On denotations of circular and non-wellfounded proofs

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based on joint work with Thomas Ehrhard and Alexis Saurin

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Tarski theorem

Let $(X, \leq)$ be a complete lattice, and F be an increasing function on X. Then the set P of all fixpoints F is a complete lattice.

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$$\frac{}{F(\mu X.F(X)) \leq \mu X.F(X)} \qquad \frac{F(S) \leq S}{\mu X.F(X) \leq S}$$

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$$\frac{\Delta, F(\nu X.F(X)) \vdash \Gamma}{\Delta, \nu X.F(X) \vdash \Gamma} \qquad \frac{S \vdash F(S)}{S \vdash \nu X.F(X)}$$

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$\Gamma \vdash \Delta \quad \rightsquigarrow \quad \vdash \Gamma^{\perp}, \Delta$:

$$\frac{\vdash F(\mu X.F(X)), \Gamma}{\vdash \mu X.F(X), \Gamma} \qquad \frac{\vdash S^{\perp}, F(S)}{\vdash S^{\perp}, \nu X.F(X)}$$

Cut-elimination fails...

$$\dfrac{\dfrac{}{\vdash 0, 0, \top}\ (\top) \qquad \dfrac{\dfrac{}{\vdash 0, \top}\ (\top)}{\vdash 0, \nu X.X}}{\vdash 0, 0, \nu X.X}\ (\text{cut})$$

$$\frac{\vdash F(\mu X.F(X)), \Gamma}{\vdash \mu X.F(X), \Gamma} \qquad \frac{\vdash S^{\perp}, F(S)}{\vdash S^{\perp}, \nu X.F(X)}$$

$$\downarrow$$

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$\mu\mathrm{LL}_\infty$ [1]

$$\frac{\vdash F(\mu X.F(X)), \Gamma}{\vdash \mu X.F(X), \Gamma} \qquad \frac{\vdash S^\perp, F(S)}{\vdash S^\perp, \nu X.F(X)}$$

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$$\frac{\vdash F(\mu X.F(X)), \Gamma}{\vdash \mu X.F(X), \Gamma}\ (\mu) \qquad \frac{\vdash \Gamma, F(\nu X.F(X))}{\vdash \Gamma, \nu X.F(X)}\ (\nu)$$

$+$ a possibility to have infinite trees.

[1] David Baelde, Amina Doumane, Alexis Saurin: Infinitary Proof Theory: the Multiplicative Additive Case.

Example

$\text{nat} = \mu X(1 \oplus X)$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\vdash 1}\,(1)
}{\vdash 1 \oplus \text{nat}}\,(\oplus_1)
}{\vdash \text{nat}}\,(\mu - \text{fold})
}{\vdash \text{nat}, \bot}\,(\bot)
\qquad\qquad
* \vdash \text{nat}, \text{nat}^{\bot}
}{
\dfrac{\vdash \text{nat}, \bot \mathbin{\&} \text{nat}^{\bot}}{* \vdash \text{nat}, \text{nat}^{\bot}}\,(\nu)
}\,(\&)
$$

But...

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \nu X.X}\,(\nu)}{\vdash \nu X.X}\,(\nu) \qquad \dfrac{\dfrac{\vdots}{\vdash \Gamma, \mu X.X}\,(\mu)}{\vdash \Gamma, \mu X.X}\,(\mu)}{\vdash \Gamma}\,(\mathbf{cut})$$

There is a validity criteria to specify “valid” proofs [2].

[2] David Baelde, Amina Doumane, Alexis Saurin: Infinitary Proof Theory: the Multiplicative Additive Case.

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Denotational semantics of non-wellfounded proofs in linear logic

Totality candidates on a set E

Given $\mathcal{T} \subseteq \mathcal{P}(E)$ we set

$$\mathcal{T}^{\perp} = \left\{u' \subseteq E \mid \forall u \in \mathcal{T}\ u \cap u' \neq \varnothing\right\}$$

Definition (Totality candidates)

$\mathcal{T}$ is a *totality candidate* for E if $\mathcal{T} = \mathcal{T}^{\perp\perp}$.

(Equivalently $\mathcal{T}^{\perp\perp} \subseteq \mathcal{T}$, equivalently $\mathcal{T} = \mathcal{S}^{\perp}$ for some $\mathcal{S} \subseteq \mathcal{P}(E)$.)

Fact

- *$\mathcal{T}$ is a totality candidate on E iff $\mathcal{T} \subseteq \mathcal{P}(E)$ and $\mathcal{T} = \uparrow\mathcal{T}$.*
- $\mathrm{Tot}(X)$ *(The set of all totality candidates on E), ordered with $\subseteq$, is a complete lattice (it is closed under arbitrary intersections).*

Non-uniform totality spaces (NUTS)

A NUTS is a pair $X = (|X|, \mathcal{T}X)$ where

- $|X|$ is a set
- $\mathcal{T}X$ is a totality candidate on $|X|$, that is, a $\uparrow$-closed subset of $\mathcal{P}(|X|)$.

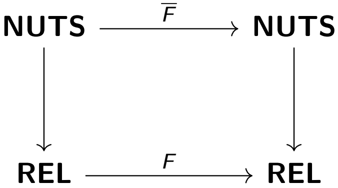
$t \in \mathbf{NUTS}(X, Y)$ if $t \in \mathbf{REL}(|X|, |Y|)$ and

$$\forall u \in \mathcal{T}X \quad t \cdot u \in \mathcal{T}Y$$

Fact

NUTS *is a model of LL where the proofs are interpreted exactly as in* **REL**.

Interpretation of $\mu X.F$ in **NUTS**



Interpretation of $\mu X.F$ in **NUTS**

$$\begin{array}{ccc}
\mathbf{NUTS} & \xrightarrow{\overline{F}} & \mathbf{NUTS} \\
\downarrow & & \downarrow \\
\mathbf{REL} & \xrightarrow{F} & \mathbf{REL}
\end{array}$$

$\overline{F} : (X, U) \mapsto (FX, \Phi U)$ where $\Phi U \in \mathcal{T}(FX)$.

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Assume μF exists.

$$g : \mathrm{Tot}(\mu F) \to \mathrm{Tot}(\mu F)$$

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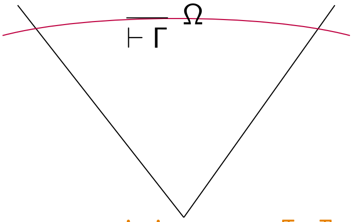
By Tarski theorem, μg exists.

$$\mu\overline{F} = (\mu F, \mu g).$$

NUTS as a denotational model of $\mu\mathrm{LL}_\infty$

$$\left[\!\!\left[\dfrac{\begin{array}{c}\vdots\ \pi \\ \vdash \Gamma, F[\mu X.F/\zeta]\end{array}}{\vdash \Gamma, \mu X.F}\ (\mu-\mathsf{fold}) \right]\!\!\right] = [\![\pi]\!] \qquad \left[\!\!\left[\dfrac{\begin{array}{c}\vdots\ \pi \\ \vdash \Gamma, F[\nu X.F/\zeta]\end{array}}{\vdash \Gamma, \nu X.F}\ (\nu-\mathsf{fold}) \right]\!\!\right] = [\![\pi]\!]$$



$$[\![\pi]\!]_{\textbf{REL}} = \bigcup_{\rho\in\mathsf{fin}(\pi)} [\![\rho]\!]_{\textbf{REL}}$$

Soundness of μLL_∞

Lemma

Let (π_i) be a Cauchy sequence. Then $[\![\lim_{n\to\infty} \pi_i]\!]_{\mathbf{REL}} = \bigcup_i \bigcap_{j>i} [\![\pi_j]\!]_{\mathbf{REL}}$.

Corollary

If π and π' are proofs of $\vdash \Gamma$ and π reduces to π' by the cut-elimination rules of μLL_∞, then $[\![\pi]\!]_{\mathbf{REL}} = [\![\pi']\!]_{\mathbf{REL}}$.

Theorem

If π is a valid proof of the sequent $\vdash \Gamma$, then $[\![\pi]\!] \in \mathcal{T}[\![\Gamma]\!]$.

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Inductive vs circular linear logic proofs

$$\text{Trans}\,() : \mu\text{LL} \to \mu\text{LL}_\infty$$

Given a $\pi \in \mu$LL, then Trans (π) can be defined by induction on π as it is done in [3].

$$\text{Trans}\left(\dfrac{\overset{\pi}{\vdash ?\Gamma, A^\perp, F[A/\zeta]}}{\vdash ?\Gamma, A^\perp, \nu\zeta\, F}\ (\nu - \text{rec}')\right) =$$

$$\dfrac{\overset{\pi}{\vdash ?\Gamma, A^\perp, F[A/\zeta]} \qquad \dfrac{\dfrac{* \vdash ?\Gamma, A^\perp, \nu\zeta\, F}{\vdash ?\Gamma, F[A/\zeta]^\perp, F[\nu\zeta\, F/\zeta]}\ (\mathfrak{F}_F)}{\vdash ?\Gamma, F[A/\zeta]^\perp, \nu\zeta\, F}\ (\nu - \text{fold})}{\dfrac{\vdash ?\Gamma, ?\Gamma, A^\perp, \nu\zeta\, F}{* \vdash ?\Gamma, A^\perp, \nu\zeta\, F}\ (\text{c})}\ (\text{cut})$$

[3] Amina Doumane. On the infinitary proof theory of logics with fixedpoints. PhD thesis, Université Paris Cité, 2017.

Inductive vs circular linear logic proofs

Theorem

Let π be a μLL proof. Then we have $[\![\pi]\!] = [\![\mathsf{Trans}(\pi)]\!]$ where the interpretation is given in a model $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ of μLL.

There is a transformation in the reverse direction for a fragment of $\mu\mathsf{LL}_\infty$ in [4].

Currently working:

Will the semantics be preserved via this reverse transformation?

[4] Rémi Nollet. Circular representations of infinite proofs for fixed-points logics : expressiveness and complexity. PhD thesis, Université Paris Cité, 2021.

An example

A syntatic-free proof that any term of booleans has a defined boolean value true or false

Consider $1 \oplus 1$ (The type of booleans).
$[\![1 \oplus 1]\!] = (\{(1, \star), (2, \star)\}, \mathcal{T}[\![1 \oplus 1]\!])$ where

$$\mathcal{T}([\![1 \oplus 1]\!]) = \mathcal{P}(|[\![1 \oplus 1]\!]|)\backslash\emptyset$$

For any proof π of $1 \oplus 1$, we have $[\![\pi]\!] \in \mathcal{T}[\![1 \oplus 1]\!]$.
Hence $[\![\pi]\!] \neq \emptyset$.

A future direction

Categorical model for circular proofs in linear logic with fixpoints.