Reducing Taylor expansion of MELL proof nets.

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Quantitative semantics (of Λ)

- Girard 88 : Normal functors
  ↳ Analyticity is the key property of the interpretation of λ-terms
  ↳ Terms denote power series

- Properties of programs
  degree of a monomial
  $\cong$ number of times a function uses its argument

Various models were proposed, allowing characterization and representation of quantitative properties, such as:

- execution time (De Carvalho, 2009)
- probability of reaching a value (Danos & Ehrhard 2011)

$\mathrm{Rel}^!$ is a degenerate, boolean valued instance of these models

Ehrhard's models:

- Köthe sequence spaces (2002)
- Finiteness spaces (2005)

Analytic maps interpreting λ-terms

↳ Analogue of Taylor expansion formula

- for λ-terms
- for linear logic proofs.

Emergence of syntax

Ehrhard & Regnier :
- Differential λ-calculus (2003)
- Differential linear logic (2005)

⇒ Syntactic version of Taylor expansion

To a λ-term / LL-proof, we associate an infinite linear combination of approximants

The dynamics (β-red / cut elim.) are dictated by the identities of quantitative semantics.

Difficulty. (in Λ)

resource λ-calculus | linear fragment of differential Λ

$$\tau(MN) = \sum_{n\in\mathbb{N}} \frac{1}{n!} \langle \tau(M) \rangle \, \tau(N)^n$$

- In order to simulate β-reduction, we define a parallel reduction $\rightrightarrows$

- How can we prevent the appearance of infinite coefficients during reduction?

Counter example :

- Let, $\forall n \in \mathbb{N}$, $M_n = \underbrace{\langle \lambda x\, x \rangle [\langle \lambda x\, x \rangle [\langle \lambda x\, x \rangle [\cdots \langle \lambda x\, x \rangle [y] \cdots]}_{n \text{ applications}}$

- For all n, $M_n \rightrightarrows y$

- Then, $\sum_{n \in \mathbb{N}} M_n \Longrightarrow \boxed{\sum_{n \in \mathbb{N}} y}$

$\downarrow$

It is not always defined !

Results : coefficients remain finite under reduction

- Ordinary λ-terms : Ehrhard & Regnier (2008)
- Non uniform, typed terms : Ehrhard (2010)
- Non uniform, strongly normalizable terms : Pagani, Tasson, Vaux (2016)
- Algebraic, weakly normalizable terms : Vaux (2017)

$$\tau(M) \ni t,\ t',\ t'',\ \ldots \qquad \mathrm{NF}(\tau(M)) \ni s$$

finite

→ has a finite coefficient in $\mathrm{NF}(\tau(M))$

... Proof nets ...

Target of Taylor expansion: resource nets

↳ linear fragment of Differential nets

$DLL_0 =$

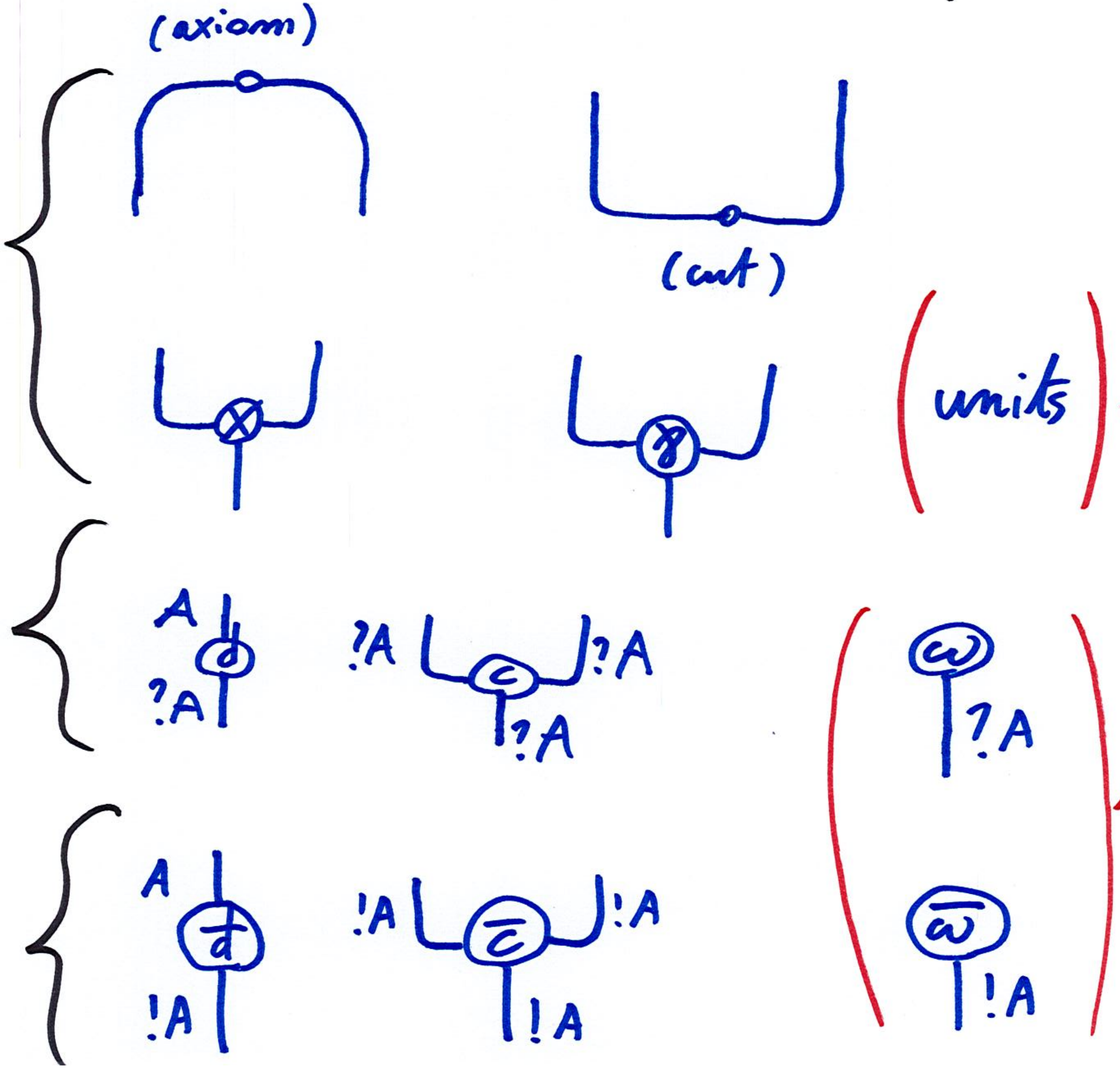
MLL

+

MELL \ promotion

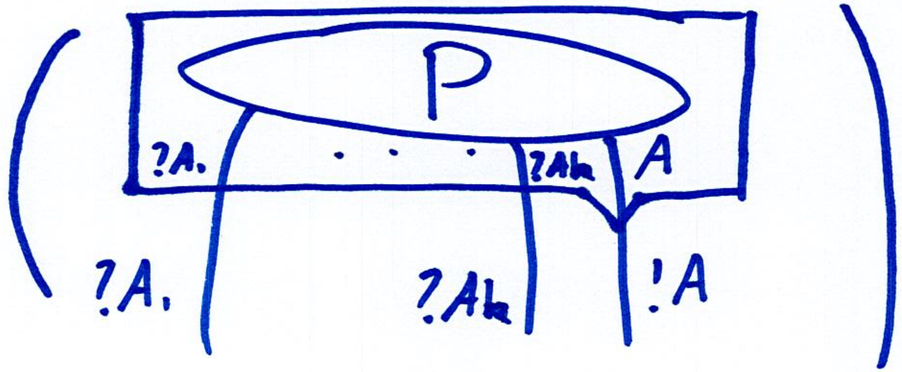
+

Differential constructs

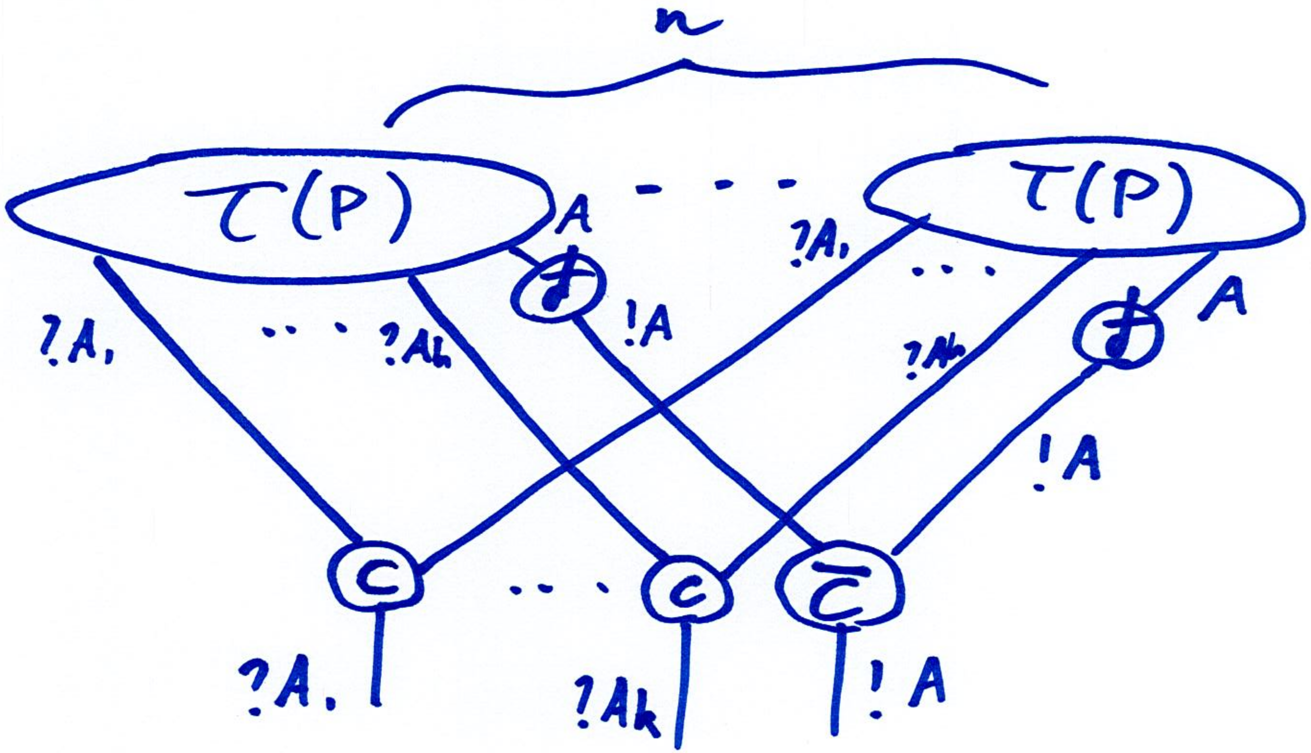


If we got time

Taylor expansion generates infinite combinations of resource nets, approximants of the exponential boxes :

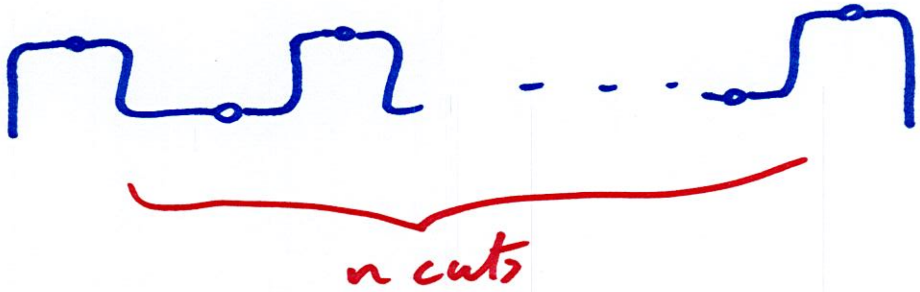


$$\mathcal{T}(\ \cdot\ ) = \sum_{n \in \mathbb{N}^{(*)}} \frac{1}{n!}$$



- In order to simulate cut elimination, we define a parallel reduction $\Rrightarrow$

- Infinite coefficients could also appear:

  - let, $\forall n \in \mathbb{N}$, $R_n =$ 

  - For all n, $R_n \rightrightarrows$ 

  - Then, $\sum_{n \in \mathbb{N}} R_n \Rrightarrow \sum_{n \in \mathbb{N}}$ 

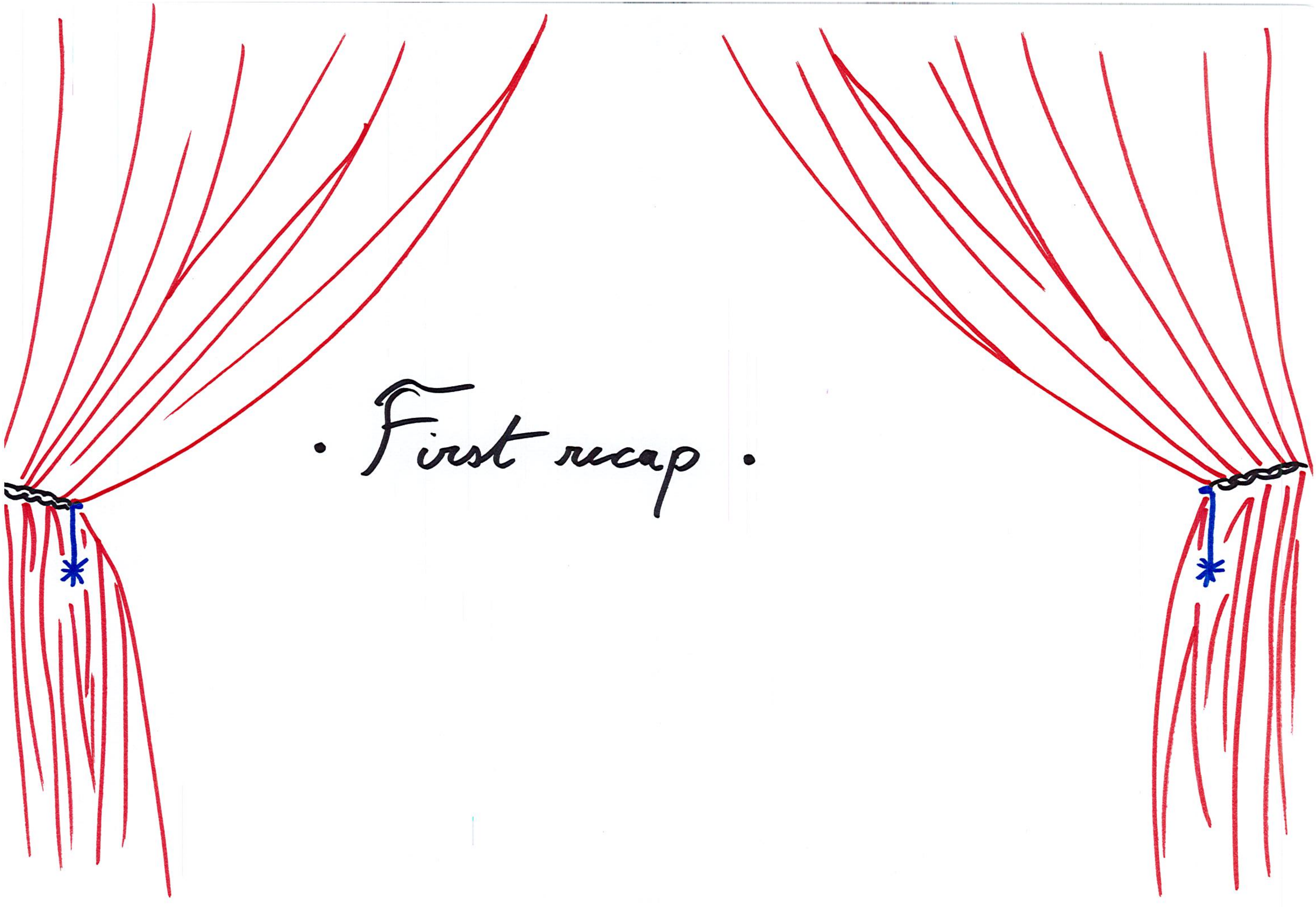
    $\rightarrow$ Not always defined

Again, this is an issue...

In order to show that it does not happen, we adapt a technique of Vaux (2017), used for algebraic λ-calculus.

That demands a close investigation of resource nets dynamics,

but before that...


· First recap ·

Syntactic Taylor expansion

↳ Infinite combination of proofs/terms

↳ We reduce in all the arguments of the sum in one step ($\Rrightarrow$)

↳ Infinite coefficients might appear

Wanted result: They don't

How to be sure?

$$\sum_{i \in I} a_i \cdot t_i \Longrightarrow \sum_{j \in J} b_j \cdot s_j$$

- Take some s_j s.t. $b_j \neq 0$
- Consider $\{ t_i \mid i \in I,\ a_i \neq 0,\ t_i \rightrightarrows s_j \}$
- Establish this set is finite.
- Our method: bound the size of its elements.

Vaux 2017 : $M \longrightarrow_\beta N$

$\text{appdepth}(M)$

let $s \in \mathcal{T}(N)$, $\forall t \in \mathcal{T}(M)$ $\text{appdepth}(t) \leq \delta$

Lemma : if $t \rightrightarrows s$, $\# t \leq \varphi(\text{appdepth}(t), \# s)$

Theorem : $\{\# t \mid t \in \mathcal{T}(M), t \rightrightarrows s\}$ is bounded by $\varphi(\delta, \# s)$

hence $\{t \in \mathcal{T}(M) \mid t \rightrightarrows s\}$ is finite

Adapting the method to proof nets

Difficulty: What is the applicative depth of a resource net ?

↳ We need a measure common to all $\rho \in \mathcal{T}(P)$, and convenient to bound the loss of size under $\Rightarrow$

Key idea of our contribution

The measure is :

$$\boxed{cc(P) \triangleq \text{Max. number of cuts crossed by any switching path of } P.}$$

• Second recap •

We want to show that, for $P \in MELL$, and $q \in DLL_0$:

$$\uparrow\uparrow q \cap \tau(P) = \{ p \in \tau(P) \mid p \rightrightarrows q \} \text{ is finite}$$

We prove that:

$$\boxed{1.\ \forall p \in \tau(P),\ cc(p) \leq \psi(P)}$$

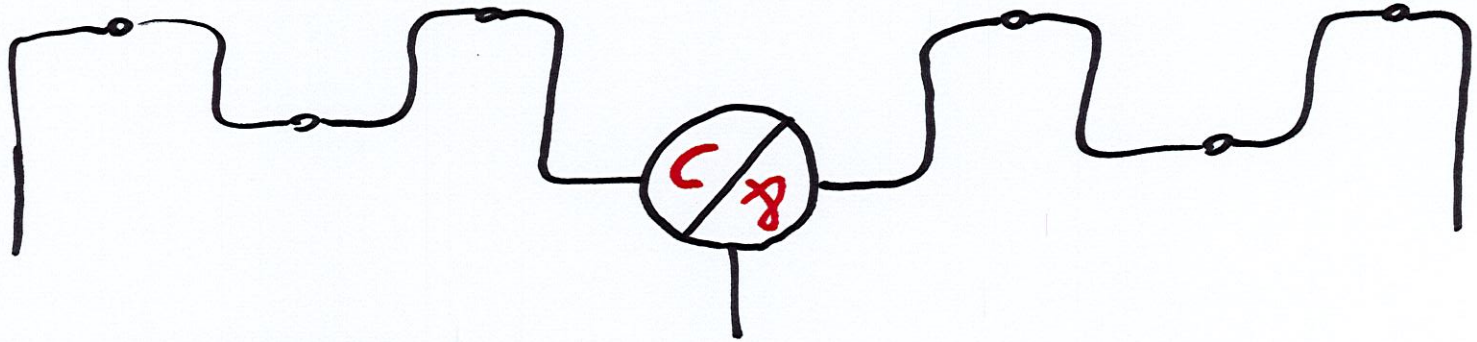
$$\boxed{2.\ \text{If } p \rightrightarrows q,\ \#p \leq \varphi(cc(p), \#q)}$$

And we can conclude

$$\boxed{1.\ \forall p \in \tau(P) \quad cc(p) \leq \psi(P)}$$

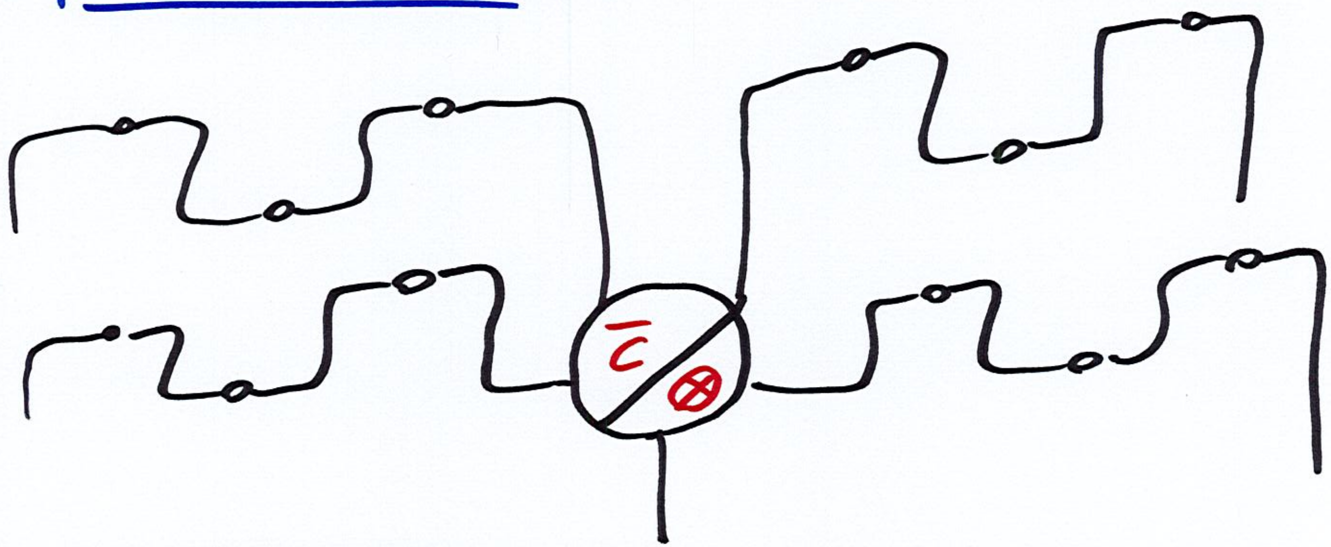
Examples

p =

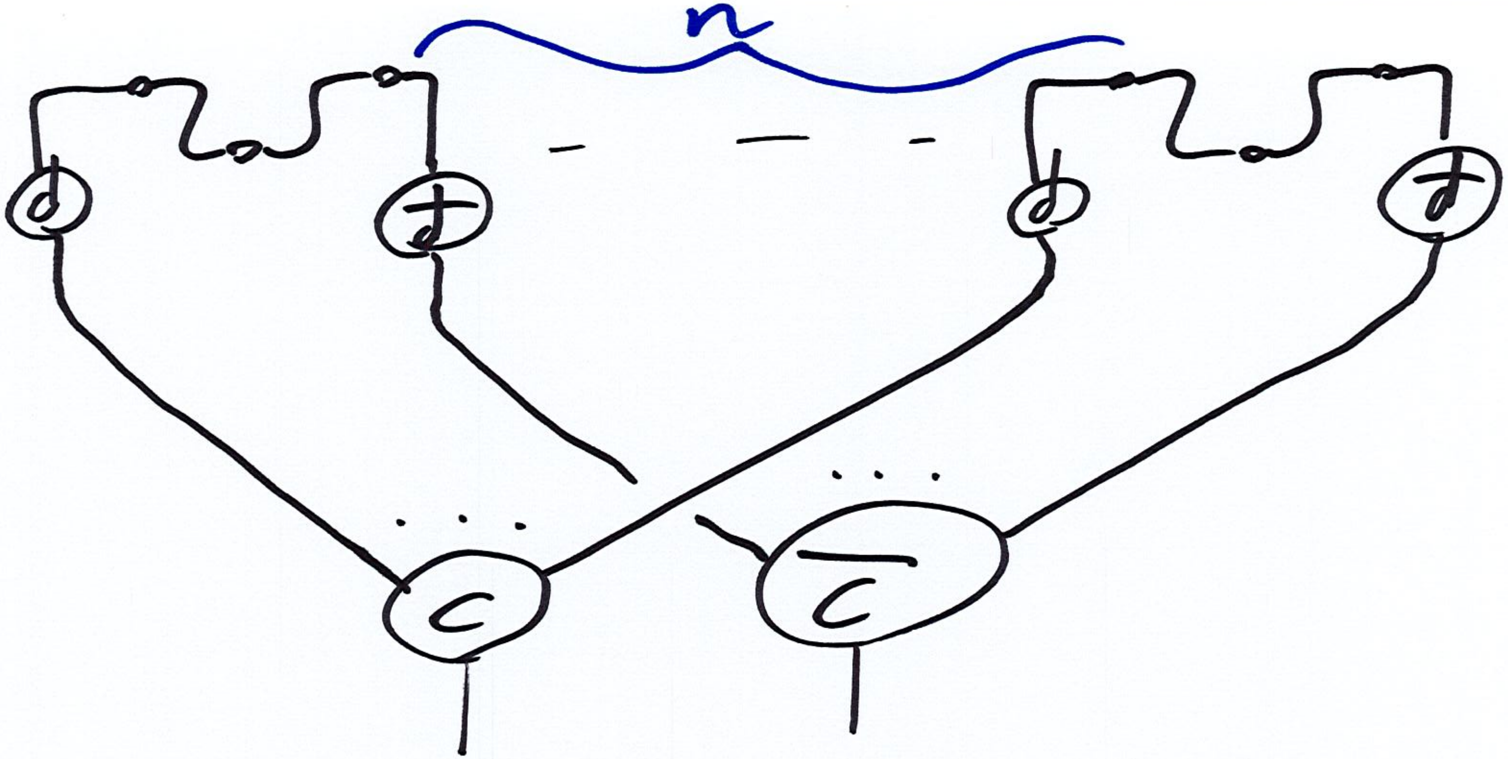


$\boxed{cc(p) = 1}$

q =



$\boxed{cc(q) = 2}$

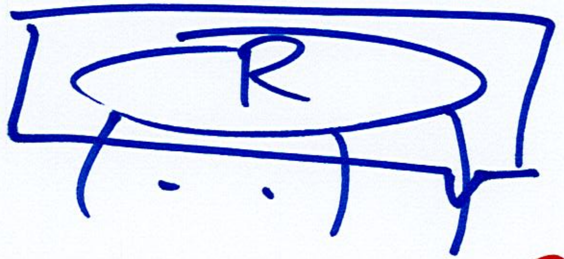
$p_n =$ 

For all $n \in \mathbb{N}$, $\alpha(p_n) = 2$

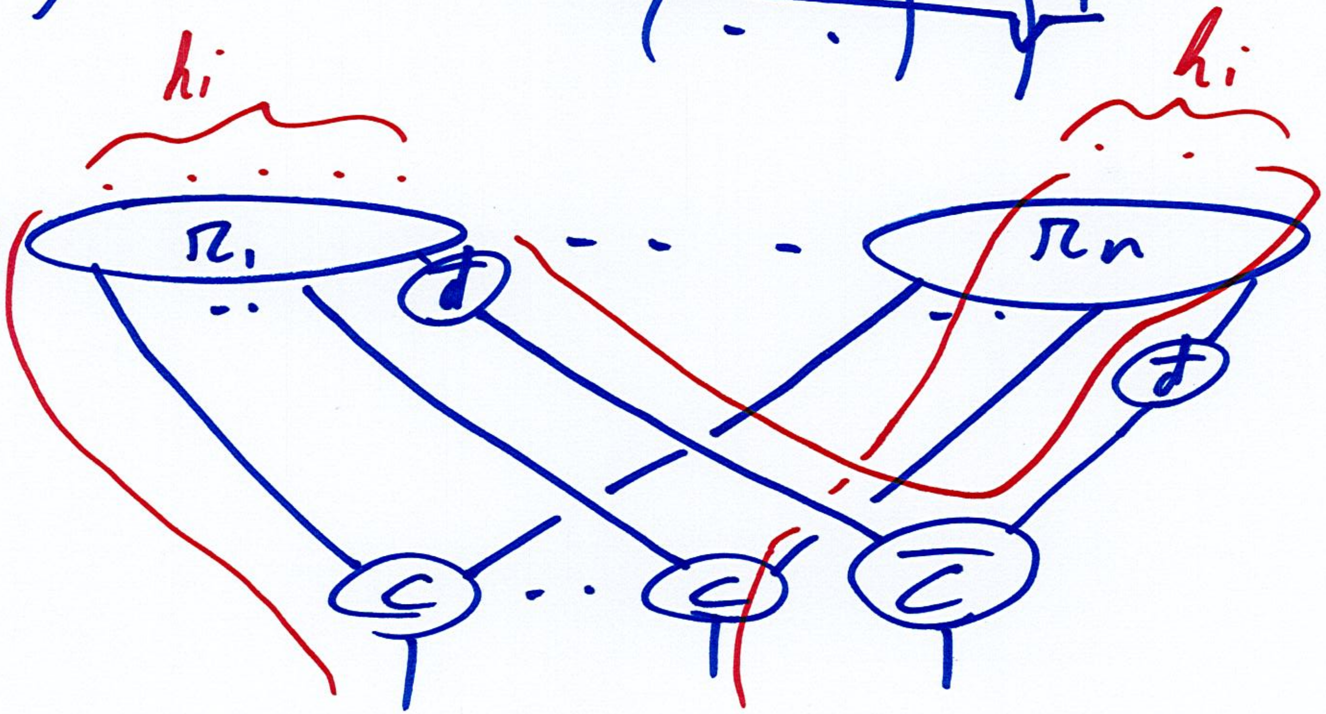
Lemma 1 $P \in MELL$, $p \in \mathcal{T}(P)$

$$\boxed{cc(p) \le 2^{\#P}}$$

Proof: $P =$



$p =$

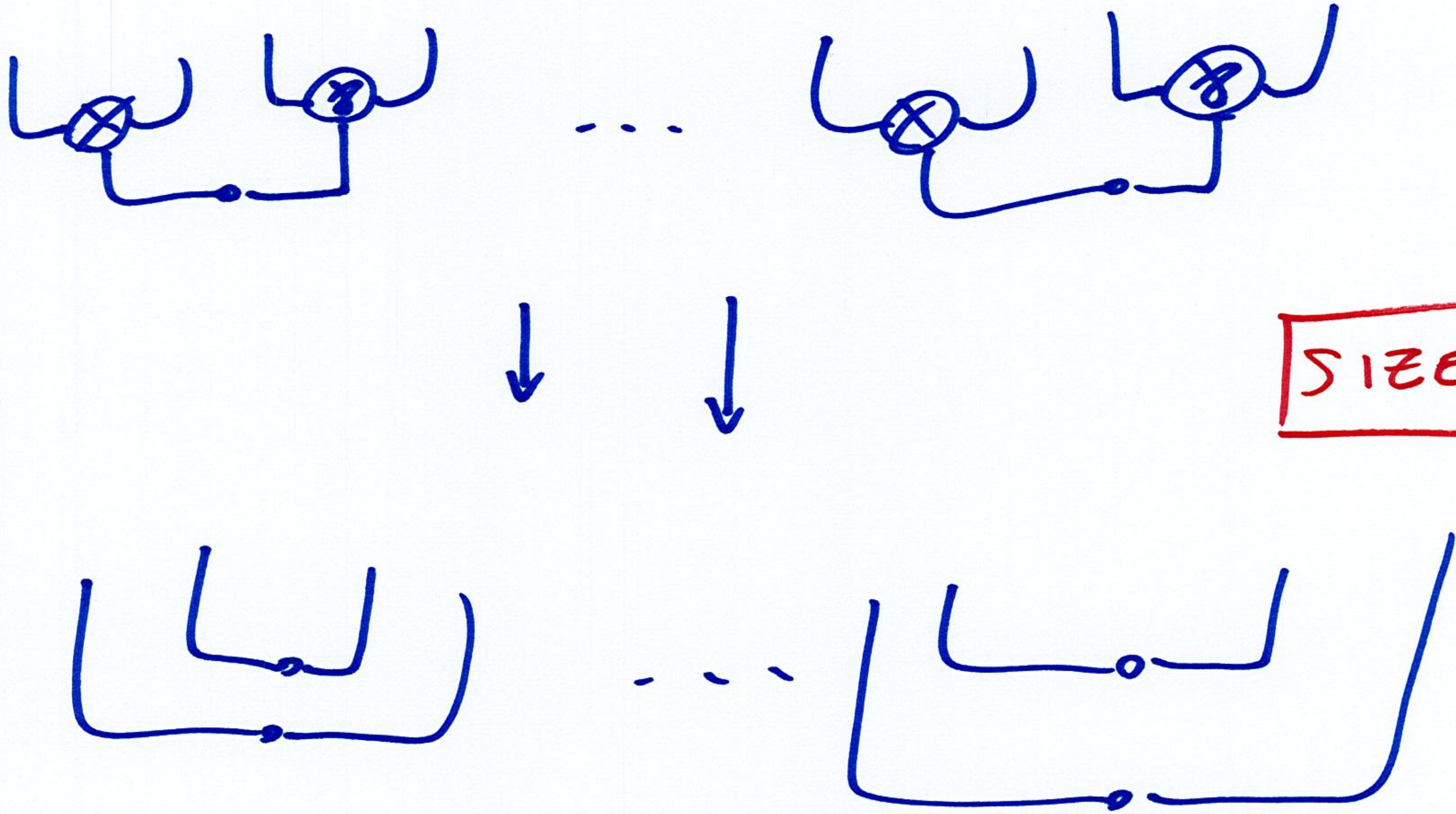


Induction on exponential depth

$h_i := \pi_1 \in \mathcal{T}(R) \Longrightarrow cc(\pi_1) \le 2^{\#R}$

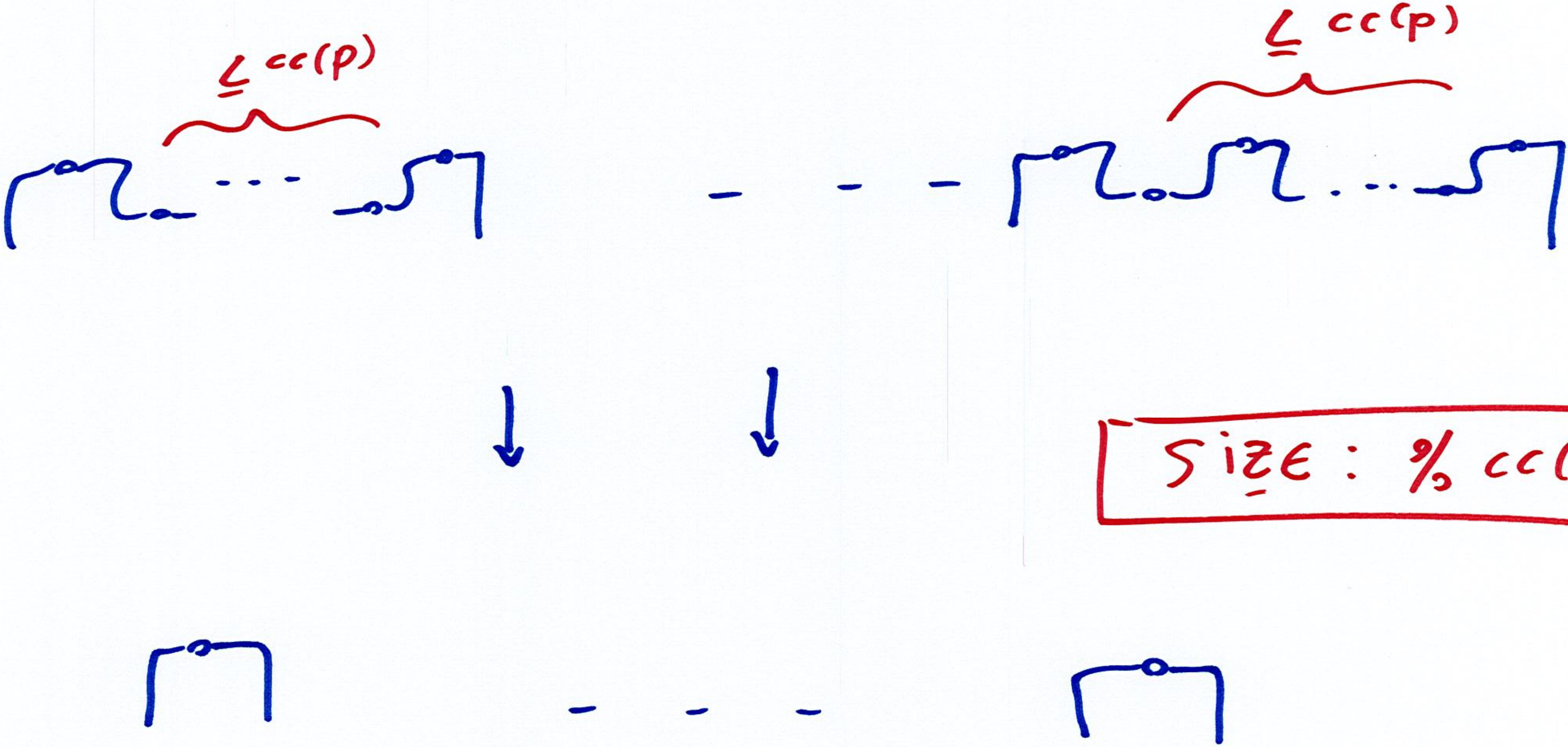
$$\boxed{2.\ p \rightrightarrows q \Longrightarrow \#p \leq \varphi(cc(p), \#q)}$$

First, observe that :



$\boxed{\text{SIZE : } \%2}$

And :



$\boxed{\text{SIZE : \% cc(p)}}$

Decomposition of the reduction of $p \rightrightarrows q$.

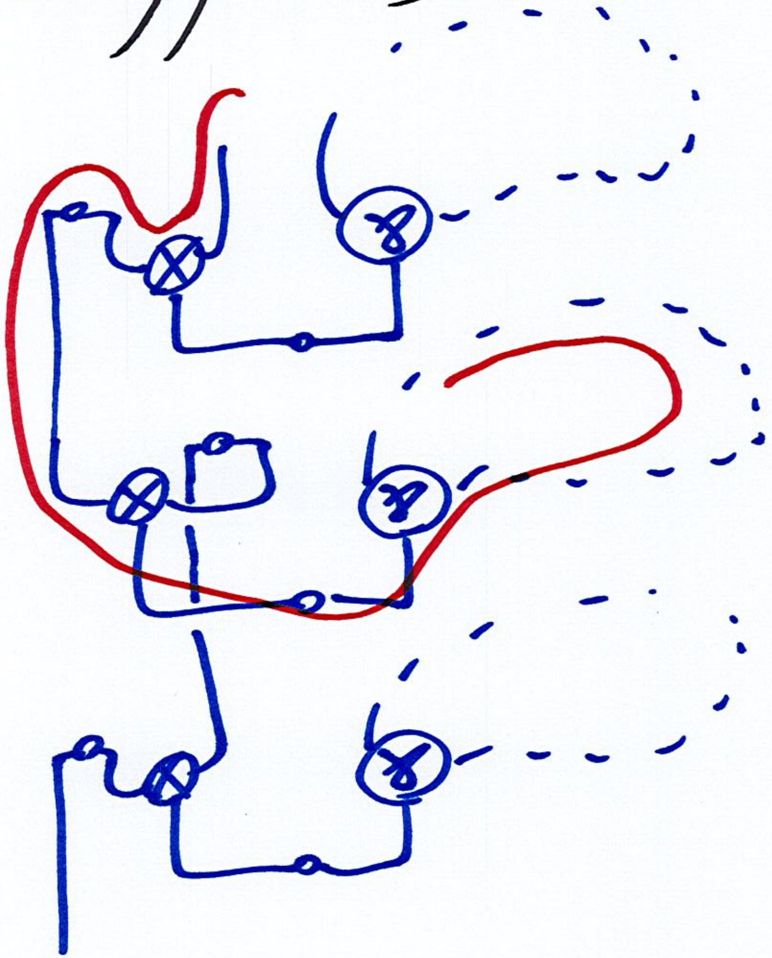
- $p \underset{\substack{\text{Multiplicative} \\ \text{phase}}}{\rightrightarrows} q^- \underset{\substack{\text{Axiomatic} \\ \text{phase}}}{\rightrightarrows} q$

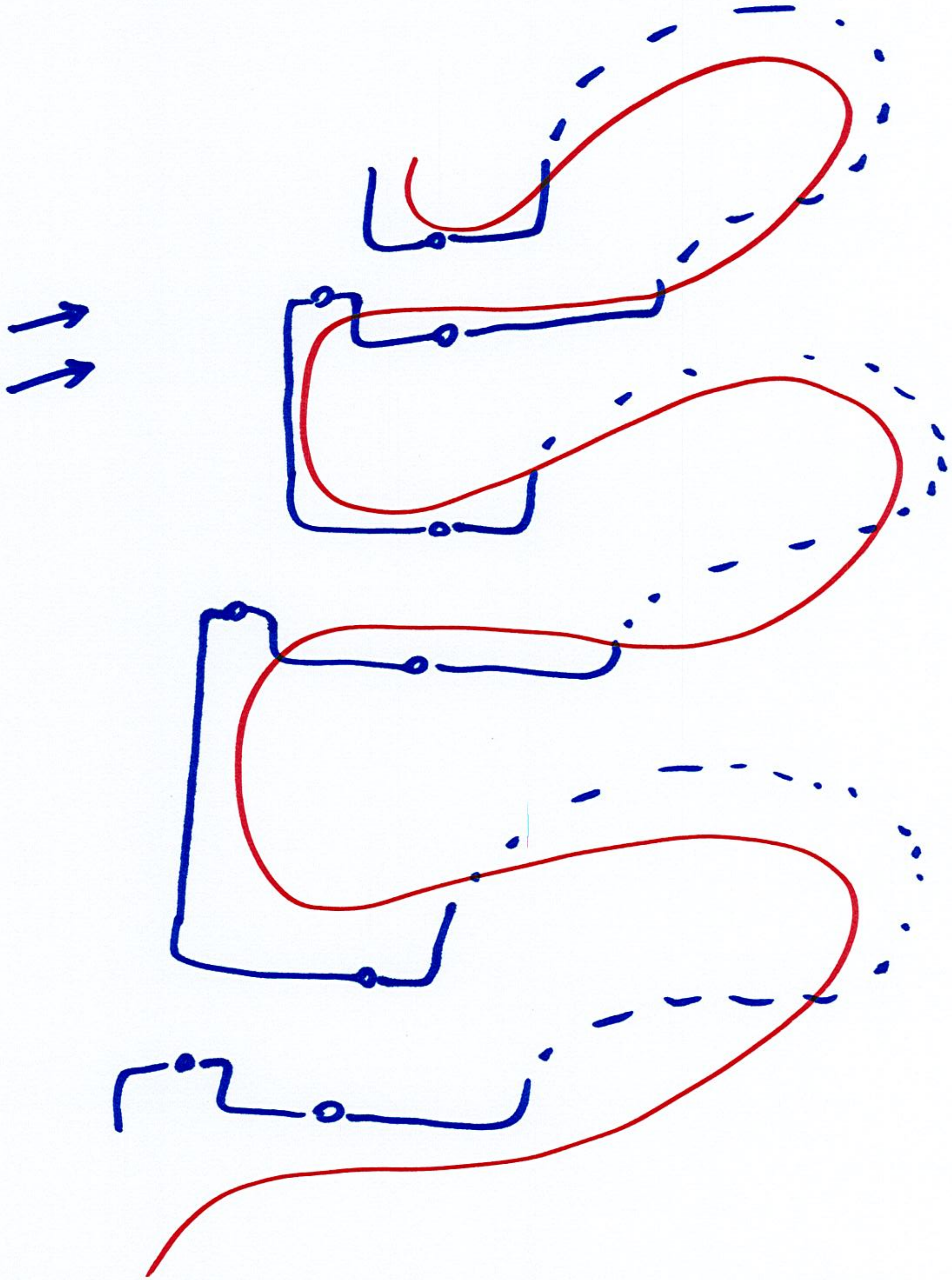
- By previous points :

  - $\underline{\#p \leq 2\#q^-}$

  - $\underline{\#q^- \leq cc(q^-) \cdot \#q}$

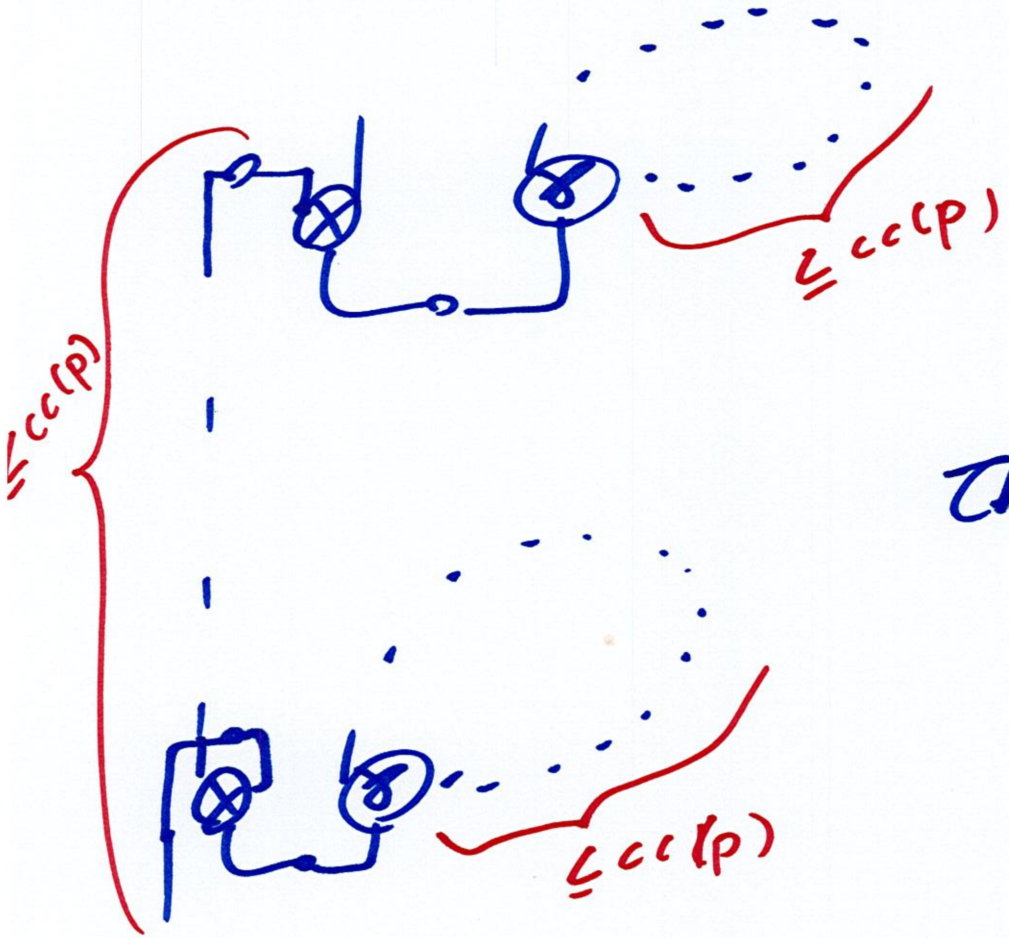
Difficulty =





CC( )
increases

Idea of the solution:



Then:

$(cc(q) \leq 2\, cc(p)!)$

Summary of the proof :

$$p \Rightarrow q^- \Rightarrow q$$

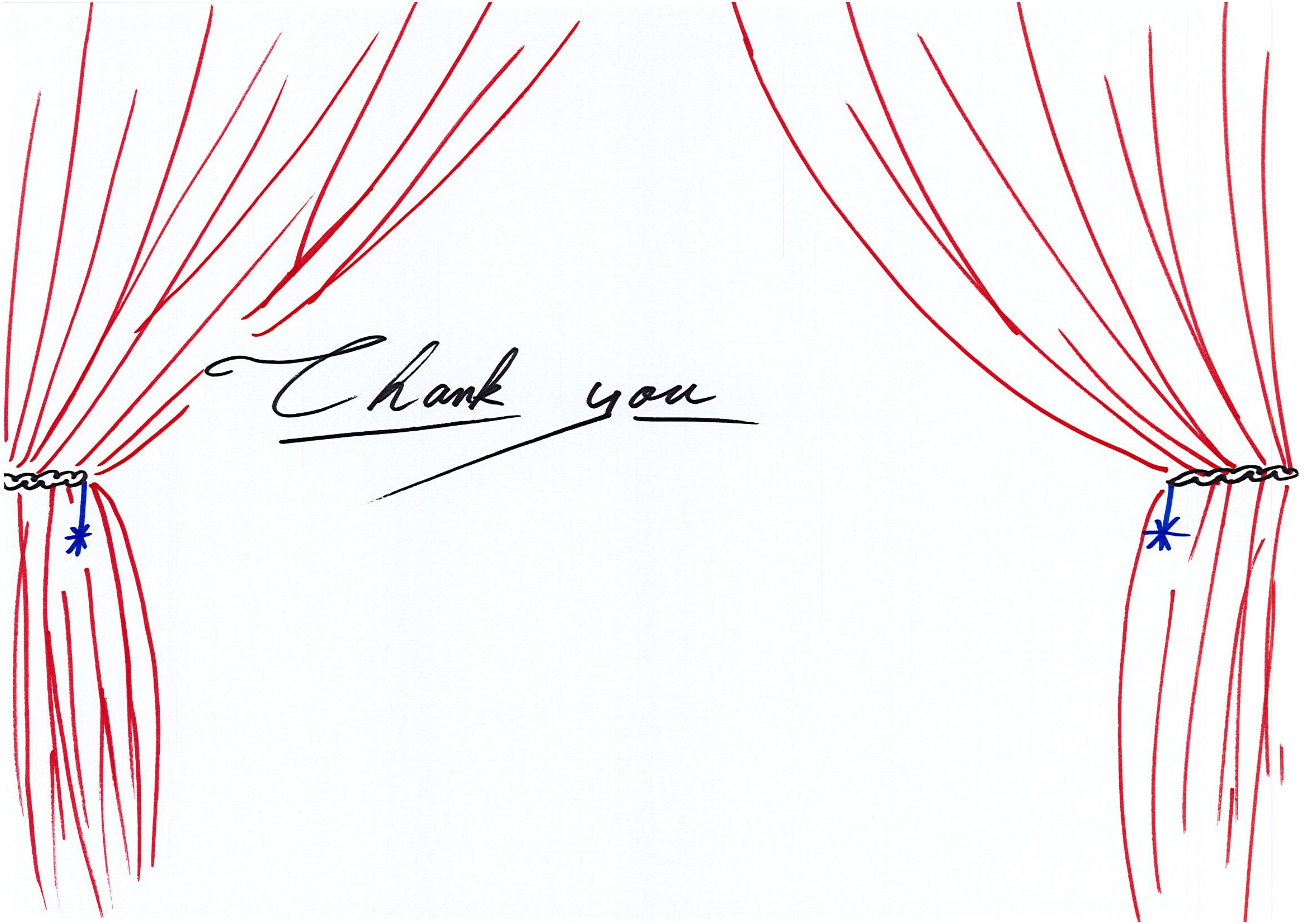
$\nearrow$ Mult. $\quad \nearrow$ Ax.

- $\# p \leq 2 \cdot \# q^-$
- $\# q^- \leq cc(q^-) \cdot \# q = 2\, cc(p)! \cdot \# q$

$$\boxed{\textbf{Theorem:} \quad \# p \leq 4\, cc(p)! \cdot \# q}$$

Summary of the talk

- $\tau(P) \Longrightarrow \Theta = \sum_{i \in I} a_i \cdot \pi_i$
- $\forall p \in \tau(P) \quad cc(p) \leq 2^{\#P}$
- $\forall p, q$ t.q $p \rightleftarrows q$, $\#p \leq 4\, cc(p)!\, \#q$
- $\boxed{\forall \pi_i \in \Theta \quad \max\left\{ \#p \mid p \in \tau(P),\, p \rightleftarrows \pi_i \right\} \leq 4\, (2^{\#P})! \cdot \#\pi_i}$
- $\forall \pi_i \in \Theta \quad \uparrow\uparrow \pi_i \cap \tau(P)$ is finite
- The coefficient of π_i in Θ is finite.


Thank you