On the geometry of validity criterion for non-wellfounded proofs

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μMALL$^\infty$ formulas & derivation rules

$$\phi, \psi ::= \phi \parr \psi \mid \phi \otimes \psi \mid \phi \& \psi \mid \phi \oplus \psi \mid \bot \mid 1 \mid \top \mid 0 \mid X \in \mathcal{V} \mid \mu X.\phi \mid \nu X.\phi.$$

μMALL$^\infty$ formulas & derivation rules

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$$\frac{}{\vdash A, A^\perp}\ \mathsf{ax} \qquad \frac{\vdash A, \Gamma \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ \mathsf{cut}$$

$$\frac{\vdash A, \Delta_1 \quad \vdash B, \Delta_2}{\vdash A \otimes B, \Delta_1, \Delta_2}\ \otimes \qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ \parr$$

$$\frac{\vdash A_1, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\ \oplus^1 \qquad \frac{\vdash A_2, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\ \oplus^2 \qquad \frac{\vdash A_1, \Gamma \quad \vdash A_2, \Gamma}{\vdash A_1 \& A_2, \Gamma}\ \&$$

$$\frac{\vdash A[X := \mu X.A], \Gamma}{\vdash \mu X.A, \Gamma}\ \mu \qquad \frac{\vdash A[X := \nu X.A], \Gamma}{\vdash \nu X.A, \Gamma}\ \nu$$

μMALL$^\infty$ formulas & derivation rules

$$\phi, \psi ::= \phi \mathbin{⅋} \psi \mid \phi \otimes \psi \mid \phi \mathbin{\&} \psi \mid \phi \oplus \psi \mid \bot \mid 1 \mid \top \mid 0 \mid X \in \mathcal{V} \mid \mu X.\phi \mid \nu X.\phi.$$

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Sequent will be set of occurrences, that are couples of a formula and an address characterizing an occurrence of a formula in the proof. We denote them by A, B, C, F, G, H.

Two-sided vs. One-sided presentation

Two sided sequent

At first, we had two-sided sequent:

$$\Gamma \vdash \Delta$$

but by orthogonality, it is possible to only consider one-sided sequent:

$$\vdash \Gamma^{\perp}, \Delta \qquad \Leftrightarrow \qquad \Gamma \vdash \Delta$$

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Transcription of rules

$$\frac{\Gamma_1, A_1 \vdash \Delta_1 \qquad \Gamma_2, A_2 \vdash \Delta_2}{\vdash \Gamma_1, \Gamma_2, A_1 \parr A_2 \vdash \Delta_1, \Delta_2} \parr_l \qquad \Leftrightarrow \qquad \frac{\vdash \Gamma_1^{\perp}, A_1^{\perp}, \Delta_1 \qquad \vdash \Gamma_2^{\perp}, A_2^{\perp} \vdash \Delta_2}{\vdash \Gamma_1^{\perp}, \Gamma_2^{\perp}, A_1^{\perp} \otimes A_2^{\perp}, \Delta_1, \Delta_2} \otimes_r$$

$$\frac{\Gamma, A[X := \mu X.A] \vdash \Delta}{\Gamma, \mu X.A \vdash \Delta} \mu_l \qquad \Leftrightarrow \qquad \frac{\vdash \Gamma^{\perp}, A^{\perp}[X := \nu X.A^{\perp}]\Delta}{\vdash \Gamma^{\perp}, \nu X.A^{\perp}, \Delta} \nu_r$$

Examples

$$\mathsf{Nat} := \mu X.1 \oplus X$$

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Inhabitant of natural number type

$$\pi_0 := \dfrac{\dfrac{\dfrac{}{\vdash 1}\,1}{\vdash 1 \oplus \mathsf{Nat}}\,\oplus_1}{\vdash \mathsf{Nat}}\,\mu \qquad \pi_{n+1} := \dfrac{\dfrac{\dfrac{\pi_n}{\vdash \mathsf{Nat}}}{\vdash 1 \oplus \mathsf{Nat}}\,\oplus_2}{\vdash \mathsf{Nat}}\,\mu$$

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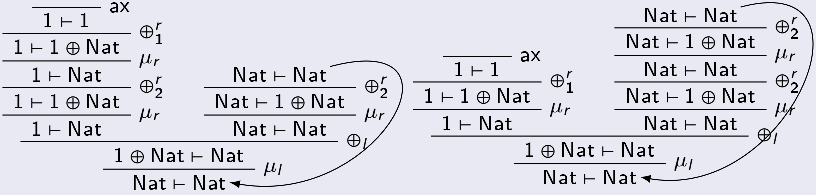
Some functions



Validity condition

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu X.X}}{\vdash \mu X.X}\,\mu \qquad \dfrac{\dfrac{\vdots}{\vdash \nu X.X, \Gamma}}{\vdash \nu X.X, \Gamma}\,\nu}{\vdash \Gamma}\,\mathsf{cut}$$

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Thread definition

A *straight* thread is a sequence of formula following a branch in a *pre-proof*.

Validity condition

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash \mu X.X\end{array}}{\vdash \mu X.X}\,\mu \qquad \dfrac{\begin{array}{c}\vdots\\ \vdash \nu X.X,\Gamma\end{array}}{\vdash \nu X.X,\Gamma}\,\nu}{\vdash \Gamma}\,\mathsf{cut}$$

Thread definition

A *straight* thread is a sequence of formula following a branch in a *pre-proof*.

Validity

A (straight) thread is said to be *valid* if its minimal formula (for sub-formula ordering) is infinitely active and is a ν-formula.
A branch is *valid* if there is a thread included in the branch which is valid.
A pre-proof is *valid* (and is a proof) if each of its infinite branch are valid.

Examples

We set :

$$\mathsf{Nat} := \mu X.1 \oplus X \qquad \mathsf{Stream\ Nat} := \nu X.\mathsf{Nat} \,\&\, X$$

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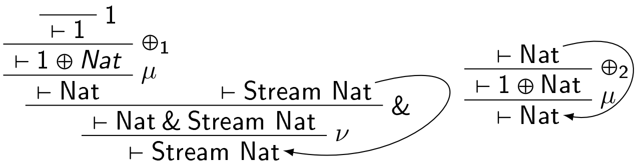

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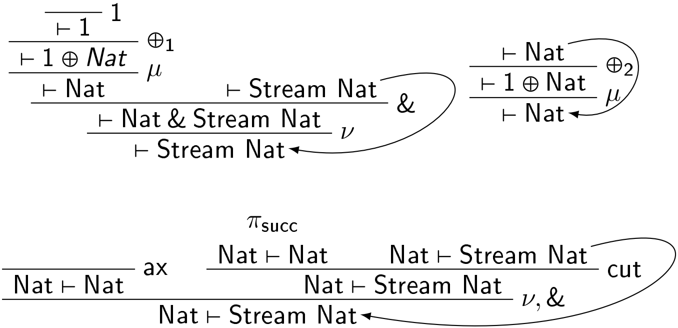

Property of the straight validity criterion

Cut-elimination (Baelde et al. 2016)

The cut-rule is admissible in $\mu\mathsf{MALL}^{\infty}$.

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The cut-rule is admissible in $\mu\mathsf{MALL}^{\infty}$.

Focalisation (Baelde et al. 2016)

$\mu\mathsf{MALL}^{\infty}$ admits the focalisation property.

Limit of the straight validity criterion

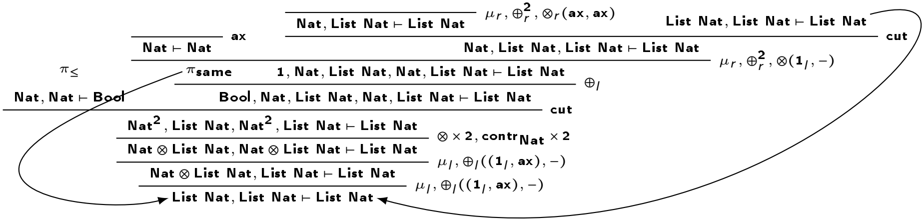
$$\dfrac{\dfrac{}{\vdash \nu X.X, \mu X.X}\,\mathsf{ax} \qquad \dfrac{\vdash \nu X.X}{\vdash \nu X.X}\,\nu}{\vdash \nu X.X}\,\mathsf{cut} \quad \equiv \quad \dfrac{\begin{array}{c}\vdots\\ \vdash \nu X.X\end{array}}{\vdash \nu X.X}\,\nu$$

Concrete example

Fusion $l_1\, l_2$ ::= match l_1 with
  | nil $\Rightarrow l_2$
  | Cons(a, l_1') $\Rightarrow$ match l_2 with
    | nil $\Rightarrow$ Cons(a, l_1')
    | Cons(b, l_2') match $\leq (a, b)$ with
      | true $\Rightarrow$ Cons$(a,$ Fusion l_1' Cons$(b, l_2'))$
      | false $\Rightarrow$ Cons$(b,$ Fusion Cons(a, l_1') $l_2')$.

Concrete example

$$\mathsf{List\ Nat} := \mu X.1 \oplus (\mathsf{Nat} \otimes X) \qquad \mathsf{Bool} := 1 \oplus 1$$



Bouncing pre-thread

Bouncing pre-thread definition

A *bouncing pre-thread* is a sequence of formula following a path in a proof-tree. The path can go from the bot to the top of the tree or from the top to the bot, but can only change its direction on an axiom or on a cut of the formula it follow.

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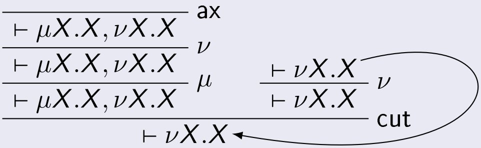
Example



h-path, b-path

A *b-path* is intuitively a pre-thread that will cancel out and be reduced during a cut-elimination procedure.
An *h-path* is a pre-thread that is an axiom followed by a *b*-path.

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Bouncing thread

A *bouncing thread* is a pre-thread that can be decomposed by a sequence $(H_i \odot V_i)_{i \in \omega}$ with H_i being an *h*-path and V_i being anything that is going upward and not meeting any axioms.

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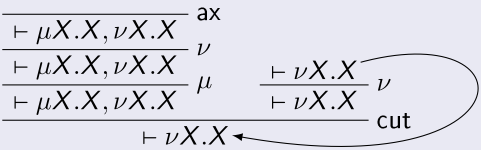
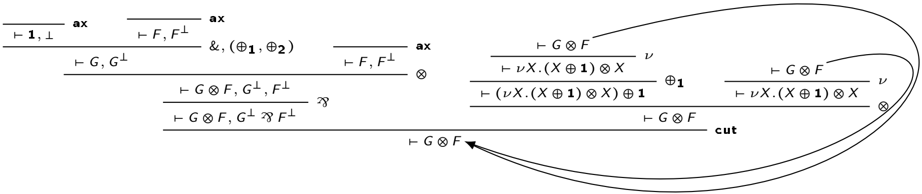
Example



Other examples

We set:

$$F := \nu X.(X \oplus 1) \otimes X \qquad \& \qquad G := F \oplus 1$$



Validity

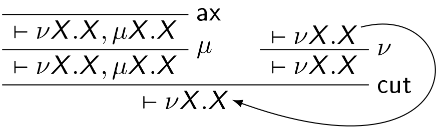
Thread validity

Let t be a bouncing-thread with the decomposition $(H_i \odot V_i)_{i\in\omega}$ with H_i being h-path. We call $(V_i)_{i\in\omega}$ *the visible part* of the thread t. ($(H_i)_{i\in\omega}$ is called the *hidden part* of the thread)
A bouncing-thread is *valid* if the least formula from (V_i) (for sub-formula ordering) is a ν-formula.

Validity

A μMALL^∞ pre-proof is *bouncing valid* if for each branches there is a valid bouncing-thread starting in it for which the visible part is completely included in the branch.

Examples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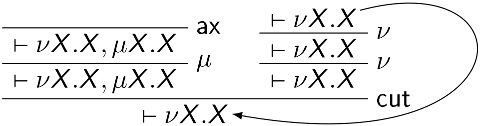
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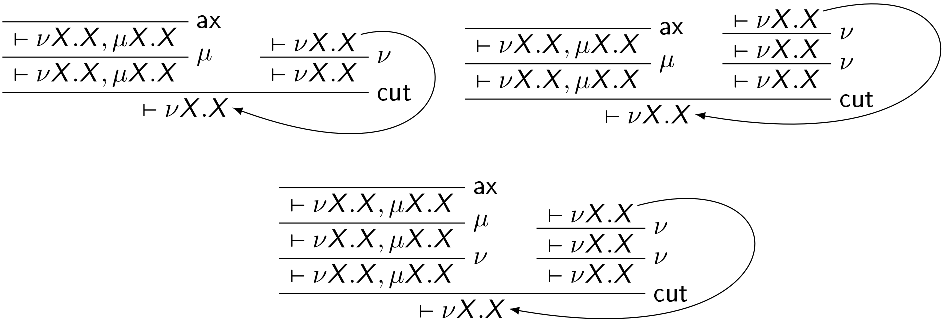
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Cut-elimination (Baelde et al. 2022)

The cut-rule is admissible.

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The cut-rule is admissible.

Multi-cut

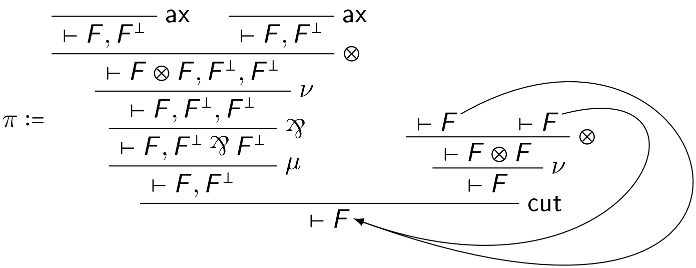
In order to prove the cut-elimination theorem, we use the multi-cut rule:

$$\dfrac{\vdash \Gamma_1, \Delta_1 \qquad \dots \qquad \vdash \Gamma_n, \Delta_n}{\vdash \Gamma_1, \dots, \Gamma_n} \text{ mcut}$$

with $\Delta_1, \dots, \Delta_n$ being the *cut-occurrences* and satisfying an acyclicity and connexity condition (relative to the cut relation).

Multi-cut in action

Taking $F := \nu X.X \otimes X$



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Multi-cut in action

Taking $F := \nu X . X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Trace of a reduction sequence

Trace

Let π be a proof and $(\pi_i)_{i\in\omega}$ be a reduction sequence starting from π, the trace of π is the sub-tree of π containing all the sequents appearing on top of a mcut-rule.

Example

For such a proof:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash (\nu X.X)_{\alpha\cdot i}, (\nu X.X)_{\gamma\cdot i}\end{array}}{\vdash (\nu X.X)_{\alpha}, (\nu X.X)_{\gamma\cdot i}}\,\nu}{\vdash (\nu X.X)_{\alpha}, (\nu X.X)_{\gamma}}\,\nu \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash (\nu X.X)_{\beta\cdot i\cdot i}, (\mu X.X)_{\gamma^\perp}\end{array}}{\vdash (\nu X.X)_{\beta\cdot i}, (\mu X.X)_{\gamma^\perp}}\,\nu}{\vdash (\nu X.X)_{\beta}, (\mu X.X)_{\gamma^\perp}}\,\nu}{\vdash (\nu X.X)_{\alpha}, (\nu X.X)_{\beta}}\,\text{cut}\;,$$

the trace would be (there is only one possible reduction sequence):

$$\dfrac{\vdash (\nu X.X)_{\alpha}, (\nu X.X)_{\gamma} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash (\nu X.X)_{\beta\cdot i\cdot i}, (\mu X.X)_{\gamma^\perp}\end{array}}{\vdash (\nu X.X)_{\beta\cdot i}, (\mu X.X)_{\gamma^\perp}}\,\nu}{\vdash (\nu X.X)_{\beta}, (\mu X.X)_{\gamma^\perp}}\,\nu}{\vdash (\nu X.X)_{\alpha}, (\nu X.X)_{\beta}}\,\text{cut}$$

Truncated proof semantic

Truncation

A truncation τ is a partial function from (occurrence) addresses to $\{0, \top\}$ satisfying $\tau(\alpha^\perp) = \tau(\alpha)^\perp$.

New rule

The rule system $\mu\mathsf{MALL}^\infty_\tau$, is the $\mu\mathsf{MALL}$ system where rules on occurrences having addresses in the domain of τ are forbidden, and where there is a new rule:

$$\frac{\vdash \Gamma, \tau(\alpha)}{\vdash \Gamma, F_\alpha}\ r_\tau$$

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Truncation associated to the trace

Let $(\pi_i)_{i\in\mathbb{N}}$ be a sequence of reduction, the truncation associated to the trace is the truncation τ where $\tau(\alpha) = \top$ if and only if α is the address of an occurrence in the trace, such that the rule on it is on the edge of the trace.

Example

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash (\nu X.X)_{\alpha\cdot i}, (\nu X.X)_{\gamma\cdot i}}}{\vdash (\nu X.X)_{\alpha}, (\nu X.X)_{\gamma\cdot i}}\ \nu}{\vdash (\nu X.X)_{\alpha}, (\nu X.X)_{\gamma}}\ \nu \qquad \dfrac{\dfrac{\dfrac{\vdots}{\vdash (\nu X.X)_{\beta\cdot i\cdot i}, (\mu X.X)_{\gamma^\perp}}}{\vdash (\nu X.X)_{\beta\cdot i}, (\mu X.X)_{\gamma^\perp}}\ \nu}{\vdash (\nu X.X)_{\beta}, (\mu X.X)_{\gamma^\perp}}\ \nu$$

$$\dfrac{\vdash (\nu X.X)_{\alpha}, (\nu X.X)_{\gamma} \qquad \vdash (\nu X.X)_{\beta}, (\mu X.X)_{\gamma^\perp}}{\vdash (\nu X.X)_{\alpha}, (\nu X.X)_{\beta}}\ \text{cut}$$

Example

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash (\nu X.X)_{\alpha\cdot i}, (\nu X.X)_{\gamma\cdot i}}}{\vdash (\nu X.X)_{\alpha}, (\nu X.X)_{\gamma\cdot i}}\,\nu}{\vdash (\nu X.X)_{\alpha}, (\nu X.X)_{\gamma}}\,\nu \quad \dfrac{\dfrac{\dfrac{\vdots}{\vdash (\nu X.X)_{\beta\cdot i\cdot i}, (\mu X.X)_{\gamma^\perp}}}{\vdash (\nu X.X)_{\beta\cdot i}, (\mu X.X)_{\gamma^\perp}}\,\nu}{\vdash (\nu X.X)_{\beta}, (\mu X.X)_{\gamma^\perp}}\,\nu$$

$$\dfrac{\vdash (\nu X.X)_{\alpha}, (\nu X.X)_{\gamma} \qquad \vdash (\nu X.X)_{\beta}, (\mu X.X)_{\gamma^\perp}}{\vdash (\nu X.X)_{\alpha}, (\nu X.X)_{\beta}}\,\text{cut}$$

By taking $\tau(\gamma) = \top, \tau(\gamma^\perp) = 0$, we have:

$$\dfrac{\dfrac{\dfrac{}{\vdash (\nu X.X)_{\alpha}, \top}\,\top}{\vdash (\nu X.X)_{\alpha}, (\nu X.X)_{\gamma}}\,r_\tau \quad \dfrac{\dfrac{\dfrac{\vdots}{\vdash (\nu X.X)_{\beta\cdot i\cdot i}, (\mu X.X)_{\gamma^\perp}}}{\vdash (\nu X.X)_{\beta\cdot i}, (\mu X.X)_{\gamma^\perp}}\,\nu}{\vdash (\nu X.X)_{\beta}, (\mu X.X)_{\gamma^\perp}}\,\nu}{\vdash (\nu X.X)_{\alpha}, (\nu X.X)_{\beta}}\,\text{cut}$$

Soundness

For a truncation τ and each μMALL$^\infty$ occurrences A, we associate a boolean $[\![A]\!]_\tau$ in $\{0, \top\}$.

Soundness

For a truncation τ and each $\mu\mathsf{MALL}^{\infty}$ occurrences A, we associate a boolean $[\![A]\!]_{\tau}$ in $\{0, \top\}$.

Soudness of bouncing $\mu\mathsf{MALL}^{\infty}$ system

For each proof of the sequent $\vdash A_1, \ldots, A_n$ of $\mu\mathsf{MALL}^{\infty}_{\tau}$, with τ coming from a truncation of a proof π, there is an i such that $[\![A_i]\!]_{\tau} = \top$.

Sketch of the proof for productivity

Lemma for productivity & validity

The truncated (pre-)proof generated by the trace of a reduction sequence is a proof

Sketch of the proof for productivity

Lemma for productivity & validity

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Suppose by contradiction that there is an infinite sequence of multicut reduction that does not produce anything and starting by π.

Sketch of the proof for productivity

Lemma for productivity & validity

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Suppose by contradiction that there is an infinite sequence of multicut reduction that does not produce anything and starting by π.
Let π_τ be the truncated proof of its trace. As no rules are produced, it means that none occurrences from the bottom sequent are principal in π_τ except on an axiom (by fairness).

Sketch of the proof for productivity

Lemma for productivity & validity

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The same is true for any occurrences bounded to those occurrences by an *h*-path, and occurrences bounded to those one by an *h*-path, and so on.

Sketch of the proof for productivity

Lemma for productivity & validity

The truncated (pre-)proof generated by the trace of a reduction sequence is a proof

Suppose by contradiction that there is an infinite sequence of multicut reduction that does not produce anything and starting by π.
Let π_τ be the truncated proof of its trace. As no rules are produced, it means that none occurrences from the bottom sequent are principal in π_τ except on an axiom (by fairness).
The same is true for any occurrences bounded to those occurrences by an *h*-path, and occurrences bounded to those one by an *h*-path, and so on.
If we replace all those occurrences by bottoms (and in particular the occurrences from the conlusion), we get a proof in $\mu\mathsf{MALL}^\infty_\tau$ of $\vdash \bot, \ldots, \bot$, contradicting the soundness of $\mu\mathsf{MALL}^\infty_\tau$.

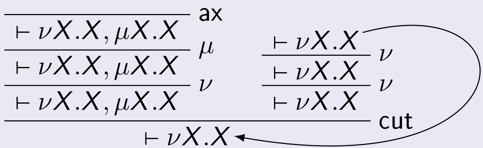
Table des matières

A new bouncing-criterion for $\mu\mathsf{MALL}^\infty$

We only change the definition of validity for proofs:

Extended validity

A $\mu\mathsf{MALL}^\infty$ pre-proof is *extended-bouncing valid* if for each branches, there is a valid bouncing-thread starting in it for which the visible part intersects the branch infinitely often.



Other example

PL'18, January 01–03, 2018, New York, NY, USA David Baelde, Amina Doumane, Denis Kuperberg, and Alexis Saurin

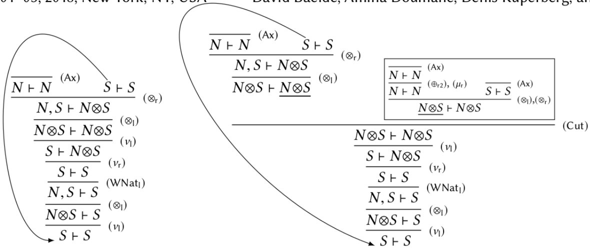


Figure 1. Example of a valid and an invalid circular pre-proof.

Cut-elimination

work in progress

The extended-bouncing system eliminates cut.

Cut-elimination

work in progress

The extended-bouncing system eliminates cut.

Lemma for cut-elimination

Let π be a proof and R a reduction sequence starting from π, if an infinite branch is contained in the trace of R, then each thread validating the branch is contained in the trace of R.

Cut-elimination

work in progress

The extended-bouncing system eliminates cut.

Lemma for cut-elimination

Let π be a proof and R a reduction sequence starting from π, if an infinite branch is contained in the trace of R, then each thread validating the branch is contained in the trace of R.

The truncated (pre-)proof associated to the trace R is extended-bouncing valid.

Table des matières

Definition of focalised proofs

A proof is focalised if the two conditions are satisfied:

- Each sequent containing a negative formula is conclusion of a negative rules.
- Each sequence of two subsequent sequents containing only positives are conclusion of rules acting on a formula and one of its sub-formula.

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a, a^\perp}\,\text{ax}}{\vdash a, a^\perp, \perp}\,\perp}{\vdash a, a^\perp \parr \perp}\,\parr \qquad \dfrac{}{\vdash 1}\,1}{\vdash a \otimes 1, a^\perp \parr \perp}\,\otimes \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash c, c^\perp}\,\text{ax}}{\vdash b \oplus c, c^\perp}\,\oplus_2}{\vdash b \oplus c, 0 \oplus c^\perp}\,\oplus_2}{\vdash a \oplus (b \oplus c), 0 \oplus c^\perp}\,\oplus_2$$

More example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash c, c^\perp}\,\mathrm{ax}}{\vdash c, c^\perp \oplus 0}\,\oplus_1}{\vdash b \oplus c, c^\perp \oplus 0}\,\oplus_2}{\vdash a \oplus (b \oplus c), c^\perp \oplus 0}\,\oplus_2 \qquad \dfrac{}{\vdash d, d^\perp}\,\mathrm{ax}}{\dfrac{\vdash (a \oplus (b \oplus c)) \otimes d, c^\perp \oplus 0, d^\perp}{\vdash (a \oplus (b \oplus c)) \otimes d, (c^\perp \oplus 0) \mathbin{⅋} d^\perp}\,⅋}\,\otimes
$$

Focalisation property for μMALL$^\infty$

Focalisation property

A logical system is said to satisfy the focalisation property if for each proof π of the system, there is a focalised proof π' such that π and π' are equivalent up to permutation of rule inferences.

Focalisation property for μMALL$^\infty$

Focalisation property

A logical system is said to satisfy the focalisation property if for each proof π of the system, there is a focalised proof π' such that π and π' are equivalent up to permutation of rule inferences.

Focalisation (Baelde et al. 2016)

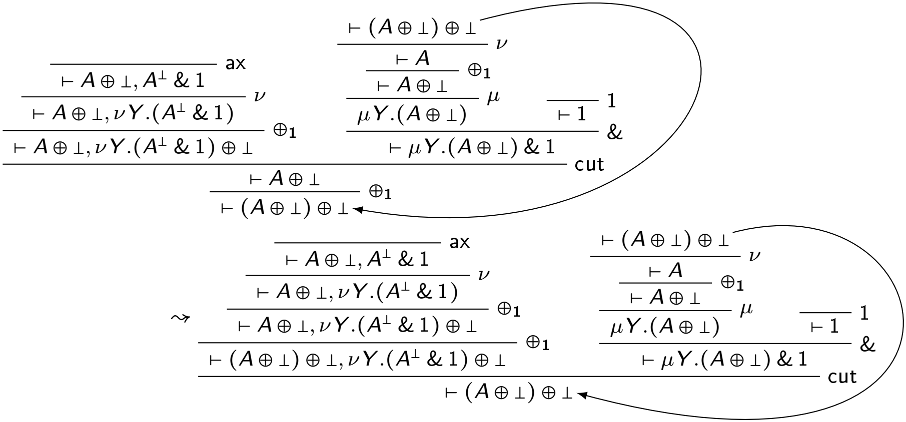
μMALL$^\infty$ with straight-thread validity admits the focalisation property.

Focalisation property for bouncing validity

Focalisation doesn't hold for bouncing validity criteria.

First counter-example to focalisation for bouncing-validity

with $A := \nu X.(X \oplus \bot) \oplus \bot$.



Second counter-example to focalisation for bouncing-validity

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash P, P^{\perp}}\,\text{ax}}{\vdash N, P^{\perp}}\,\nu}{\vdash N \oplus q, P^{\perp}}\,\oplus_1 \qquad \dfrac{\dfrac{}{\vdash q, q^{\perp}}\,\text{ax}}{\vdash N \oplus q, q^{\perp}}\,\oplus_2}{\vdash N \oplus q, P^{\perp} \,\&\, q^{\perp}}\,\& \qquad \dfrac{\dfrac{\vdash N \oplus q}{\vdash P}\,\mu}{\vdash P \oplus q}\,\oplus_1}{\vdash N \oplus q}\,\text{cut}$$

An intermediate validity criterion

A μMALL$^{\infty}$pre-proof is *intermediate-bouncing valid* if for each branches, there is a valid bouncing-thread starting in it for which the negative part of its visible part intersect the branch infinitely often.

Third counter-example to focalisation for bouncing-validity

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\dfrac{}{\vdash A_1^\perp, N'[N]}\,\textbf{ax}
}{\vdash A_1^\perp, N}\,\nu
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\dfrac{}{\vdash N^\perp, N}\,\textbf{ax}
}{\vdash A_1^\perp \,\&\, N^\perp, N}\,\&
}{\vdash (A_1^\perp \,\&\, N^\perp) \otimes (p \parr p^\perp), N \parr (p \otimes p^\perp)}\,\parr, \otimes(-, \textbf{ax})
}{\vdash ((A_1^\perp \,\&\, N^\perp) \otimes (p \parr p^\perp)) \parr (q \otimes q^\perp), N'[N]}\,\parr, \otimes(-, \textbf{ax})
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\dfrac{\vdash (N \parr (p \otimes p^\perp)) \otimes (q \parr q^\perp), p}{\vdash N, p}\,\nu
}{\vdash A_1 \oplus N, p}\,\oplus_2
}{\vdash (A_1 \oplus N) \parr (p \otimes p^\perp), p}\,\parr, \otimes(-, \textbf{ax})
}{\vdash ((A_1 \oplus N) \parr (p \otimes p^\perp)) \otimes (q \parr q^\perp), p}\,\otimes(-, (\parr, \textbf{ax}))
}{\vdash (N \parr (p \otimes p^\perp)) \otimes (q \parr q^\perp), p}\,\textbf{cut}
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\dfrac{}{\vdash N'^\perp[N^\perp], N'[N]}\,\textbf{ax}
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}{\vdash A_1^\perp \,\&\, N^\perp, N'[N]}\,\&
}{\vdash A_1^\perp \,\&\, N^\perp, N}\,\nu
}{\vdash (A_1^\perp \,\&\, N^\perp) \otimes (p \parr p^\perp), N \parr (p \otimes p^\perp)}\,\parr, \otimes(-, \textbf{ax})
}{\vdash ((A_1^\perp \,\&\, N^\perp) \otimes (p \parr p^\perp)) \parr (q \otimes q^\perp), N'[N]}\,\parr, \otimes(-, \textbf{ax})
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\dfrac{\vdash (N \parr (p \otimes p^\perp)) \otimes (q \parr q^\perp), p}{\vdash N, p}\,\nu
}{\vdash A_1 \oplus N, p}\,\oplus_2
}{\vdash (A_1 \oplus N) \parr (p \otimes p^\perp), p}\,\parr, \otimes(-, \textbf{ax})
}{\vdash ((A_1 \oplus N) \parr (p \otimes p^\perp)) \otimes (q \parr q^\perp), p}\,\otimes(-, (\parr, \textbf{ax}))
}{\vdash (N \parr (p \otimes p^\perp)) \otimes (q \parr q^\perp), p}\,\textbf{cut}
$$

Conclusion

Two new bouncing-criteria admitting cut-elimination property

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Counter-example for focalisation property for all those bouncing criteria

Conclusion

Two new bouncing-criteria admitting cut-elimination property

Counter-example for focalisation property for all those bouncing criteria

Not in this talk: We use those criteria to prove cut-elimination for bouncing-validity criteria in μLL

Conclusion

Two new bouncing-criteria admitting cut-elimination property

Counter-example for focalisation property for all those bouncing criteria

Not in this talk: We use those criteria to prove cut-elimination for bouncing-validity criteria in μLL

Future works: Adapting the infinitary proof system for reactive system typed with temporal logic (like the one from Cave et al. 2014)