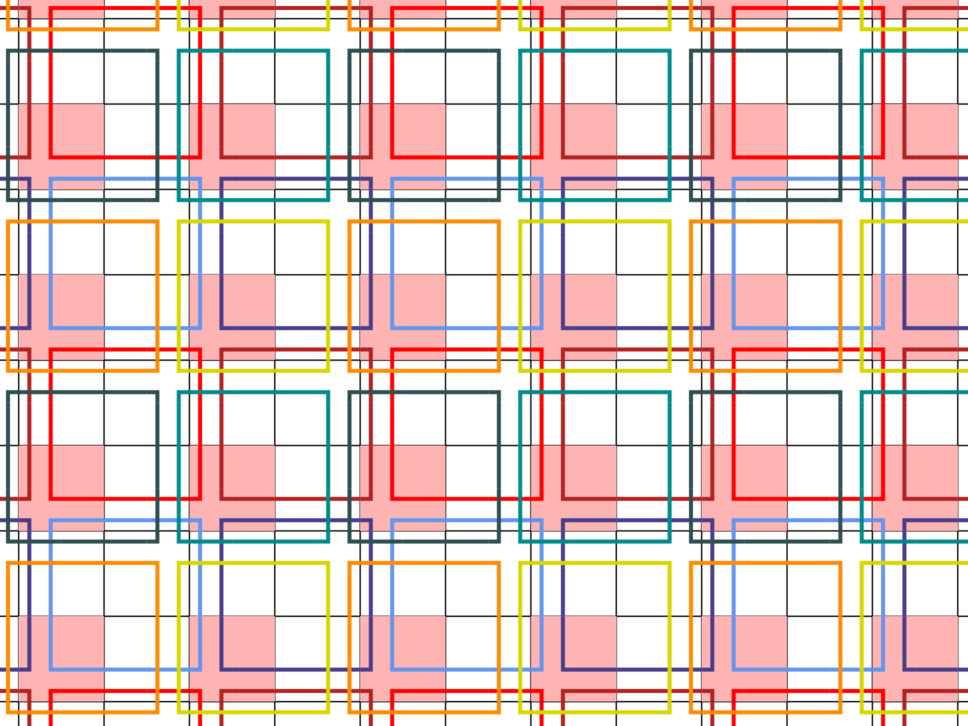
We synchronize the parity layer in order to **code the same coloring** on the level 0 of both components.

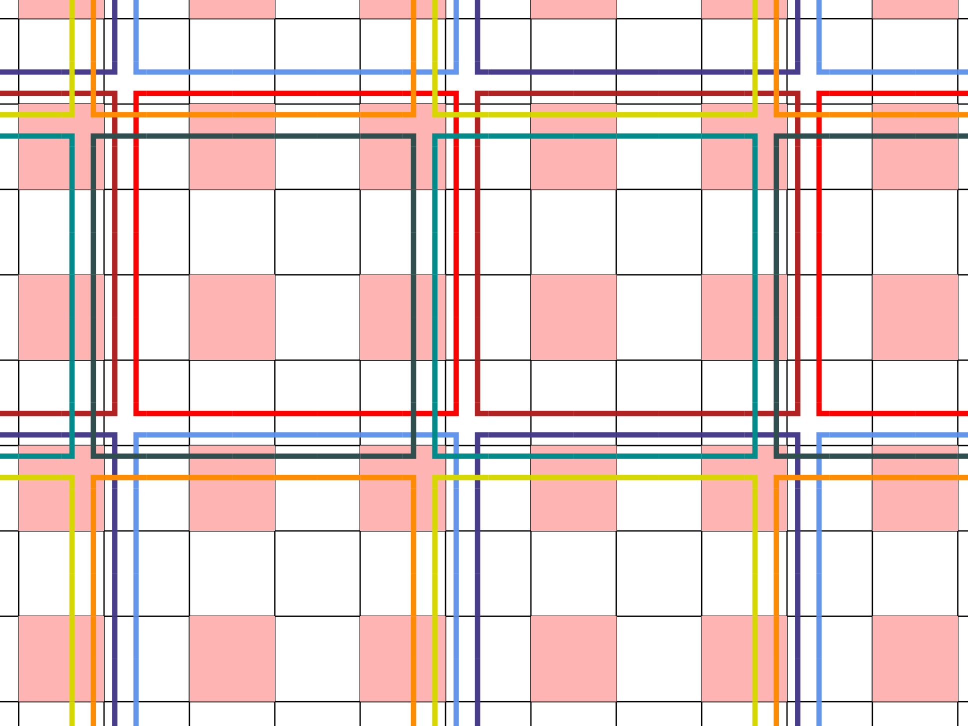
The component 1 (resp. 2) is NE-deterministic (resp. SW-deterministic).

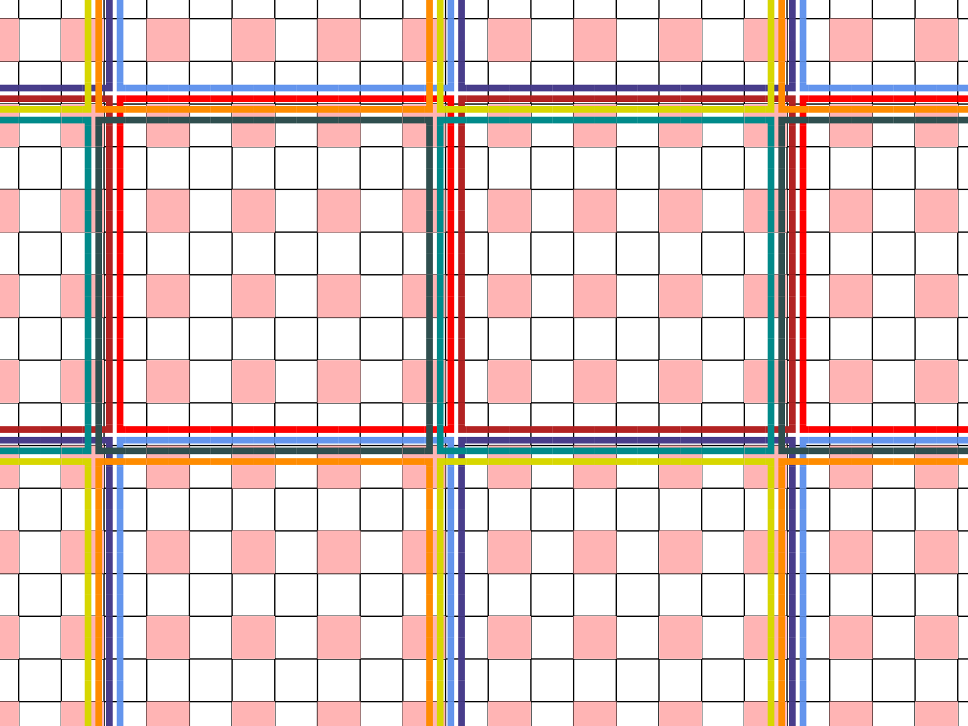
We need to synchronize the whole history on both components.





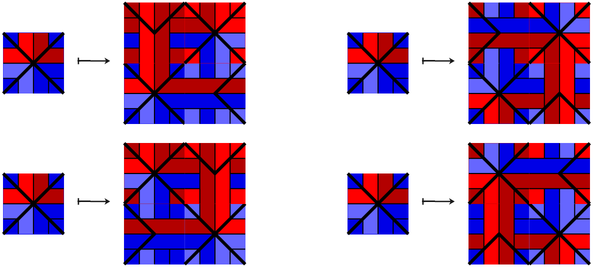






104 4-way + Substitutions

We consider this time the cartesian product of the four **104 1-way + substitution** tilesets obtained from the following four **symmetrical substitution schemes**.



Similar analysis and same solution (3×3 grouping) as before.