Linear Recurrence Sequence Automata

and the Addition of Abstract Numeration Systems

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We want to compute on a word

$$abbcbcabcaabbcaababbcbca \cdots$$

some combinatorial properties.

Maybe we want to know the possible lengths of cubes.

We want to compute* on a word

$$abbcbcabcaabbcaababbcbca\cdots$$

* Compute **first-order properties** using the **Büchi-Bruyère** framework. In practice, with tools like **Walnut**.

We want to compute on a **word***

$$abbcbcabcaabbcaababbcbca\cdots$$

* **fixpoint** $\tau^{\omega}(a)$ of $\tau : \begin{array}{l} a \to ab \\ b \to bc \\ c \to a \end{array}$

Pick **your** favorite fixpoint.

We want to compute on a word

$$abbcbcabcaabbcaababbcbca\cdots$$

fixpoint $\tau^\omega(a)$ of some τ.

We need a **numeration system** with **regular addition**!

Positional Numeration Systems

Here, the substitution $\tau : a \to ab,\ b \to bc,\ c \to a$ is **Pisot**.

$$M_\tau = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad P_\tau(X) = X^3 - 2X^2 + X - 1$$

We could use the canonical **Bertrand** PNS associated to its **Pisot root**.

$$\psi : a \to ab,\ b \to c,\ c \to ac \qquad U = (1, 2, 3, \cdots)$$ **[OEIS A005314]**

From **Bruyère-Hansel** and **Frougny-Solomyak**, we know that this PNS has **regular addition** and $\tau^\omega(a)$ is **U-automatic**.

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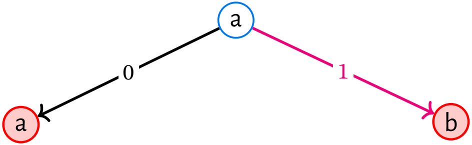
a

$$\tau : a \mapsto ab, \; b \mapsto bc, \; c \mapsto a$$



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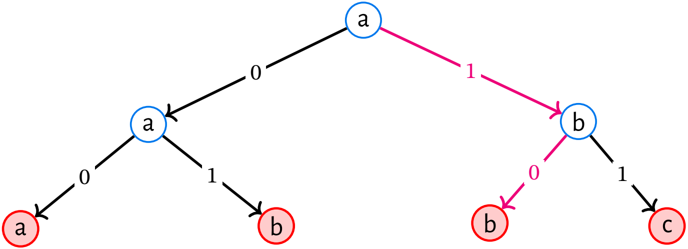

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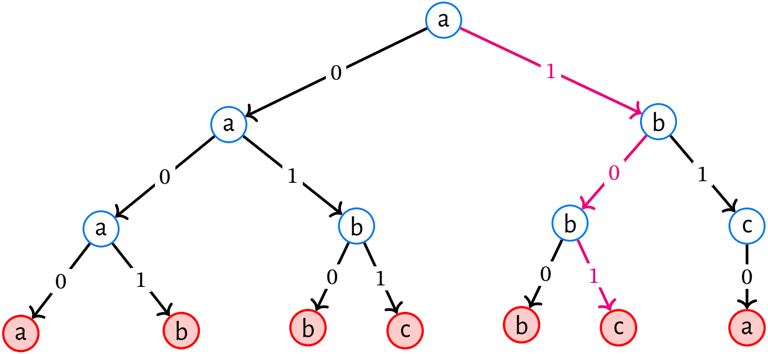

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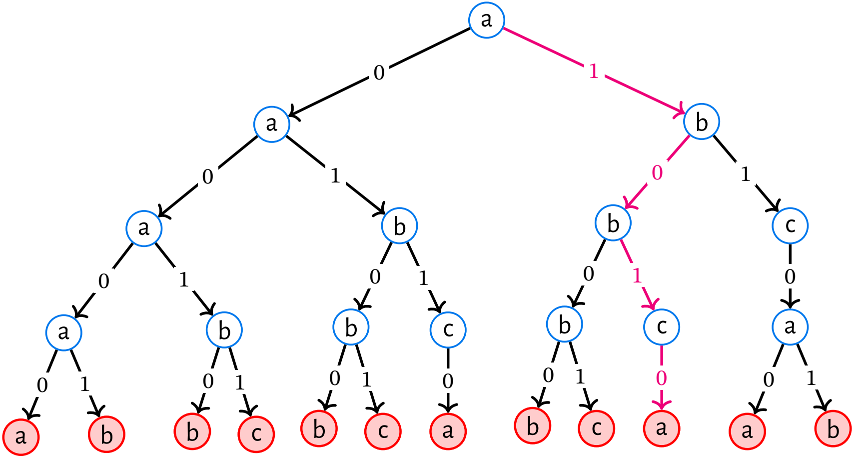

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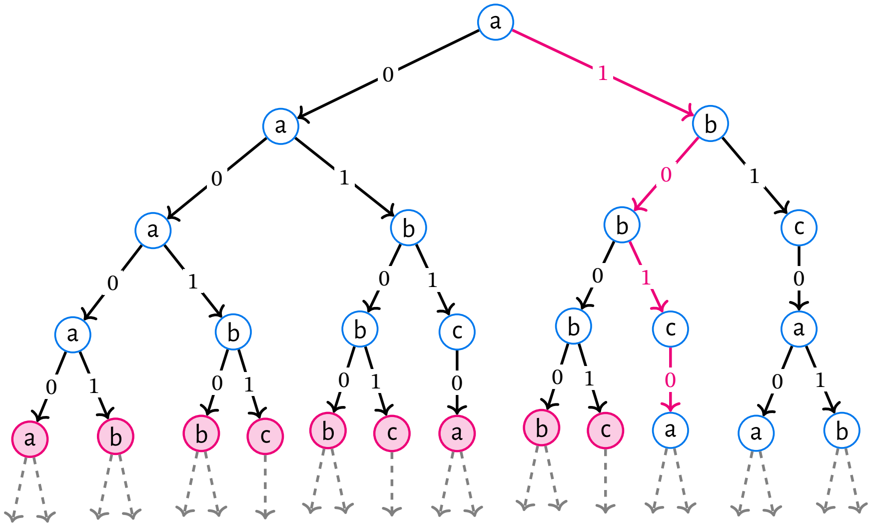


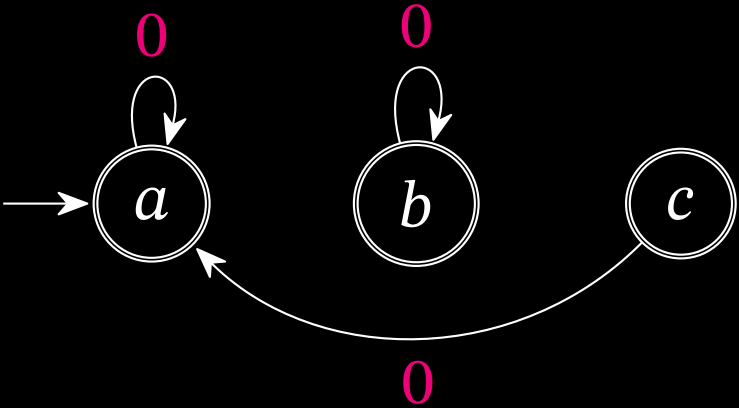
$$\tau : a \mapsto ab,\ b \mapsto bc,\ c \mapsto a$$

Dumont-Thomas numeration system

$$\tau : \begin{aligned} a &\to ab \\ b &\to bc \\ c &\to a \end{aligned}$$

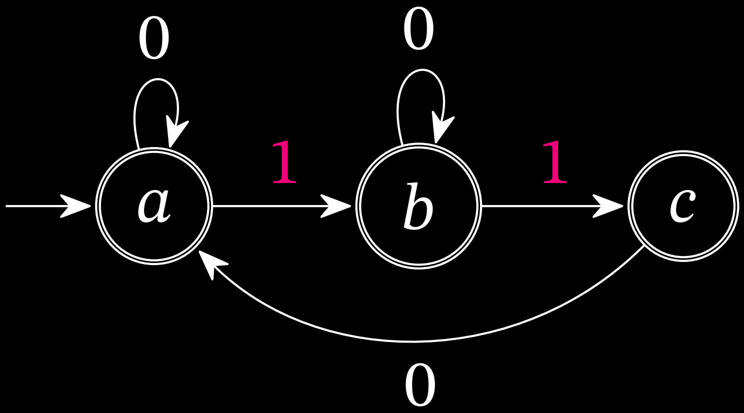
Dumont-Thomas numeration system

$$\tau : \begin{array}{l} a \to \overset{\color{magenta}{0}}{a}\overset{\color{magenta}{1}}{b} \\ b \to bc \\ c \to a \end{array}$$



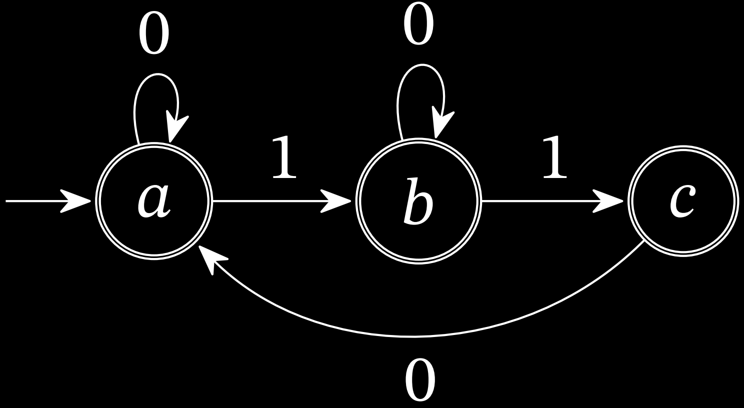
Dumont-Thomas numeration system

$$\tau : \begin{array}{l} \\ a \to \\ b \to \\ c \to \end{array} \begin{array}{l} 0\ 1 \\ ab \\ bc \\ a \end{array}$$



Dumont-Thomas numeration system

$$\begin{array}{rl} & \;\;0\;1 \\ \tau: & a \to ab \\ & b \to bc \\ & c \to a \end{array}$$



Dumont-Thomas numeration system

$$\begin{array}{rl} & \ \ {}^{0\,1} \\ \tau : a & \to ab \\ b & \to bc \\ c & \to a \end{array}$$

As a DFA this automaton gives an **Abstract Numeration System** for radix order.

As a DFAO the same automaton gives an **automatic representation** for the fixpoint.

We still need a **regular addition** to compute.

ANS and rational series

Theorem (Lecomte-Rigo) The **valuation series** of an **abstract numeration system** (ANS) is **$\mathbb{N}$-rational**.

$$\nu_{\mathcal{S}} : u \mapsto \begin{cases} \mathrm{val}_{\mathcal{S}}(u) & \text{if defined} \\ 0 & \text{if not} \end{cases}$$

Can we use this to compute the addition?

First we need to define **synchronized addition** on series.

Notation The canonical **isomorphism** between $\bigcup_{n\geqslant 0} (\Sigma^n \times \Gamma^n)$ and $(\Sigma \times \Gamma)^*$ is denoted as $\langle .,. \rangle$ for every alphabets Σ, Γ:

$$\langle u, v\rangle = (u_1, v_1)\cdots(u_m, v_m) \quad \forall u, v \in \Sigma^m \times \Gamma^m$$

Synchronized addition

Definition The **synchronized addition** $f \oplus g$ between two series $f : \Sigma^* \to \mathbb{K}$ and $g : \Gamma^* \to \mathbb{K}$ is defined as

$$\left(f \oplus g\right)(\langle u, v\rangle) = f(u) + g(v) \quad \forall (u, v) \in \Sigma^m \times \Gamma^m$$

The **support** $\mathrm{supp}(f)$ of a series f is the language $\Sigma^* \setminus f^{-1}(0)$.

Proposition An **ANS** $\mathcal{S}$ has **regular addition** if and only if $\mathrm{supp}(\nu_{\mathcal{S}} \oplus \nu_{\mathcal{S}} \oplus -\nu_{\mathcal{S}})$ is **regular**.

Unfortunately, $\nu_{\mathcal{S}} \oplus \nu_{\mathcal{S}} \oplus -\nu_{\mathcal{S}}$ is a **rational $\mathbb{Z}$-series** and testing if the support of a $\mathbb{Z}$-series is regular is generally **undecidable**…

$\mathcal{S}$-automatic

DFAO

$f(n) = \pi(\delta(q_0, \mathrm{repr}_{\mathcal{S}}(n)))$

$\mathcal{S}$-synchronized

$\mathcal{S}$-automatic

DFAO

$f(n) = \pi(\delta(q_0, \mathrm{repr}_{\mathcal{S}}(n)))$

DFA, closed by **FO formulaes**

$\langle \mathrm{repr}_{\mathcal{S}}(n), \mathrm{repr}_{\mathcal{S}}(f(n)) \rangle \in L(\mathcal{A})$

$\mathcal{S}$-regular

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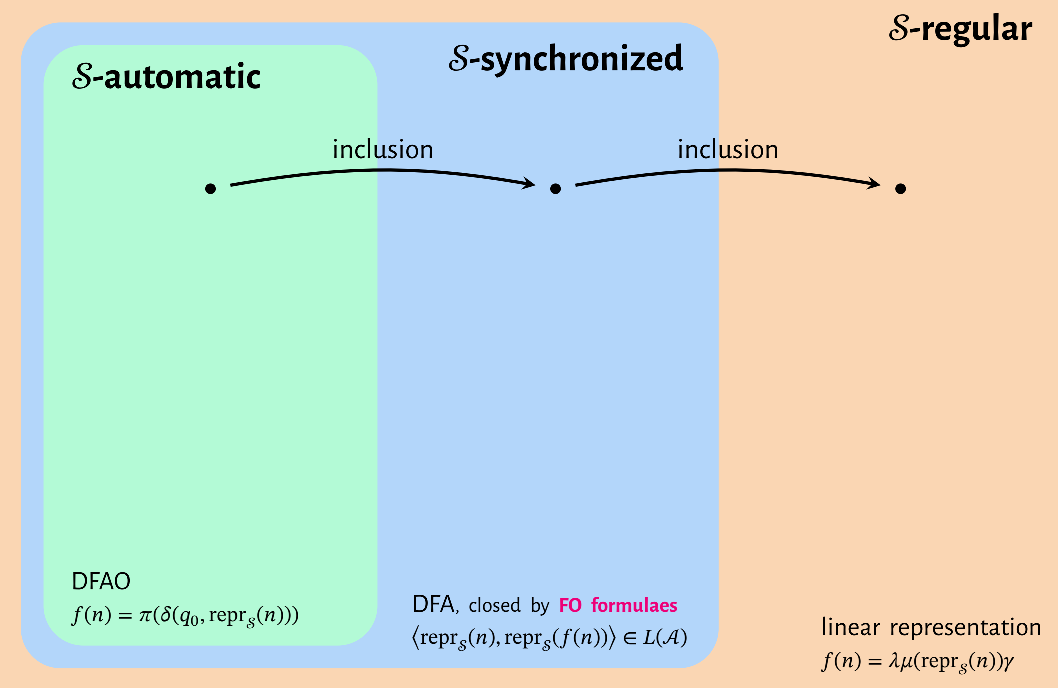
linear representation

$f(n) = \lambda\mu(\mathrm{repr}_{\mathcal{S}}(n))\gamma$


$\mathcal{S}$-regular
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inclusion
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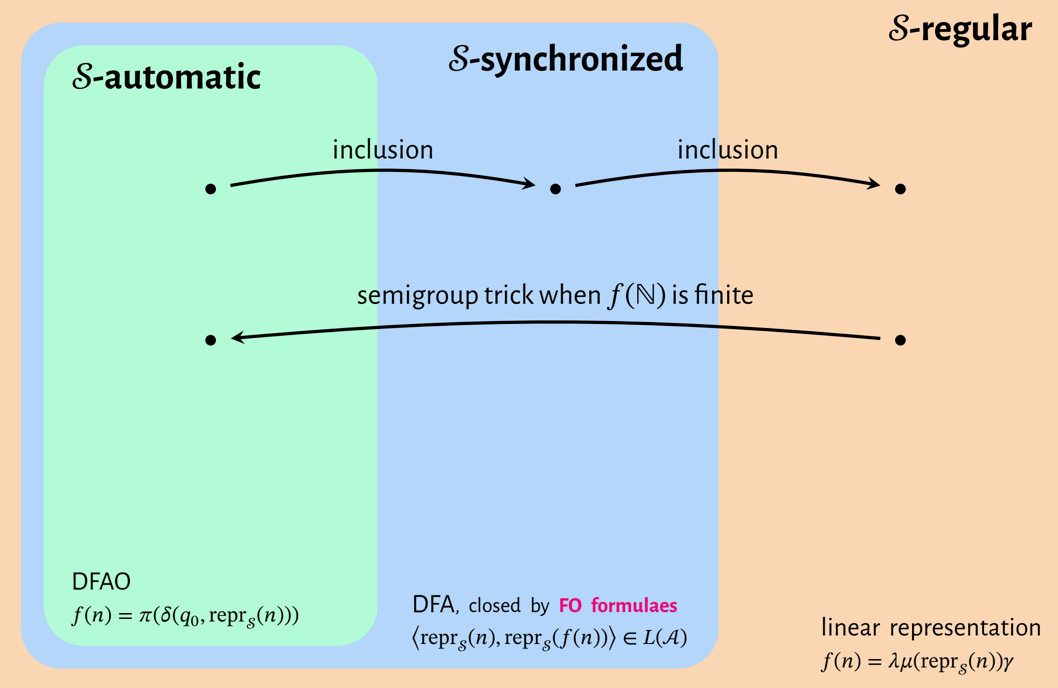
𝒮-regular
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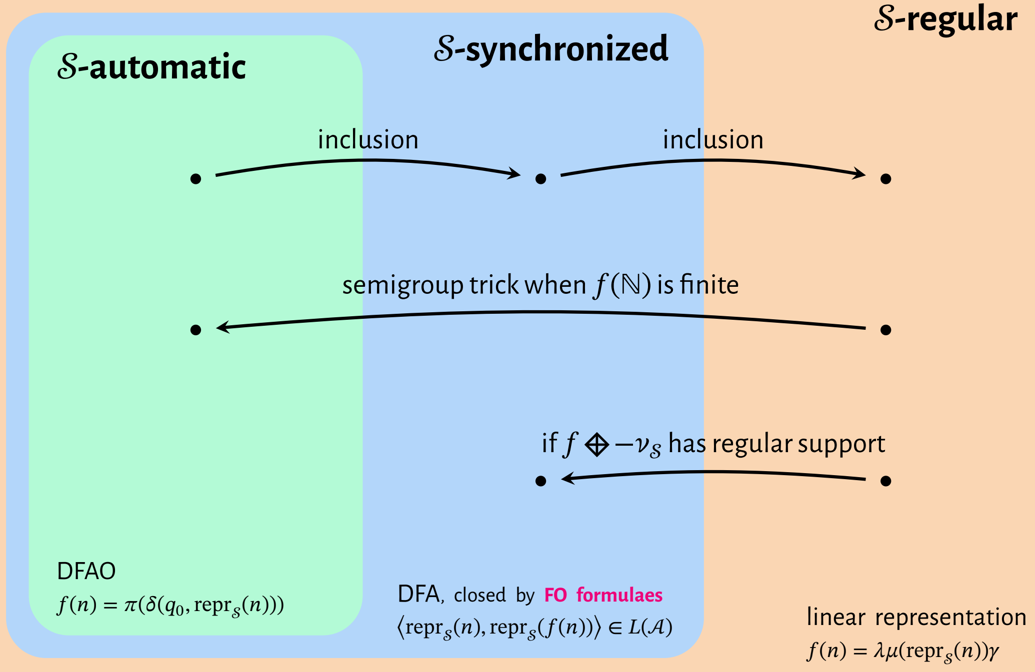
$\mathcal{S}$-regular
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semigroup trick when $f(\mathbb{N})$ is finite
if $f \oplus -\nu_{\mathcal{S}}$ has regular support
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In this talk

We introduce **Sequence Automata** as a generic way to manipulate similar constructions of $\mathbb{Z}$-series related to **linear recurrence sequences**.

We lift the **Bruyère-Hansel** and **Frougny-Solomyak** proofs to obtain **regular support** for the associated series under **Pisot** conditions.

It comes with an implementation and **practical tools** for Walnut users.

```
[1]: %DT dmua "a->ab, b->bc, c->a"

Studying substitution a->ab, b->bc, c->a
abbcbcabcaabbcaababbcbcaababbcabbcbcabcaababbcabbcbcaabbcbcabcaabbcaab...
Substitution polynomial: X^3-2X^2+X-1
Minimized substitution a->ab, b->bc, c->a
Substitution polynomial: X^3-2X^2+X-1
Addition polynomial: X^3-2X^2+X-1 (n=3)
θ=1.754877666246692
Substitution automaton: 3 states, 3 final states, 5 transitions.
>>> Writing /Users/nopid/Downloads/explosubst/Word Automata Library/Dmua.txt in format Walnut

>>> Writing /Users/nopid/Downloads/explosubst/Word Automata Library/DmuaParent.txt in format Walnut
```

Dumont-Thomas prefix factoring

Dumont and Thomas proved that **every prefix** p of $\tau^{\omega}(a)$ can be represented using a **unique sequence** $(p_i, a_i)_{i=0}^{k} \in (\Sigma^* \times \Sigma)^*$ as

$$p = \prod_{i=0}^{k} \tau^{k-i}(p_i) \quad \text{where } p_i a_i \text{ is a prefix of } \tau(a_{i-1}) \qquad \forall i \leqslant k$$

$$\text{(with } a_{-1} = a\text{)}$$

Thus every natural number can be decomposed as a sum of elements of **prefix length sequences**:

$$|p| = \sum_{i=0}^{k} \left|\tau^{k-i}(p_i)\right|$$

Linear Recurrence Sequences

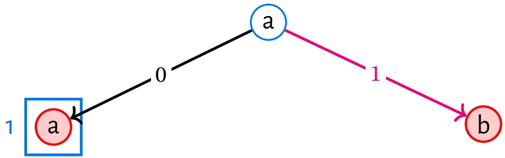
$$\tau : \begin{aligned} a &\to ab \\ b &\to bc \\ c &\to a \end{aligned} \qquad\qquad M_\tau = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

By **Cayley-Hamilton theorem**, all the sequences $(|\tau^n(u)|)$ are **linear recurrence sequences** (LRS) satisfying the recurrence relation given by the **monic characteristic polynomial** of M_τ, here $X^3 - 2X^2 + X - 1$.

$$\begin{array}{rcccccc} |\tau^n(a)| & = & 1, & 2, & 4, & 7, & \dots \\ |\tau^n(b)| & = & 1, & 2, & 3, & 5, & \dots \\ |\tau^n(c)| & = & 1, & 1, & 2, & 4, & \dots \end{array}$$

Prefix length sequences $(|\tau^n(p_i)|)$ are **linear combinations** of these sequences: they satisfy the same recurrence relation.

$$\tau : a \mapsto ab,\ b \mapsto bc,\ c \mapsto a$$



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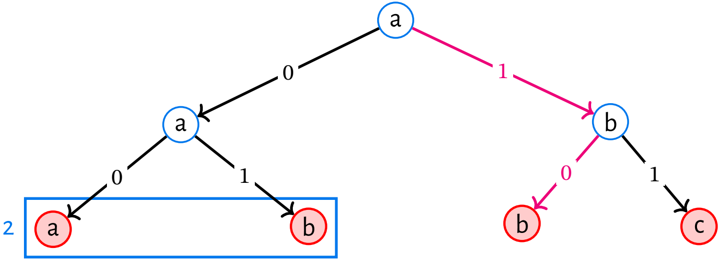
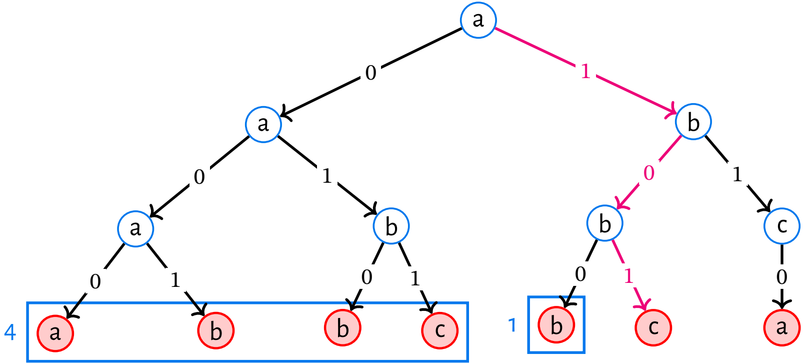


$$\tau \,:\, a \mapsto ab,\; b \mapsto bc,\; c \mapsto a$$



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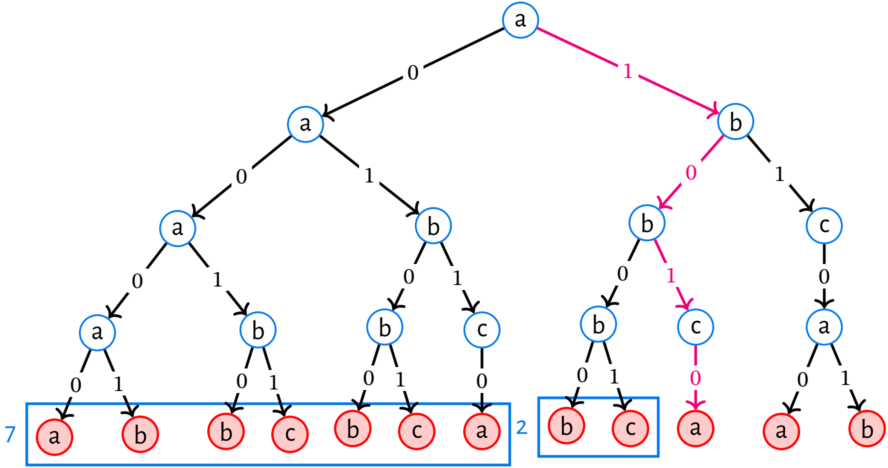
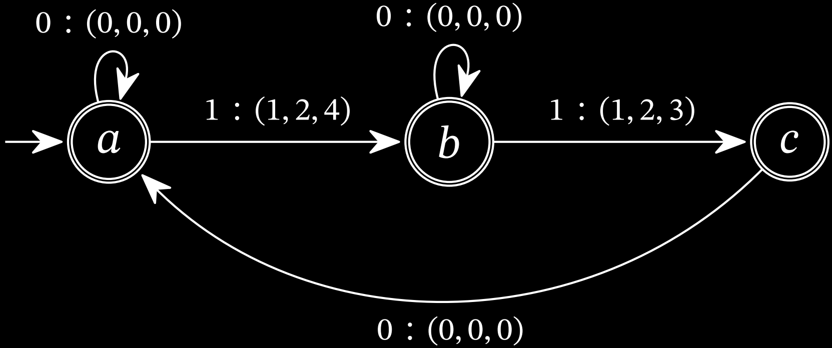


$$\tau : a \mapsto ab,\ b \mapsto bc,\ c \mapsto a$$

$X^3 - 2X^2 + X - 1$



Sequence Automaton

Sequence Automata

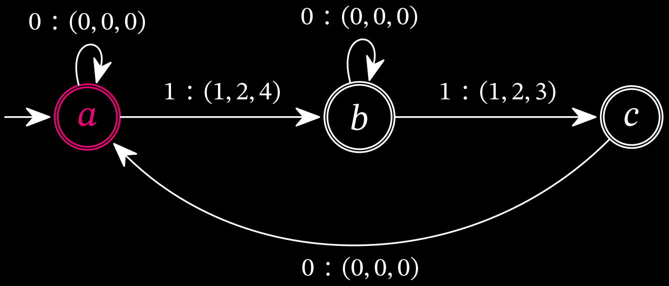
Definition A **sequence automaton** is a **partial DFA** $(Q, \Sigma, \delta, q_0, F)$ equipped with a **partial vector map** $\pi : Q \times \Sigma \to \mathbb{Z}^{\mathbb{N}}$ sharing a common domain with the **partial transition map** $\delta : Q \times \Sigma \to Q$.

Let (0) denote the zero sequence and σ denote the **shift map** on sequences: $\sigma(u)[n] = u[n+1]$ for all $u \in \mathbb{Z}^{\mathbb{N}}$ and $n \geqslant 0$.

The **transition map** and **vector map** are inductively extended from symbols to words as follows, for all $q \in Q$, $u \in \Sigma^*$ and $a \in \Sigma$:

$$\begin{aligned} \delta(q, \varepsilon) &= q & \pi(q, \varepsilon) &= (0) \\ \delta(q, ua) &= \delta(\delta(q, u), a) & \pi(q, ua) &= \sigma\pi(q, u) + \pi(\delta(q, u), a) \end{aligned}$$

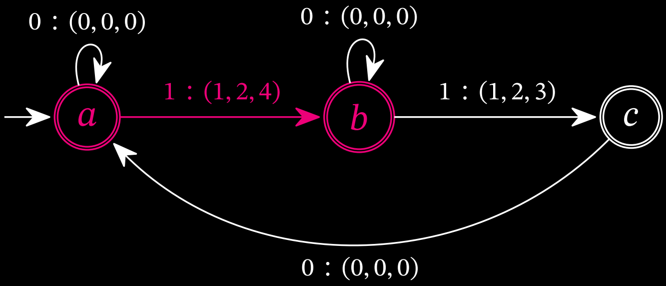
$X^3 - 2X^2 + X - 1$



$(0, 0, 0)$

$\pi(a, {}_{\blacktriangle}1010) = \ldots$

$$X^3 - 2X^2 + X - 1$$



$$\sigma(0,0,0) + (1,2,4) = (0+1, 0+2, 0+4)$$

$$\pi(a, 1010) = \ldots$$

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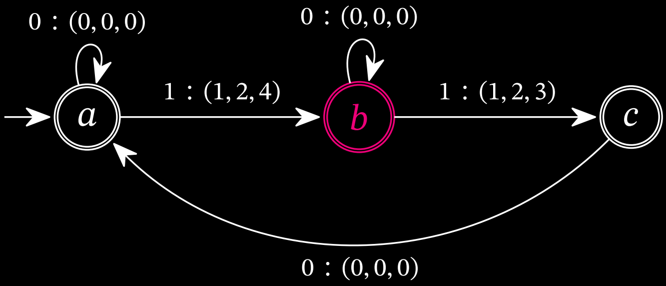
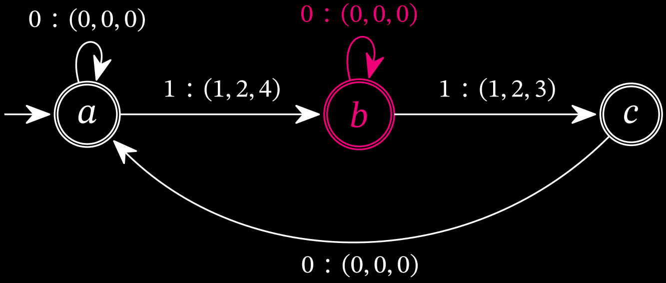


$$(1, 2, 4)$$

$$\pi(a, 1_{\blacktriangle}010) = \ldots$$

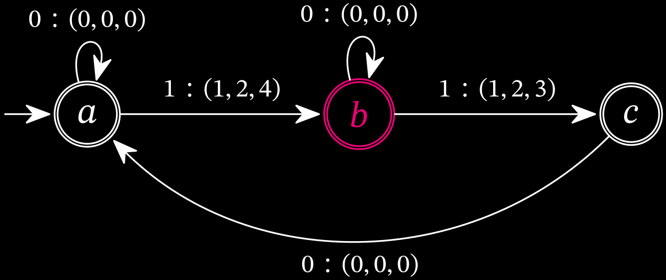
$$X^3 - 2X^2 + X - 1$$



$$\sigma(1,2,4) + (0,0,0) = (2+0, 4+0, 7+0)$$

$$\pi(a, 1010) = \ldots$$

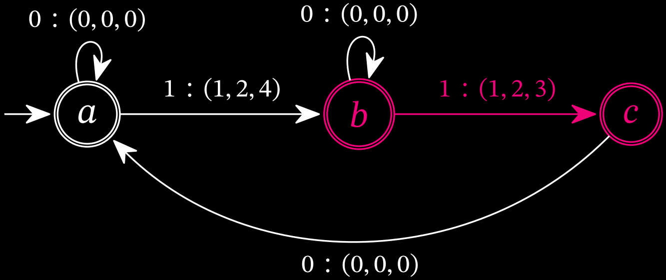
$X^3 - 2X^2 + X - 1$



$(2, 4, 7)$

$\pi(a, 10\,10) = \ldots$

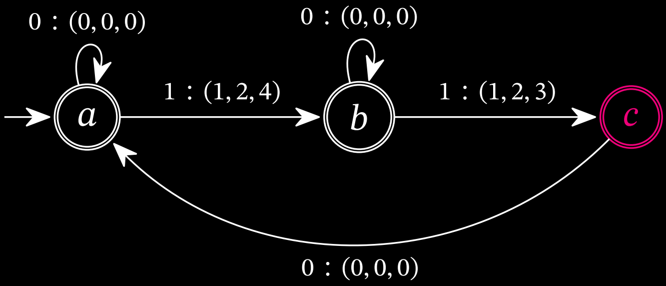
$$X^3 - 2X^2 + X - 1$$



$$\sigma(2,4,7) + (1,2,3) = (4+1, 7+2, 12+3)$$

$$\pi(a, 1010) = \dots$$

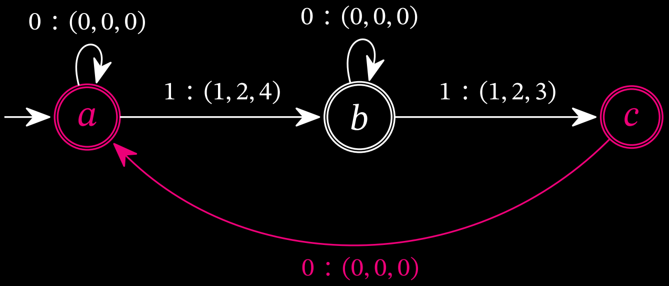
$$X^3 - 2X^2 + X - 1$$



$$(5, 9, 15)$$

$$\pi(a, 101\,0) = \dots$$

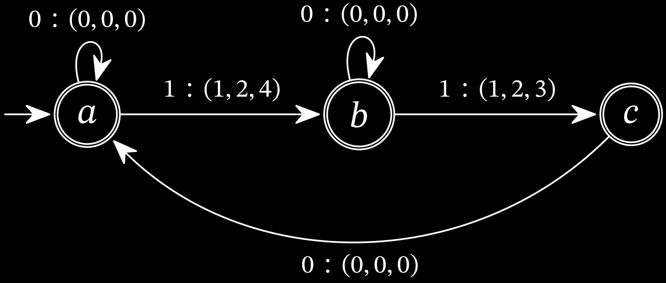
$$X^3 - 2X^2 + X - 1$$



$$\sigma(5, 9, 15) + (0, 0, 0) = (9 + 0, 15 + 0, 26 + 0)$$

$$\pi(a, 1010) = \ldots$$

$$X^3 - 2X^2 + X - 1$$



$$(9, 15, 26)$$

$$\pi(a, 1010) = (9, 15, 26)$$

Series and Linear Combinations

Definition The **series** $s_{\mathcal{A}}$ of a **sequence automaton** $\mathcal{A}$ maps every word $u \in \Sigma^*$ to the first element of its vector $\pi(q_0, u)[0]$.

Using **product DFA** and linear combination of vector maps, one can combine **sequence automata** to produce **sequence automata**:

- $\mathcal{A} + \mathcal{B}$ of series $s_{\mathcal{A}} \oplus s_{\mathcal{B}}$;
- $\alpha\mathcal{A}$ of series $\alpha s_{\mathcal{A}}$ for every scalar α.

Proposition The **valuation series** $\nu_{\mathcal{S}}$ of an ANS is the **series of a sequence automaton**. As a consequence $\nu_{\mathcal{S}} \oplus \nu_{\mathcal{S}} \oplus -\nu_{\mathcal{S}}$ is also the **series of a sequence automaton**.

Linear Recurrence Sequence Automata

Definition When all sequences in the **vector map** of a sequence automaton are **LRS**, the **recurrence polynomial** of the automaton is the **minimal polynomial** of all these sequences.

Remark When all LRS come from **incidence matrices**, the **recurrence polynomial** is **monic**.

Theorem The **support** of the series of a sequence automaton with **(ultimately) Pisot recurrence polynomial** is **regular**.

The proof lift bounds and techniques from **Bruyère-Hansel** and **Frougny-Solomyak** into sequence automata series.

Case 1. starting from a Pisot substitution

When starting from a (ultimately) **Pisot substitution**, **sequence automata** let us compute numerous DFA and DFAO:

- **addition DFA** for **Dumont-Thomas numeration systems** of fixpoints;

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- **abelian complexity DFAO** for Pisot substitutions [arXiV 2504.13584];

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- **abelian complexity DFAO** for Pisot substitutions **[arXiV 2504.13584]**;
- **addition DFA** for **Abstract Numeration Systems** built as morphic images of fixpoints;
- …

Case 2. the Non Pisot substitutions

In general, we can still **construct the sequence automata**.

On a **case-by-case** basis for each application:

- sometimes the **recurrence polynomial** of the automaton is **Pisot**;

$$a \to aba,\ b \to bab$$

- sometimes it is not but a **manual bound** let us derive a valid DFA;

$$a \to aba,\ b \to b$$

- sometimes a **pumping argument** let us prove there is no DFA.

$$a \to ab,\ b \to bc,\ c \to c$$

In Practice

We maintain a collection of **Python tools** to interact with **Walnut**:

- **licofage** — sequence automata library generating Walnut files:
  - basic usage: give a substitution, get the addition DFA and word DFAO;
  - advanced usage: generic sequence automata manipulation in Python;
- **ratser** — missing library for rational power series and semigroup trick;
- **walnut-kernel** — a Jupyter kernel to combine Walnut and all this (hello Walnut Notebooks!);
- **jupywalnut** — a Docker container for a simplified use of everything;
- **walnut-cli** — Walnut + Java runtime packaged together.

Everything is available on **GitHub**, install Python libs from **Pypi**.

```
$ pip install walnut-cli
$ walnut
```